Path Kernels and Partitions

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Abstract

Let $\tau(G)$ denote the number of vertices in a longest path of the graph $G = (V, E)$. A subset K of V is called a P_n-*kernel* of G if $\tau(G[K]) \leq n-1$ and every vertex $v \in V(G-K)$ is adjacent to an end-vertex of a path of order $n-1$ in $G[K]$. A partition $\{A, B\}$ of V is called an (a, b)-partition if $\tau(G[A]) \leq a$ and $\tau(G[B]) \leq b$. We show that any graph with girth greater than $n-3$ has a P_n-kernel and that every graph has a P_7-kernel. As corollaries of these results we show that if $\tau(G) = a + b$ and G has girth greater than $a - 2$ or $a \leq 6$, then G has an (a, b)-partition.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

1 Introduction

Let $G = (V, E)$ be a finite simple graph. We denote the number of vertices in a longest path (which need not be an induced path) in G by $\tau(G)$. The *girth* $g(G)$ and the *circumference* $c(G)$ are, respectively, the length of a shortest and a longest cycle in G. The cycle of order n and the path of order n are denoted by C_n and P_n respectively. We shall call a vertex $v \in V$ a P_n-*terminal vertex of* G if v is an end-vertex of a P_n but not of a P_{n+1} in G.

If S is any subset of the vertex set $V(G)$, we denote the subgraph of G induced by S by $G[S]$. We denote the distance between two vertices v and

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w by $d(v, w)$, and we define the *distance between a vertex v of G and a subset S of $V(G)$* by $d(v, S) = min\{d(v, x) | x \in S\}$. The *open neighbourhood* of a vertex v is defined as the set of vertices $N(v) = \{u \in V(G) | uv \in E(G)\}$. Further, for any subset A of $V(G)$, the *open neighbourhood of A* is the set $N(A) = \cup_{a \in A} N(a)$, and the *closed neighbourhood of A* is the set $N[A] = N(A) \cup A$. A *block* of a graph G is a maximal nonseparable subgraph of G. A block with exactly one cut-vertex of G is an *end block* of G. Further graph theoretic definitions may be found in [5].

A class $\mathcal{P}$ of graphs is called a *hereditary class* if every subgraph of a graph in $\mathcal{P}$ is also in $\mathcal{P}$.

Given any pair of positive integers (a, b), we call a partition $\{A, B\}$ of $V(G)$ an *(a, b)-partition* if $\tau(G[A]) \leq a$ and $\tau(G[B]) \leq b$. If G can be (a, b)-partitioned for every pair of positive integers (a, b) satisfying $a + b = \tau(G)$, we say that G is *τ-partitionable*. The following conjecture is stated in [1], [2] and [3].

Conjecture 1 *Every graph is τ-partitionable.*

In 1968 Chartrand, Geller and Hedetniemi [4] defined the k-chromatic number of a graph G, $\chi_k(G)$, as the smallest number of sets in a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $\tau(G[V_i]) \leq k$ for each i. This is clearly related to our problem. In fact, the upper bound $\chi_k(G) \leq \lfloor (\tau(G) - 1 - k)/2 \rfloor + 2$, given in Theorem 2 of [4] can be improved to $\chi_k(G) \leq \lceil \tau(G)/k \rceil$ if Conjecture 1 is true.

For any graph G, a subset K of $V(G)$ is called a *P_n-kernel* of G if

1. $\tau(G[K]) \leq n - 1$ and
2. every vertex $v \in V(G - K)$ is adjacent to a P_{n-1}-terminal vertex of $G[K]$.

Note that if $\tau(G) < n$, then $V(G)$ is a P_n-kernel of G, and if $\tau(G) \geq n$ equality is necessary in (1) above. Further, a maximal independent set of vertices is a P_2-kernel for any graph, and similarly the vertices of a maximal matching, together with any isolates, furnish any graph with a P_3-kernel.

The following conjecture is stated in [3] and in [6].

Conjecture 2 *Every graph has a P_n-kernel for every integer $n \geq 2$.*

We shall call a subset S of $V(G)$ a *P_n-semikernel* of G if

(1) $\tau(G[S]) \leq n - 1$ and

(2) every vertex in $N(S) - S$ is adjacent to a P_{n-1}-terminal vertex of $G[S]$.

Conjecture 3 *Every graph has a P_n-semikernel for every integer $n \geq 2$.*

Again we see that equality is necessary in (1) above if $\tau(G) \geq n$. Note that a graph has a P_n-kernel if and only if each of its (connected) components has a P_n-kernel; therefore we shall only consider connected graphs in the sequel.

In this paper we investigate relationships among path partitions, kernels and semikernels as well as relationships among the three conjectures stated above. We provide two classes of graphs for which Conjecture 2 holds, and obtain results concerning path semikernels which have important consequences for both path kernels and path partitions. Our main result (which is proved via P_n-semikernels) is that every graph with girth greater than $n-3$ has a P_n-kernel. A consequence of this result is that every graph G with girth greater than $\lfloor \frac{\tau(G)}{2} \rfloor - 2$ is τ-partitionable. We also prove that every graph has a P_n-kernel for $n \leq 7$, which implies that every graph G with $\tau(G) \leq 13$ is τ-partitionable.

2 Relationships among path kernels, semikernels and partitions

We have the following relationship between path kernels and path partitions.

Proposition 2.1 *Let G be a graph with $\tau(G) = a + b$, where a and b are positive integers. If G has a P_{a+1}-kernel or a P_{b+1}-kernel, then G is (a, b)-partitionable.*

Proof. Suppose G has a P_{a+1}-kernel K. Since b is positive we know that $\tau(G) > a$, and thus $\tau(G[K]) = a$. Now suppose $G - K$ has a path of order greater than b. Let x be an end-vertex of such a path. Then x is adjacent to a P_a-terminal vertex of $G[K]$. But then G has a path of order greater than $a + b$, contradicting our assumption that $\tau(G) = a + b$. This proves that $\tau(G - K) \leq b$, and hence $\{K, V(G) - K\}$ is an (a, b)-partition of G. □

Corollary 2.2 *If Conjecture 2 is true, then Conjecture 1 is true.*

We do not know whether the converse of Proposition 2.1 is true.

Obviously, any P_n-kernel of a graph is also a P_n-semikernel of the graph, but the converse does not hold. However, in [7] it is proved that if, for some n, every graph has a P_n-semikernel, then every graph has a P_n-kernel. We prove a slightly stronger result.

Proposition 2.3 *Let $\mathcal{P}$ be a hereditary class of graphs and $n \geq 2$ any integer. If every graph in $\mathcal{P}$ has a P_n-semikernel, then every graph in $\mathcal{P}$ has a P_n-kernel.*

Proof. The proof is by induction on the order of the graph. Suppose G is a graph in $\mathcal{P}$ and S is a P_n-semikernel of G. If $|V(G)| < n$, then $\tau(G) < n$ and the vertex set of G is a P_n kernel of G. Now assume $\tau(G) \geq n$.

If $G - N[S]$ is empty, S is a P_n-kernel of G. Otherwise, since the graph $G - N[S]$ is in $\mathcal{P}$, it also has a P_n-semikernel and hence, by the induction hypothesis, $G - N[S]$ has a P_n-kernel K. Since there are no edges between S and K, the set $S \cup K$ is a P_n-kernel of G. □

Corollary 2.4 *If every graph has a P_n-semikernel for $n \geq 2$, then every graph has a P_n-kernel for $n \geq 2$.*

We also have the following useful relationship between semikernels and path partitions.

Proposition 2.5 *Suppose G is a graph with $\tau(G) = a + b$; $1 \leq a \leq b$. If G has a P_{b+1}-semikernel, then G is (a, b)-partitionable.*

Proof. Let S_0 be a P_{b+1}-semikernel of G. Again, since a is positive, we know that $\tau(G) > b$, and hence that $\tau(G[S_0]) = b$. Put

$$S_i = \{v \in V(G) | d(v, S_0) = i\}.$$

Recall that we assume G is connected. Suppose that for some j, $G[S_j]$ has a path of order greater than a. Let v_j be an end-vertex of such a path. Then there is a path $v_j, v_{j-1}, \ldots v_1$ in G with $v_i \in S_i$; $i = 1, \ldots j$. Since $S_1 = N(S_0) - S_0$, the vertex v_1 is adjacent to a P_b-terminal vertex in S_0. But then G contains a path of order greater than $a + b$. This contradiction proves that

$$\tau(G[S_i]) \leq a \text{ for all } i \geq 1.$$

Now put

$$A = \bigcup_{i \text{ odd}} S_i \text{ and } B = \bigcup_{i \text{ even}} S_i.$$

Then $\{A, B\}$ is an (a, b)-partition of G. □

We do not know whether the converse of Proposition 2.5 is true.

3 Graphs that have P_n-kernels for all n

In [2] we proved that several classes of graphs are τ-partitionable. Our attempts at finding classes of graphs that have P_n-kernels for all n were less successful. For example, it is obvious that Hamiltonian graphs are τ-partitionable, but we know very little about the existence of P_n-kernels in Hamiltonian graphs. In [2] we also proved that if every block of a graph G is either K_2 or is Hamiltonian, then G is τ-partitionable. For path kernels we have the following analogous result.

Theorem 3.1 *If every block of a graph G is either a complete graph or a cycle, then G has a P_n-kernel for every integer $n \geq 2$.*

Proof. Consider any integer $n \geq 2$. The proof is by induction on the order of G. If $|V(G)| < n$, then the vertex set of G is a P_n-kernel of the graph.

If G has only one block, then G is a complete graph or a cycle. If G is a complete graph, then any $n-1$ vertices furnish a P_n-kernel. If G is a cycle, a P_n-kernel can be constructed by taking subpaths of length $n-1$ (or less for the last) and skipping every n^{th} vertex.

Now suppose G has more than one block. Let B be an end-block of G. We consider two cases:

Case 1. B is a complete graph: Let v be any vertex in B except the cut-vertex of G in B. By the induction hypothesis, $G - v$ has a P_n-kernel K. If v is adjacent to a P_{n-1}-terminal vertex in K, then K is a P_n-kernel of G. If not, then v is not a P_n-terminal vertex of $G[K \cup \{v\}])$. But if there were a P_n in $G[K \cup \{v\}]$ then, since B is an end-block and B is a complete graph, there would be a P_n in $G[K \cup \{v\}]$ with v as end-vertex. Thus $\tau(G[K \cup \{v\}]) \leq n-1$, and hence $K \cup \{v\}$ is a P_n-kernel of G.

Case 2. B is a cycle: Let x be the cut-vertex of G in B. By the induction hypothesis, $G - (B - x)$ has a P_n-kernel K'. If $x \notin K'$, let P be the path $B - x$, and let K consist of all the vertices in K' and all the vertices in P except for every n'th vertex of P. If $x \in K'$, let k be the order of a longest path in K' with end-vertex x. If $|B| \leq n-k$, let $K = K' \cup B$. If $|B| > n-k$, let Q be the path $B - x = b_1, b_2, \ldots, b_r$, with b_1 adjacent to x. Initialize K with the union of K' and the first $n-1-k$ vertices of the path Q. Skip the next vertex of Q. If there are more than $n-1$ subsequent vertices in Q, add the next $n-1$ vertices to K and skip the next vertex of Q. Continue in this way until the number of vertices remaining in Q is less than or equal to $n-1$. Let $t = \min\{k-1, n-1-k\}$. If there are no more than t vertices remaining in Q, add them to K. Since $t \leq k-1$, these vertices together with x and the first subpath of B will not create a path of order n. If there are more than t vertices remaining in Q, add the last t vertices of Q to K. Skip b_{r-t}, then add those remaining vertices of Q not previously skipped to K. In each case, K is a P_n-kernel of G. □

We have two immediate corollaries to Theorem 3.1.

Corollary 3.2 *If G is a graph with circumference $c(G) \leq 3$, then G has a P_n-kernel for every integer $n \geq 2$.*

Corollary 3.3 *If G is a graph without even cycles, then G has a P_n-kernel for every integer $n \geq 2$.*

Proof. If a connected graph has no even cycles then each of its blocks is either a K_2 or an odd cycle. □

In [2] we proved that a join of any two graphs is τ-partitionable. We now prove that a join of edgeless graphs has a P_n-kernel for every $n \geq 2$.

Theorem 3.4 *Let G be a complete multipartite graph. Then G has a P_n-kernel for every integer $n \geq 2$.*

Proof. Every graph has a P_2-kernel and a P_3-kernel (recall the comment following Conjecture 2). So we may assume $4 \leq n \leq \tau(G)$. Let $V_1, \ldots V_m$ be the partite sets of G. If G has a path P of order $n-1$ with its end-vertices in two different V_i, then every vertex of $G - V(P)$ is adjacent to an end-vertex of P, and so $V(P)$ is a P_n-kernel. We may therefore assume that every path of order $n-1$ in G has its two end-vertices in the same partite set. Let P be a path of order $n-1$ with both end-vertices in V_1, say. If every vertex of V_1 is in P, then $V(P)$ is a P_{n-1}-kernel of G. Therefore we may assume that there is a vertex $u \in V_1$ that is not in P. Let P be the path $x_1 \ldots x_{n-1}$. Suppose, for some $i \leq n-3$, neither x_i nor x_{i+1} is in V_1. Then the path $x_1 \ldots x_i u x_{i+1} \ldots x_{n-2}$ is a path of order $n-1$ with end-vertices in different V_i, contradicting our assumption. This proves that $x_1, x_3, \ldots x_{n-3}, x_{n-1}$ are all in V_1. Therefore $n-1$ is odd and more than half of the vertices of P lie in V_1. Now put $K = V(P) \cup V_1$. Then $\tau(G[K]) = n-1$ and every vertex in $G - K$ is adjacent to an end-vertex of P. Thus K is a P_n-kernel of G. □

4 Cycle lengths and path kernels

We see in Corollary 3.2 that if a graph has no large cycles, then it has a P_n-kernel for every integer $n \geq 2$. In this section we explore further connections between the cycle structure of a graph and the existence of path kernels. First we note an obvious result.

Proposition 4.1 *If C is an $(n-1)$-cycle in a graph G, then C is a P_n-semikernel of G.*

As an immediate corollary of the above result and Proposition 2.5, we have the following result, which appeared as a theorem in [2].

Corollary 4.2 *Let G be a graph with $\tau(G) = a + b$; $a \leq b$. If G has a b-cycle, then G is (a,b)-partitionable.*

Our next goal is to show that if a graph has no small cycles, then the graph has path kernels. We begin with some lemmas on semikernels.

Lemma 4.3 *If G is a graph with girth $g(G) = n-2$, then G has a P_n-semikernel.*

Proof. We may assume that $\tau(G) \geq n$. It is straightforward to prove that every graph has a P_n-kernel for $n \leq 5$ and it is proved in [7] that every graph has a P_6-kernel. We therefore assume that $n > 6$. Let C be a (chordless) $(n-2)$-cycle in G. Initially, we let S be the set of vertices of C, we let $B = V(G) - S$ and $A = \emptyset$. We now describe a method of moving vertices from B to S and to A without creating a P_n in $G[S]$, until eventually $N(S) \cap B = \emptyset$, while every vertex in A is adjacent to a P_{n-1}-terminal vertex of S.

STEP 1: If $N(S) \cap B = \emptyset$, then stop. Otherwise, choose one vertex, say x, of C that has a neighbour in B and move all B-neighbours of x to S.

STEP 2: Mark all the P_{n-1}-terminal vertices of $G[S]$ and then move all their B-neighbours to A. Then return to STEP 1.

Since $g(G) = n-2$ and we are assuming that $n-2 > 4$, no vertex in $V(G)-V(C)$ is adjacent to two different vertices of C and no two neighbours of the same vertex of C are adjacent to one another. Thus moving vertices to S in the prescribed manner will not create a P_n in $G[S]$, since every vertex that we move to S will be adjacent to only one vertex of S and that vertex is a P_{n-2}-terminal vertex of $G[S]$. (Recall that the B-neighbours of all P_{n-1}-terminal vertices of $G[S]$ were moved to A in Step 2.) When $B \cap N(S)$ becomes empty, then $N(S) - S = A$, and so S will be a P_n-semikernel of G. □

Lemma 4.4 *If G is a graph and $g(G) > n-1$, then G has a P_n-semikernel.*

Proof. We may assume $\tau(G) \geq n-1$, and let P be a path of order $n-1$ in G with vertices $x_1, \ldots, x_{n-1}$. For $i = 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, put

$$A_i = \{v \in G - P | d(v, x_i) \leq i-1\}$$

and for $i = \lceil \frac{n}{2} \rceil, \ldots, n-2$, put

$$A_i = \{v \in G - P | d(v, x_i) \leq n-1-i\}.$$

Note that $A_i \cap A_j = \emptyset$ for $i \neq j$, since G has no cycles of length less than n.

Now put

$$S = \cup_{i=2}^{n-2} A_i \cup P$$

Then the distance between any two vertices in S is less than $n-1$ and (since $g(G) > n-1$) there are no cycles in S. Thus S is a tree and $\tau(G[S]) = n-1$.

Also, every vertex in $N(S) - S$ is adjacent to a P_{n-1}-terminal vertex in S. Thus S is a P_n-semikernel of G. □

Since the class of graphs with girth equal to $n - 2$ is not a hereditary class, we cannot apply Proposition 2.3 directly to the result of Lemma 4.3. However, graphs with girth *not less than* $n - 2$ do form a hereditary class so we can now prove:

Theorem 4.5 *If G is a graph with $g(G) \geq n-2$, then G has a P_n-kernel.*

Proof. It follows from Lemma 4.3, Lemma 4.4 and Proposition 4.1 that if G is any graph with $g(G) \geq n - 2$, then G has a P_n-semikernel. Now we apply Proposition 2.3 and the theorem is proved. □

Since every tree has infinite girth we have

Corollary 4.6 *If T is a tree, then T has a P_n-kernel for all n.*

Applying Proposition 2.1, we further obtain

Corollary 4.7 *Let G be a graph and suppose $\tau(G) = a+b$ with $1 \leq a \leq b$. If $g(G) \geq a - 1$, then G is (a, b)-partitionable.*

Corollary 4.8 *If G is a graph such that $g(G) \geq \lfloor \frac{\tau(G)}{2} \rfloor - 1$, then G is τ-partitionable.*

The following result is proved in [2].

Theorem 4.9 *Let G be a graph and suppose $\tau(G) = a+b$ with $1 \leq a \leq b$. If $c(G) \leq a + 1$, then G is (a, b)-partitionable.*

As a consequence of Theorem 4.9 and Corollary 4.7 we have

Corollary 4.10 *If G is a graph with $c(G) - g(G) \leq 3$, then G is τ-partitionable.*

Proof. Suppose $\tau(G) = a + b$ with $1 \leq a \leq b$. If $g(G) \geq a - 1$, then G is (a, b)-partitionable by Corollary 4.7. If $g(G) < a - 1$, then $c(G) \leq a + 1$ and then G is (a, b)-partitionable by Theorem 4.9. □

5 The existence of P_n-kernels for small values of n

Recall that every graph has a P_2-kernel and a P_3-kernel. It is also straightforward to prove that every graph has a P_4-kernel and a P_5-kernel. It is proved in [7], by means of an elaborate 10-page proof, that every graph has a P_6-kernel. We can construct a shorter proof for the latter by making use of Proposition 2.3, Lemma 4.4, and Proposition 4.1. Using these results we can also prove

Theorem 5.1 *Every graph has a P_7-kernel.*

Proof. In light of Proposition 2.3 it is sufficient to prove that every graph has a P_7-semikernel. Let G be any graph with $\tau(G) \geq 7$. In view of Theorem 4.5 and Proposition 4.1, we may assume that G has girth less than 5 and that G does not have a C_6.

Consider the graphs in Figures 1-3 below. In each sketch the darkened vertices represent P_6-terminal vertices. (There may be more, depending on whether other adjacencies occur, e.g. in H_1 the vertex x_4 will also be P_6-terminal if x_5 is adjacent to x_3.) We first present the cases that we need to consider and then we describe a procedure for constructing a P_7-semikernel that applies to all cases.

Case 1. G has a C_5, but no C_6: Since G is connected and has a P_7, the graph of Figure 1 must be a subgraph of G.

Case 2. G has a C_4, but no C_5 or C_6: Again since G is connected and has a path of order seven, at least one of the graphs in Figure 2 will be a subgraph of G.

Case 3. G has a C_3, but no C_4, C_5 or C_6: At least one of the graphs in Figure 3 will be a subgraph of G.

Now let $S = H_s$, where s is the smallest integer such that H_s is a subgraph of G.

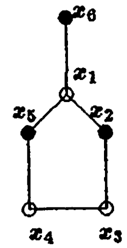
Thus if $S = H_3$, we may assume that x_2 and x_4 have no neighbours in G other than each other and x_1 and x_3, since otherwise G would have H_2 as a subgraph. Similarly, when considering H_7, H_8 or H_9, we assume that x_1 and x_4 have no common neighbours, otherwise H_6 is a subgraph of G. And in subgraph H_8, x_4 and x_5 have no common neighbours, since otherwise H_7 is a subgraph of G.

For subgraphs H_2, H_3, H_4 or H_5, we may assume that no vertex outside the subgraph is adjacent to two consecutive vertices of the 4-cycle (otherwise a C_5 occurs). Finally, in H_5, we may assume that no vertex is adjacent to both x_1 and x_5 (otherwise H_4 occurs). Also in this case x_3 is not adjacent to x_5, otherwise H_2 occurs.

Initially we let $B = V(G) - S$ and $A = \emptyset$. Then we move vertices from B to S and to A according to the following procedure.

STEP 1: Identify all the P_6-terminal vertices of S and move all their B-neighbours to A. If $N(S) \cap B = \emptyset$, then stop. Otherwise proceed to STEP 2.

STEP 2: If two vertices x and y in S have a common B-neighbour, then move one common B-neighbour of x and y to S and return to STEP 1.



H_1

Figure 1: C_5 but no C_6

Otherwise proceed to STEP 3.

STEP 3: If some P_5-terminal vertex x of S has a B-neighbour, then move one B-neighbour of x to S and return to STEP 1. Otherwise proceed to STEP 4.

STEP 4: If some P_4-terminal vertex x of S has a B-neighbour, then move one B-neighbour of x to S and return to STEP 1. Otherwise go to STEP 2.

We claim that when $N(S) \cap B$ becomes empty, S will be a P_7-semikernel of G. Clearly, every vertex in A will be adjacent to a P_6-terminal vertex in S. It remains only to show that $\tau(G[S]) = 6$, so our next goal is to demonstrate that no P_7 could have been created in S.

STEPS 3 and STEPS 4 are only performed when no vertex in B is adjacent to more than one vertex of S or to any P_6-terminal vertex of S. Performing STEPS 3 or 4 will therefore not create a P_7 in $G[S]$. A close scrutiny of the different cases also shows that a P_7 will not be created in STEP 2:

In Case 1, if x_2 is adjacent to x_5 or x_4, then x_3 is a P_6-terminal vertex. Suppose that at STEP 2 there is a vertex b in B that is adjacent to two vertices in S. Then since STEP 1 has been executed at least once and since G has no C_8, b must be adjacent to x_1 and to x_3 or x_4, say to x_3. And x_2 is not adjacent to x_5 or to x_4 (otherwise the B-neighbours of x_3 would have been moved to A in STEP 1). We can therefore add b to S without creating a P_7. And b as well as x_4 become P_6-terminal vertices of S, so all their B-neighbours will then be moved to A. Thus in subsequent iterations STEP 2 will only be executed if a vertex in B is adjacent to x_1 as well as to x_3. So performing STEP 2 will never create a P_7 in $G[S]$.

In Case 2, suppose at STEP 2 that B has a vertex b adjacent to more than one vertex of S. In H_2, we may assume that b is adjacent to x_1 and x_3.

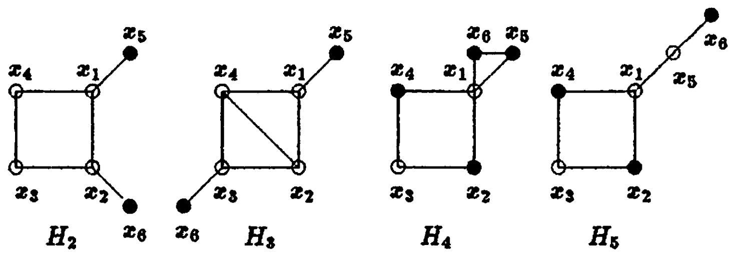


Figure 2: C_4 but no C_5 or C_6

We can therefore add b to S without creating a P_7. And b as well as x_4 become P_6-terminal vertices of S, so all their B-neighbours will then be moved to A. Thus in subsequent iterations STEP 2 will only be executed if a vertex in B is adjacent to x_1 as well as x_3. So performing STEP 2 will never create a P_7 in $G[S]$. If S is H_3, STEP 2 will not be performed since we assume in that case that H_2 does not occur and that the graph has no C_5. If S is H_4, b must be adjacent to x_1 and x_3 and the proof that STEP 2 does not create a P_7 in $G[S]$ is the same as that for the case when $S = H_2$. If S is H_5, b must be adjacent to x_1 and x_3, for otherwise S is H_2 or H_4. And again the proof that STEP 2 does not cause a P_7 is identical to the proof for $S = H_2$.

Suppose S is any of the graphs in Case 3. Initially, no vertex in B is adjacent to two different vertices of S, since we are assuming that G has no C_4. However, we note that x_1 is a P_4-terminal vertex, so suppose that in STEP 4, a B-neighbour, say b_1, of x_1 is moved to S. If S is H_6, H_7, or H_8, there may then be a vertex in B that is adjacent to x_1 as well as to b_1. If this is the case, when we execute STEP 2, we move one common B-neighbour (call it b_2) of x_1 and b_1 to S. This will not create a P_7 in $G[S]$, and both b_1 and b_2 will then become P_6-terminal vertices of S. If S is H_9, no vertex of B will be adjacent to both b_1 and x_1, for otherwise S would contain H_6. Thus in all cases our procedure will not create a P_7 in $G[S]$. □

Corollary 5.2 *If G is a graph with $\tau(G) \leq 13$, then G is τ-partitionable.*

Proof. If $\tau(G) = a + b$ with $a \leq b$, then $a \leq 6$. Since G has a P_n-kernel for all $n \leq 7$, it now follows from Proposition 2.1 that G is (a, b)-partitionable. □

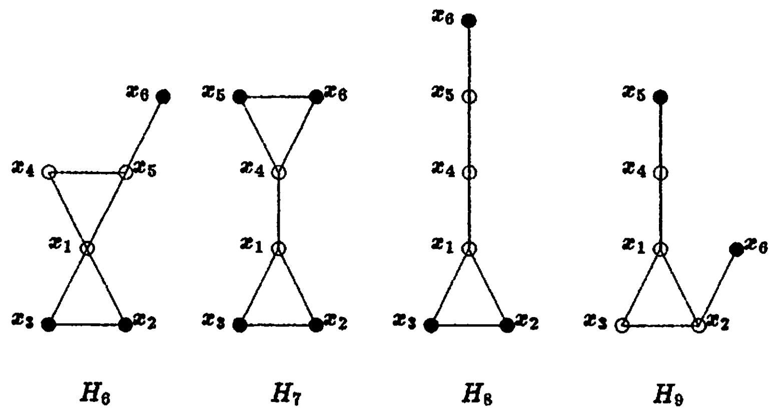


Figure 3: C_3 but no C_4, C_5 or C_6

6 Summary of conjecture status

Settling any of the three conjectures is an interesting problem which is still in process. Actually proving the truth of Conjecture 3 would, of course, suffice to finish the other two problems. On the other hand, since the proof of the truth of these conjectures is extremely elusive, the authors alternate between the search for a proof and the search for a counterexample. Since a counterexample to Conjecture 1 would suffice, suppose there exists a graph G that is not τ-partitionable. Then there exists a pair (a, b) of positive integers such that $\tau(G) = a + b$ and G is not (a, b)-partitionable. We may assume $a \leq b$. Then results in this paper together with results from [2] give the following properties of G:

1. G has no Hamilton path and is not the join of any two graphs,
2. $a > 6$,
3. $13 < \tau(G) < |V(G)| - 1$,
4. $3 < \Delta(G) < |V(G)| - a - 1$ (where Δ denotes the maximum degree),
5. G has a cyclic block which is not Hamiltonian,
6. G has an even cycle and an odd cycle, but G does not have a b-cycle,
7. G has sufficiently large cycles: $c(G) > a + 1$, and
8. G has sufficiently small cycles: $g(G) < a - 1$.

References

[1] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, A survey of hereditary properties of graphs. Discussiones Mathematicae Graph Theory **17** (1) (1997)5–50.

[2] I. Broere, M. Dorfling, J. E. Dunbar, and M. Frick, A path(ological) partition problem. Discussiones Mathematicae Graph Theory **18** (1) (1998).

[3] I. Broere, P. Hajnal and P. Mihók, Partition problems and kernels of graphs. Discussiones Mathematicae Graph Theory **17** (2) (1997) 51–56.

[4] G.Chartrand, D.P. Geller and S.T. Hedetniemi, A generalization of the chromatic number. Proc. Camb. Phil. Soc. **64** (1968) 265 - 271.

[5] G. Chartrand and L. Lesniak, *Graphs & Digraphs*. 3rd ed. (1996) Chapman & Hall.

[6] P. Mihók, Problem 4, p. 86 in *Graphs, Hypergraphs and Matroids*. M.Borowiecki and Z. Skupien (eds), Zielona Góra (1985).

[7] J. Vronka, Vertex sets of graphs with prescribed properties (in Slovak). Thesis, supervised by P. Mihók, P.J. Safárik University, Košice (1986).