The ratio of three domination parameters in some classes of claw-free graphs

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Abstract

A graph G is claw-free if it does not contain any complete bipartite graph $K_{1,3}$ as an induced subgraph, and closed claw-free if it is the line-graph of a triangle-free graph. The inflation H_I of a graph H is obtained from H by replacing each vertex x of degree $d(x)$ by a clique $X \simeq K_{d(x)}$. Every inflated graph $G = H_I$ is closed claw-free. The minimum cardinalities $\gamma(G)$, $\mathrm{ir}(G)$ and rai(G)) of respectively a dominating set, a maximal irredundant set and an R-annihilated irredundant set of any graph G satisfy $\mathrm{rai(G)} \leq \mathrm{ir}(G) \leq \gamma(G)$. The motivation of this paper is that for inflated graphs, it is known that the difference $\gamma(G) - \mathrm{ir}(G)$ can be arbitrarily large but not how large the ratio $\gamma(G)/\mathrm{ir}(G)$ can be. We show that $\gamma(G) \leq 3\mathrm{rai(G)}/2$ for every claw-free graph G and study the sharpness of the bounds $1 \leq \gamma(G)/\mathrm{ir}(G) \leq \gamma(G)/\mathrm{rai(G)} \leq 3/2$ in the four classes of claw-free graphs, closed claw-free graphs, inflated graphs and line graphs of bipartite graphs.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

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1. Introduction

The graphs $G = (V(G), E(G))$ we consider here are simple and finite of order $|V(G)| = n(G)$. The degree, neighborhood, closed neighborhood of a vertex x of G are respectively denoted by $d(x)$, $N(x)$, $N[x]$ (with $N[x] = N(x) \cup \{x\}$). If $S \subseteq V(G)$, $G[S]$ is the subgraph induced in G by S and $N[S] = \bigcup_{x \in S} N[x]$.

A set X of vertices of G is *dominating* if every vertex of $V(G) \setminus X$ has at least one neighbor in X. The minimum cardinality of a dominating set is denoted by $\gamma(G)$. If x is a vertex of a subset X of $V(G)$, the set $N[x] \setminus N[X \setminus \{x\}]$ is called the *X-private neighborhood* of x and its elements are the *X-private neighbors* of x. Note that x belongs to its X-private neighborhood if and only if it is isolated in $G[X]$. The other X-private neighbors of x are in $V(G) \setminus X$ and are called *external.* The vertex x of X is *irredundant* in X if its X-private neighborhood is not empty, *redundant* otherwise. The set X is *irredundant* in G if all its vertices are irredundant. If an irredundant set X is *maximal* for the inclusion, then for every vertex u which is not dominated by X there exists a non-isolated vertex y of X which is redundant in $X \cup \{u\}$. That means that u dominates every X-private neighbor of y. We say that u *annihilates* y. We denote by R the set of vertices of G which are not dominated by X. The irredundant set X is *R-annihilated* if every vertex of R annihilates some vertex of X. The concept of R-annihilated irredundant set was considered in [7] (where it was called semi-maximal irredundant set) and in [3]. The minimum cardinality of a maximal (resp. R-annihilated) irredundant set of G is denoted by $\mathrm{ir}(G)$ (resp. rai(G)). Since every maximal irredundant set is R-annihilated, we have $\mathrm{rai(G)} \leq \mathrm{ir}(G)$ for any G. Moreover, it is well known that every minimal dominating set is a maximal irredundant set. Hence $\mathrm{rai(G)} \leq \mathrm{ir}(G) \leq \gamma(G)$ and thus $1 \leq \dfrac{\gamma(G)}{\mathrm{ir}(G)} \leq \dfrac{\gamma(G)}{\mathrm{rai(G)}}$ for all graphs G.

The *inflation* or *inflated graph* H_I of a simple graph H is obtained as follows: each vertex x_i of degree $d(x_i)$ of H is replaced by a clique $X_i \simeq K_{d(x_i)}$ and each edge (x_i, x_j) of H is replaced by an edge (u, v) of H_I in such a way that $u \in X_i$, $v \in X_j$, and two different edges of H are replaced by non-incident edges of H_I. There are two different kinds of edges in H_I. The edges of the cliques X_i are colored red and the X_i's are called the *red cliques* (a red clique X_i is reduced to a vertex if x_i is a pendant vertex of H). The other ones, which correspond to the edges of H, are colored blue and form a perfect matching of H_I (note that every clique of order more than two is red). Every vertex of H_I belongs to exactly one red clique and one blue edge. Since the graph H is simple, two red cliques are linked by at most one blue edge. We adopt the following notation: if x_i and x_j are two adjacent vertices of H, the endvertices of the blue edge of H_I replacing the edge (x_i, x_j) of H are called x_ix_j in X_i and x_jx_i in X_j, and this blue edge is (x_ix_j, x_jx_i).

A graph is *claw-free* if it does not contain any claw, i.e. any complete bipartite graph $K_{1,3}$, as an induced subgraph. The line graphs of simple graphs are claw-free and among them, the class of line graphs of triangle-free graphs has played a particularly interesting role since the introduction of Ryjáček's closure in claw-free graphs [11]. These line graphs, which are called *closed claw-free graphs*, are characterized by the fact that the neighborhood of each vertex induces one or two vertex-disjoint cliques. From this characterization, we can see that inflated graphs are closed claw-free graphs. More precisely, the inflated graph H_I of H is the line graph of the subdivision $S(H)$ of H obtained by inserting a new vertex on each edge of H. Since $S(H)$ is bipartite, the inflated graphs are not only closed claw-free but are more precisely line graphs of bipartite graphs. Line graphs of triangleless graphs and of bipartite graphs have been studied by Hedetniemi and Slater in [9]. We denote by $\mathcal{CF}$ (resp. $\mathcal{CCF}$, $\mathcal{LB}$, $\mathcal{I}$) the class of claw-free graphs (resp. closed claw-free graphs, line graphs of bipartite graphs, inflated graphs). Clearly $\mathcal{I} \subset \mathcal{LB} \subset \mathcal{CCF} \subset \mathcal{CF}$.

In [6], it is shown that, contrary to a conjecture given in [5], the difference $\gamma(G) - \mathrm{ir}(G)$ can be arbitrarily large even if G is an inflated graph. In the family constructed to prove this result, the ratio $\gamma(G)/\mathrm{ir}(G)$ is arbitrarily near to 5/4. A natural question is to ask how large this ratio can be. This problem has already been considered in several other classes of graphs. It is known [1, 2] that $\gamma(G)/\mathrm{ir}(G) < 2$ for any graph. Damaschke [4] and Volkmann [12] respectively proved that $\gamma(G)/\mathrm{ir}(G) < 3/2$ in any tree and $\gamma(G)/\mathrm{ir}(G) \leq 3/2$ in any block graph and in any graph with cyclomatic number at most 2. Later, V. E. Zverovich [13] obtained the bound 8/5 in the class of block-cactus graphs, which was conjectured in [12]. The references of related results can be found in [8].

In this paper, we prove that in the class of claw-free graphs, 3/2 is an upper bound not only on $\gamma(G)/\mathrm{ir}(G)$ but also on $\gamma(G)/\mathrm{rai}(G)$ (the example of the inflation of a clique K_{3k} given at the end of the next section, and for which $\mathrm{ir}(G) = 3k - 1$ and $\mathrm{rai}(G) \leq 2k$, shows that it is relevant to distinguish between $\gamma(G)/\mathrm{ir}(G)$ and $\gamma(G)/\mathrm{rai}(G)$ in all our classes of claw-free graphs). We also show that in the class $\mathcal{LB}$ of line graphs of bipartite graphs, and thus in $\mathcal{CCF}$ and in $\mathcal{CF}$, the value 3/2 can be attained even by $\gamma(G)/\mathrm{ir}(G)$. While in the subclass $\mathcal{I}$ of inflated graphs, this value is never attained by $\gamma(G)/\mathrm{rai}(G)$, but can be arbitrarily close, even for $\gamma(G)/\mathrm{ir}(G)$. So in a sense, the bound is sharp.

2. The ratio $\gamma(G)/\mathrm{rai}(G)$ in the class of claw-free graphs

For an R-annihilated irredundant set X of G, let Z be the set of isolated vertices in $G[X]$ and $Y = X \setminus Z$, $B(x)$ the set of the external private neighbors of the vertex x of X and $B = \bigcup_{x \in X} B(x)$, Q the set of the vertices of G having at least two neighbors in X, and $R = V(G)\setminus(B \cup Q \cup X)$ the set of the vertices of G which are not dominated by X. Since X is irredundant, $B(y) \neq \emptyset$ for every $y \in Y$, and since the irredundant set X is R-annihilated, every vertex $u \in R$ is adjacent to every vertex of $B(y)$ for some $y \in Y$. We denote by $R(y)$ the set of vertices of R annihilating y.

Lemma 2.1 Let X be an R-annihilated irredundant set of a claw-free graph G. With the above notation, the following properties hold.

1. Every component of $G[Y]$ is a clique. For $y \in Y$, we denote by C_y the component of $G[Y]$ containing y.
2. For every $y \in Y$, the set $B(y)$ is a clique.
3. For every $y \in Y$, the set $B(y) \cup R(y)$ is a clique.
4. If a vertex q of Q is adjacent to $y \in Y$, then q is adjacent to every vertex of C_y or to every vertex of $B(y)$.

Proof : These properties are easy consequences of the fact that G is claw-free, and have already been observed elsewhere (see for instance [6]). We just give a short proof of them.
1. If (y_1, y, y_2) is an induced path of Y and $y' \in B(y)$, then $G[y, y_1, y_2, y']$ is a claw.
2. Suppose y' and y'' are two nonadjacent vertices of $B(y)$ and let $y_1 \in C_y \setminus \{y\}$. Then $G[y, y', y'', y_1]$ is a claw.
3. Suppose u and v are two nonadjacent vertices of $R(y)$ and let $y' \in B(y)$. Then $G[y', y, u, v]$ is a claw. Hence $R(y)$ is a clique, and since each vertex of $R(y)$ dominates each vertex of $B(y)$, $R(y) \cup B(y)$ is also a clique.
4. Suppose the vertex q of Q is adjacent to $y \in Y$ but not to $y_1 \in C_y \setminus \{y\}$. Then q is adjacent to every vertex $y' \in B(y)$ for otherwise $G[y, q, y_1, y']$ is a claw. ■

Theorem 2.2 : The inequalities $1 \leq \frac{\gamma(G)}{\text{ir}(G)} \leq \frac{\gamma(G)}{\text{rai}(G)} \leq \frac{3}{2}$ hold for any graph G in $\mathcal{CF}$.

Proof: Let G be a claw-free graph. Since $\text{ir}(G) \leq \gamma(G)$ for any graph, the lower bound is obvious.

To get the upper one, we consider an R-annihilated irredundant set X of G and use the notation defined above. For every component C of $G[Y]$, we choose a vertex $y_C \in C$ and for every vertex $y \in C$, we choose a vertex $y' \in B(y)$. Let $A_C = \{y_C\} \cup \{y', y \in C\}$ and $A = \bigcup_C A_C$. Then $|A_C| = |C| + 1 \leq 3|C|/2$ and thus $|A| \leq 3|Y|/2$.

Claim : The set $D = A \cup Z$ is a dominating set of G.
Proof of the Claim : The set $Z \cup \{y'; y \in Y\}$ dominates X, and also $B \cup R$ since each $B(y) \cup R(y)$ is a clique by Lemma 2.1. Let q be any vertex of Q. If q is not dominated by D, then q has at least two neighbors in $Y \setminus A$. Let y_1 be such a neighbor of q, and C the component of Y containing y_1. Then, by Lemma 2.1(4), the vertex q is adjacent to y_C or to y'_1, a contradiction. Therefore, D also dominates Q. □

We have constructed a dominating set D of G of order $|Z| + |A| \leq |Z| + 3|Y|/2 \leq 3|X|/2$ and thus $\gamma(G) \leq 3|X|/2$. When X is a minimum R-annihilated irredundant set of G, that is, when $|X| = \text{rai(G)}$, we get $\gamma(G) \leq 3\text{rai(G)}/2$. ■

The lower bound 1 on $\gamma(G)/\text{rai(G)}$ is clearly attained in the class $\mathcal{LB}$ as shown for instance by cliques which are line graphs of stars and for which $\gamma(G) = \text{rai(G)} = 1$.

In the class $\mathcal{I}$, the observation that the equality $\gamma(H_I) = \text{ir}(H_I)$ was often satisfied was the grounding of the conjecture of [5] claiming that this equality held for every graph H. Although this conjecture has been disproved, it has been shown that the equality $\gamma(H_I) = \text{ir}(H_I)$ holds if H is any tree [10] or any clique [6]. Hence for the inflations G of cliques or trees, $\gamma(G)/\text{ir}(G) = 1$, but this is not necessarily true for $\gamma(G)/\text{rai(G)}$. Consider for instance the tree T obtained from three paths P_3 of vertex sets $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$ by adding the two edges (c_3, a_3) and (c_3, b_3). Its inflation T_I satisfies $\gamma(T_I) = \text{ir}(T_I) = 6$ and as $\{c_3a_3, c_3b_3, a_2a_1, b_2b_1, c_2c_1\}$ is an R-annihilated irredundant set, $\text{rai}(T_I) \leq 5$. For inflated cliques, the ratio $\gamma(G)/\text{rai(G)}$ can even be nearer to 3/2 than to 1 as shown by the following example. Consider the inflation G of a clique K_{3k} of vertex set $\{x_1, x_2, \cdots, x_{3k}\}$: it is known [6] that $\text{ir}(G) = \gamma(G) = 3k-1$; on the other hand, the set $\{x_{3i}x_{3i-1}, x_{3i}x_{3i+1}\}_{1 \leq i \leq k}$, where the subscripts are taken modulo $3k$, is an R-annihilated irredundant set and thus $\text{rai(G)} \leq 2k$. However, the lower bound $1 \leq \gamma(G)/\text{rai(G)}$ is also attained in $\mathcal{I}$ as can be seen by the inflation of a cycle C_{3k}, for which it is easy to check that $\gamma = rai = 2k$.

The sharpness of the upper bound 3/2 is studied in the following two sections.

3. Sharpness of the upper bound 3/2 on $\frac{\gamma(G)}{\text{ir}(G)}$ in the class $\mathcal{LB}$

Theorem 3.1 Infinitely many graphs of $\mathcal{LB}$ satisfy $\frac{\gamma(G)}{\text{ir}(G)} = \frac{3}{2}$.

Proof Consider the graph L_k obtained from a cycle C_{6k} of vertex set $x_1y_1z_1u_1v_1w_1x_2y_2z_2u_2v_2w_2\cdots x_ky_kz_ku_kv_kw_k$ by adding $3k$ vertices $\{r_i, s_i, t_i\}$ for $1 \le i \le k$ with each r_i adjacent to x_i and to y_i, each s_i adjacent to z_i and to u_i, and each t_i adjacent to v_i and to w_i. The graph L_k is the line graph of the graph M_k obtained from a cycle C_{6k} of vertex set $a_1b_1a_2b_2\cdots a_{3k}b_{3k}$ by attaching a pendant vertex at each vertex a_i. Since M_k is bipartite, L_k belongs to $\mathcal{LB}$.

For L_k, $\{r_i, s_i, t_i\}_{1\le i\le k}$ is a minimum dominating set and $\{z_i, u_i\}_{1\le i\le k}$ a maximal irredundant set. Therefore $\dfrac{\gamma(L_k)}{ir(L_k)} \ge \dfrac{3}{2}$ and thus $\dfrac{\gamma(L_k)}{ir(L_k)} = \dfrac{3}{2}$ by Theorem 2.1. ■

4. Sharpness of the upper bound 3/2 on $\dfrac{\gamma(G)}{\mathrm{ir}(G)}$ in the class $\mathcal{I}$

First we prove that if G is an inflation, the ratio $\dfrac{\gamma(G)}{\mathrm{rai(G)}}$ cannot be exactly 3/2.

Theorem 4.1 If G is an inflated graph, then $\dfrac{\gamma(G)}{\mathrm{rai(G)}} < \dfrac{3}{2}$.

Proof : Let $G = H_I$ and suppose that $\gamma(G) = 3\mathrm{rai(G)}/2$. Let X be an R-annihilated irredundant set of rai(G) elements, and $D = A \cup Z$ the dominating set of at most $3|X|/2$ elements constructed in Theorem 2.2. By the assumption $\gamma(G) = 3|X|/2$, D has $\gamma(G) = 3|X|/2$ elements. From the proof of Theorem 2.2, this implies that $Z = \emptyset$ and that for every component C of Y, $|C| + 1 = 3|C|/2$, i.e. $|C| = 2$. If every neighbor in Q of some vertex y of Y is adjacent to a vertex of $X \setminus C_y$, or if y has no neighbor in Q, then, by Lemma 2.1(4), $A \setminus \{y\}$ dominates G, in contradiction to $|D| = \gamma(G)$. Hence for every vertex y of Y, some neighbor q of y in Q is adjacent to the two vertices of C_y. In the inflated graph G, the clique $\{q\} \cup C_y$ has order at least three and is thus a part of a red clique. Hence the edge (y, y') is blue and since it forms a maximal clique, $B(y) = y'$. Therefore the clique $R(y) \cup B(y)$ is red and all the neighbors of y in Q belong to the red clique containing $\{q\} \cup C_y$. Similarly, the second vertex of C_y is joined by a blue edge to its unique private neighbor. The same situation holds for every component of X. Since the red cliques of G are disjoint, the set Q is partitioned into parts of red cliques, each of them containing one component of X. Hence the number of red cliques of G is at most three times the number of components of X, that is at most $3|X|/2$. On the other hand, if $G = H_I$ then the number of red cliques of G is equal to the order $n(H)$ of H, and it is proved in [5] that $\gamma(H_I) \le n(H) - 1$ for any graph H. We get thus $\gamma(G) < 3|X|/2$, in contradiction to the assumption $\gamma(G) = 3|X|/2$. Therefore $\gamma(G)$ is strictly less than 3rai(G)/2. ■

However the following example shows that in the class $(\mathcal{I})$, the ratio $\gamma(G)/\mathrm{ir}(G)$ can be arbitrarily near to 3/2. In this sense we can say that the upper bound 3/2 on $\gamma(G)/\mathrm{ir}(G)$ is sharp.

For an integer $k \geq 2$, the graph H_k consists of $k+1$ internally disjoint paths (a, c, b) and (a, x_i, y_i, z_i, b) for $1 \leq i \leq k$, plus all the edges (y_i, y_j) for $1 \leq i \neq j \leq k$ (so that $H_k[y_1, y_2, \cdots, y_k]$ is a clique). Its inflation G_k consists of $2k+1$ disjoint paths $(ax_i, x_ia, x_iy_i, y_ix_i)$ for $1 \leq i \leq k$, $(y_iz_i, z_iy_i, z_ib, bz_i)$ for $1 \leq i \leq k$ and (ac, ca, cb, bc), $k(k-1)$ other vertices y_iy_j, $1 \leq i \neq j \leq k$, all the red edges forming the $k+2$ red cliques $G_k[ac, ax_1, ax_2, \cdots, ax_k]$, $G_k[bc, bz_1, bz_2, \cdots, bz_k]$ and $G_k[y_ix_i, y_iz_i, \{y_iy_j\}_{1\leq i\neq j\leq k}]$ for $1 \leq i \leq k$, and finally the $k(k-1)/2$ blue edges (y_iy_j, y_jy_i) for $1 \leq i \neq j \leq k$. Figure 1 shows the graphs H_3 and G_3 where the red (blue resp.) edges are represented by thin (thick resp.) lines.

Theorem 4.2 The inflated graph G_k described above satisfies $\dfrac{\gamma(G_k)}{\mathrm{ir}(G_k)} \geq \dfrac{3}{2} - \dfrac{1}{k+1}$.

Proof The set $\{ca, cb, y_1x_1, y_1z_1, y_2x_2, y_2z_2, \cdots, y_kx_k, y_kz_k\}$ is obviously a maximal irredundant set. Hence, $\mathrm{ir}(G_k) \leq 2k+2$.

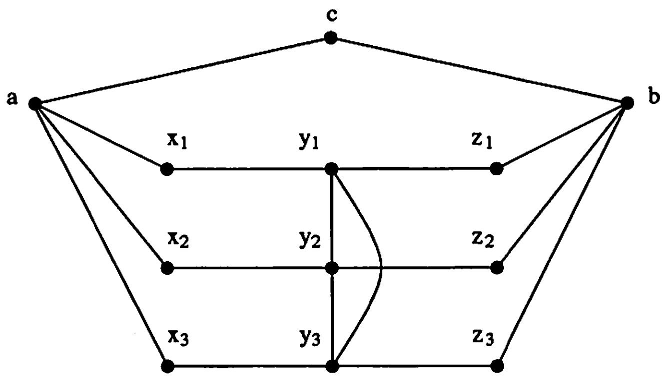
On the other hand, let D be any dominating set of G_k. We denote by B_i the set of the $k+1$ vertices of G_k coming from the vertex y_i of H_k, that is $B_i = \{y_ix_i, y_iz_i, \{y_iy_j\}_{1\leq j\neq i\leq k}\}$.

If $D \cap B_i = \emptyset$ for some i, say $D \cap B_1 = \emptyset$, then, in order to dominate each vertex y_1y_j, $2 \leq j \leq k$, D must contain $\{y_2y_1, y_3y_1, \cdots, y_ky_1\}$. The set D must also contain x_1y_1 and z_1y_1 to dominate y_1x_1 and y_1z_1, and at least one vertex of each set $\{x_ja, x_jy_j, y_jx_j\}$ and $\{z_jb, z_jy_j, y_jz_j\}$, $2 \leq j \leq k$, to dominate the vertices x_jy_j and z_jy_j. Finally, D contains at least two more vertices to dominate $\{ax_1, ac, ca, cb, bc, bz_1\}$. Hence $|D| \geq 3k+1$.

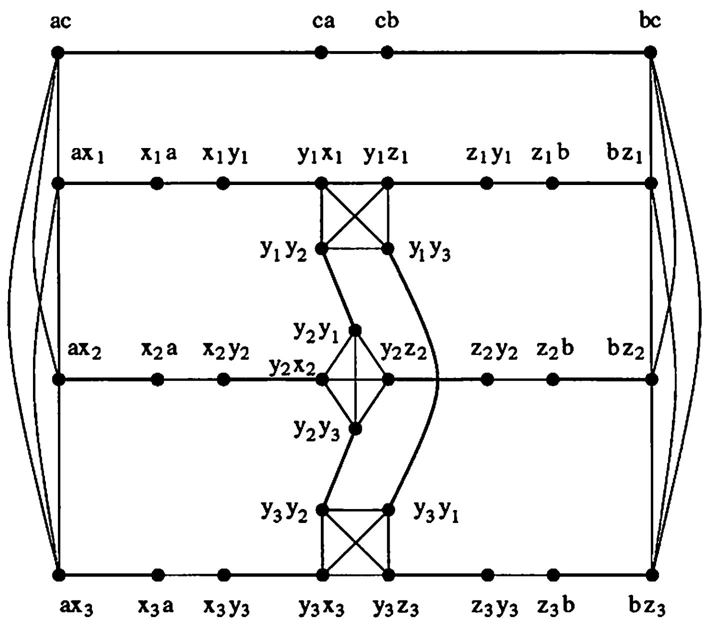
If $D \cap B_i \neq \emptyset$ for all $1 \leq i \leq k$, then, to dominate x_ia and z_ib, D must contain at least one vertex in each set $\{ax_i, x_ia, x_iy_i\}$ and $\{bz_i, z_ib, z_iy_i\}$ for $1 \leq i \leq k$. Moreover, D contains at least one more vertex to dominate $\{ca, cb\}$. Hence, as above, $|D| \geq 3k+1$.

Therefore, taking for D a minimum dominating set of G_k, we find $\gamma(G_k) \geq 3k+1$ and thus, $\dfrac{\gamma(G_k)}{\mathrm{ir}(G_k)} \geq \dfrac{3k+1}{2k+2} = \dfrac{3}{2} - \dfrac{1}{k+1}$. ■

Since $\gamma(G_k)/\mathrm{ir}(G_k)$ tends to 3/2 when k (that is, when the order of G) tends to $+\infty$, this proves that the bound 3/2 on $\gamma(G)/\mathrm{ir}(G)$, established in the class of claw-free graphs, is best possible even in the class of inflated graphs.



The graph H_3



The inflated graph G_3 of H_3

Figure 1:

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