A Note on Relations Between a Graph and Its Line Graph Which Involve Domination Concepts

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Abstract

Sharp invariant relationships involving various types of domination numbers are found between a graph and its line graph.

Dedicated to Professor Stephen T. Hedetniemi on the occasion of his 60th birthday

1 Introduction

The *line graph* $L(G)$ of graph $G = (V, E)$ has vertex set $E(G)$ with two vertices adjacent if and only if the corresponding edges in G are adjacent. In this note we relate graphical parameters of G and $L(G)$, with domination invariants playing a central role. Line graphs have been characterized by Beineke [2] in terms of nine forbidden induced subgraphs, one of which is the *claw* $K_{1,3}$. Faudree, Flandrin, and Ryjáček [4] have produced an extensive survey of claw-free graphs and their properties.

A subset S of $V(G)$ *dominates* G if every vertex not in S is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the size of a smallest dominating set. Other types of domination involved in this paper are *connected domination* in which S induces a connected subgraph, *independent domination* in which S induces an empty subgraph, *total domination* in which every vertex is adjacent to a vertex in S, *paired domination* in which each vertex of S is adjacent to a unique other vertex of S, and *2-domination* in which every vertex not in S is adjacent to at least two vertices of S. The corresponding domination numbers are designated γ_c,

γ_i, γ_t, γ_p, and γ_2, respectively. Information about these invariants can be found in greater detail in Haynes, Hedetneimi, and Slater [5]. Allan and Lasker [1] showed that the domination and independent domination numbers of claw-free graphs (and hence line graphs) are equal, a characteristic which will be useful in discussions below.

Other invariants employed here include the *vertex cover number* α_0 representing the smallest number of vertices such that each edge is incident to at least one of them, the *vertex independence number* β_0 representing the largest size of an independent set of vertices, and the *edge independence number* β_1 representing the largest size of an independent set of edges. Additional invariants will be defined as needed.

2 Results

Throughout this section we usually represent sets of edges of graph G by capital letters with the corresponding set of vertices in $L(G)$ represented by the same letters primed. The initial result relates the domination and independent domination numbers of the line graph of G to both the vertex cover and edge independence numbers of G. The symbol $i_1(G)$ is the smallest size of a maximal independent set of edges in G.

Theorem 1 *For any graph G, $\alpha_0(G)/2 \leq \gamma[L(G)] = \gamma_i[L(G)] = i_1(G) \leq \beta_1(G) = \beta_0[L(G)]$.*

Proof: Let X' be a minimum dominating set of $L(G)$. In G the edge set X has the property that any edge not in X is adjacent to an edge in X. Thus the $2|X| = 2|X'| = 2\gamma[L(G)]$ vertices which are endpoints of the edges of X form a vertex cover of G, thus showing the leftmost inequality. The fact that $\gamma[L(G)] = \gamma_i[L(G)]$ follows from the Allan and Lasker result mentioned in the Introduction, and that $\beta_1(G) = \beta_0[L(G)]$ from the definition of line graph. Suppose now that X' is a minimum independent dominating set of $L(G)$. Then X is an independent set of edges in G, and it is maximal since every edge not in X is adjacent to an edge of X. Furthermore $|X|$ must be the minimum size of such a maximal set of edges since any maximal independent set X of edges in G results in X' being an independent dominating set of $L(G)$. This establishes the equality between $\gamma_i[L(G)]$ and $i_1(G)$. The proof is completed by noting that certainly $i_1(G) \leq \beta_1(G)$. □

Corollary 2 *If G has no isolated vertices, $\gamma(G)/2 \leq \gamma_p(G)/2 \leq \gamma_i[L(G)]$.*

Proof: The proof to Theorem 1 shows that a minimum independent dominating set X' of $L(G)$ corresponds to an independent set of edges X in G

whose end vertices form a vertex cover. They also are a paired dominating set if no vertex of G is isolated. □

Sharpness for the upper bounds of Theorem 1 is given by K_{2n}. The *corona* of graphs G and H, written $G \circ H$, is obtained by creating a copy of H for each vertex of G and connecting that vertex of G to every vertex of its associated copy of H. Sharpness for the lower bounds for both Theorem 1 and Corollary 2 is given by $G = P_{2k} \circ K_1$ where P_{2k} is the path on $2k$ vertices. It is easy to see that $\alpha_0(G) = \gamma(G) = \gamma_p(G) = 2k$ and $\gamma[L(G)] = k$.

No upper bound for $\gamma[L(G)]$ is possible in terms of $\gamma(G)$. Consider the wheel W_{2k+1} with $2k$ vertices on the cycle. It has $\gamma(W_{2k+1}) = 1$ but $\gamma[L(W_{2k+1})] = k$.

When X is a set of vertices of G, $\langle X \rangle$ represents the subgraph of G induced by X, $c(X)$ is the number of components in $\langle X \rangle$, and $\hat{c}(X)$ is the number of those components containing at least one edge. With this notation, an upper bound for $\gamma[L(G)]$ based on the vertex cover number of G can be determined.

Lemma 3 *Let X be a vertex cover of a graph G having no isolated vertices. Then $\gamma[L(G)] \le |X| - \hat{c}(X)$.*

Proof: Consider a spanning forest of the $\hat{c}(X)$ nontrivial components of $\langle X \rangle$. The number of vertices in these components is $|X| - [c(X) - \hat{c}(X)]$ so the number of edges in the forest is $\{|X| - [c(X) - \hat{c}(X)]\} - \hat{c}(X) = |X| - c(X)$. These edges correspond in $L(G)$ to a collection of vertices which will be a part of a dominating set. The only vertices of $L(G)$ not dominated by this collection correspond in G to edges covered by the trivial components, of which there are $c(X) - \hat{c}(X)$. For each of the vertices comprising these trivial components, an edge can be found in G incident to the vertex which, in turn, corresponds to a vertex in $L(G)$. The set of such vertices of $L(G)$ dominates any remaining vertices. Thus $\gamma[L(G)] \le [|X| - c(X)] + [c(X) - \hat{c}(X)]$. □

Corollary 4 *For any graph G, $\gamma[L(G)] \le \alpha_0(G) - max[\hat{c}(X)]$ where the maximum is taken over all minimum vertex covers X of G.*

Sharpness for Lemma 3 and Corollary 4 is shown by the graph G_k constructed from a path $P_{3k+2} = \langle a_1, b_1, c_1, a_2, \ldots, c_k, a_{k+1}, b_{k+1} \rangle$ by adding pendant edges $a_i x_i$ and $b_i y_i$ for $1 \le i \le k+1$. Figure 1 shows G_3 and its line graph. In this figure, if G has an edge joining vertices u and v, we designate the corresponding vertex in $L(G)$ by the symbol uv. It is straightforward to show that a minimum vertex cover of G_k is $X = \{a_i, b_i : 1 \le i \le k+1\}$

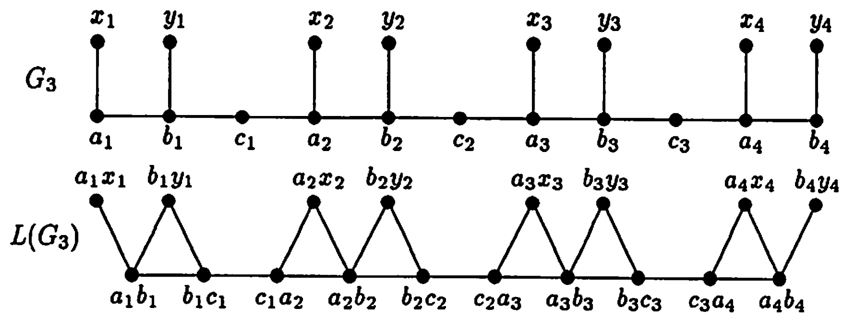

Figure 1: The graph G_3 and its line graph

and that $|X| = 2k+2$, $E(\langle X \rangle) = \{a_ib_i : 1 \leq i \leq k+1\}$, $\hat{c}(X) = k+1$, and $\gamma[L(G_k)] = k+1$ with the vertices a_ib_i forming a dominating set.

Several bounds can be found for the connected domination number of $L(G)$.

Theorem 5 *For any connected graph G having n vertices, $\alpha_0(G) - 1 \leq \gamma_c[L(G)] \leq n - 2$.*

Proof: Let X' be a minimum connected dominating set of $L(G)$. The subgraph induced by X in G is a tree whose vertices form a vertex cover of G. Thus the number of these vertices is at least $\alpha_0(G)$ so $|X|$, the number of edges in the tree, is at least $\alpha_0(G) - 1$, establishing the first inequality. Consider any spanning tree of G and remove from it a degree one vertex. Every edge of G, including the removed one, has at least one endvertex in the reduced tree. Let X be the $n-2$ edges of this tree. Then X' is a connected dominating set of $L(G)$. □

The complete graph K_n, since $\alpha_0(K_n) - 1 = n-2$, demonstrates sharpness of both bounds and incidently shows $\gamma_c[L(K_n)] = n-2$.

Theorem 5 can be improved. The symbol $\kappa(G)$ represents the vertex connectivity number of graph G.

Theorem 6 *For any connected graph G having n vertices, either $\gamma_c[L(G)] = \alpha_0(G) - 1$ or $\alpha_0(G) \leq \gamma_c[L(G)] \leq n - \kappa(G)$.*

Proof: Assume $\gamma_c[L(G)] \neq \alpha_0(G) - 1$. Then Theorem 5 gives $\alpha_0(G) \leq \gamma_c[L(G)]$. Let B be a set of $\beta_0(G)$ independent vertices of G, and let Y be a largest subset of B for which $G - Y$ is connected. Such a Y exists since G is connected. Suppose $Y = B$. Then $G - Y$ is connected and has $\alpha_0(G)$ vertices. Thus the $\alpha_0(G) - 1$ edges in any spanning tree of $G - Y$ correspond to a connected dominating set of vertices in $L(G)$. It follows that $\gamma_c[L(G)] = \alpha_0(G) - 1$, a contradiction.

Thus we must have $t = |Y| < |B| = \beta_0(G)$. Then $B - Y$ is a set of $\beta_0(G) - t$ vertices in $G - Y$ which are independent and none is adjacent to any vertices of Y. Because Y is a maximum subset of B which does not disconnect $G - Y$, the removal of any vertex of $B - Y$ must disconnect $G - Y$, implying $\kappa(G) \leq t + 1$. Since the $n - t - 1$ edges in a spanning tree of $G - Y$ correspond to a connected dominating set of vertices in G, it follows that $\gamma_c[L(G)] \leq n - t - 1 \leq n - \kappa(G)$. □

The complete bipartite graph $K_{m,m}$ yields sharpness for Theorem 6 and also shows $\gamma_c[L(K_{m,m})] = m$. The following corollary results from the fact that $n = \alpha_0(G) + \beta_0(G)$.

Corollary 7 *For any connected graph G, $\gamma_c[L(G)] = \alpha_0(G) - 1$ if $\kappa(G) > \beta_0(G)$.*

A relationship exists between the connected domination numbers of G and $L(G)$. The minimum degree of G is $\delta(G)$.

Theorem 8 *Let G be a connected graph. Then $\gamma_c(G) \leq \gamma_c[L(G)] + 1$. Furthermore, if $\delta(G) \geq 2$, then $\gamma_c(G) \leq \gamma_c[L(G)]$.*

Proof: As before, consider the tree in G induced by the edges X corresponding to a minimum connected dominating set X' of $L(G)$. The vertices of this tree, of which there are $\gamma_c[L(G)] + 1$, form a connected dominating set of G. Further, if $\delta(G) \geq 2$, any single degree one vertex in this tree is unneeded to dominate G. □

The path P_n on $n \geq 4$ vertices shows sharpness for the first bound since $\gamma_c(P_n) = n - 2$ and $\gamma_c[L(P_n)] = \gamma_c(P_{n-1}) = n - 3$. Any cycle demonstrates sharpness for the second bound.

We have seen that lower bounds for $\gamma_c[L(G)]$ can be based on the vertex cover number of G. An upper bound in terms of $\alpha_0(G)$ also exists. The following lemma is the basis for this result.

Lemma 9 *Let G be a connected graph and X be any vertex cover of G. Then $\gamma_c[L(G)] \leq |X| + c(X) - 2$.*

Proof: Let $C_1, C_2, \ldots, C_m$ be the components of $\langle X \rangle$ where $m = c(X)$. Since G is connected, adding no more than $m - 1$ vertices to X can create a new connected vertex cover $\hat{X}$ of G with at most $|X| + m - 1$ vertices. The edges in any spanning tree of the subgraph induced by these $|\hat{X}|$ vertices correspond to $|\hat{X}| - 1 = |X| + m - 2$ vertices of $L(G)$ which form a connected dominating set. □

The graph G_k described earlier demonstrates sharpness of this lemma since $|X| = 2k + 2$, $c(X) = k + 1$, and $\gamma_c[L(G)] = 3k + 1$. A series of results are now easy corollaries.

Corollary 10 *Let G be a connected graph. Then $\gamma_c[L(G)] \leq \alpha_0(G) + min[c(X)] - 2$ where the minimum is taken over all minimum vertex covers X.*

Corollary 11 *Let G be a connected graph. Then $\gamma_c[L(G)] \leq 2\alpha_0(G) - 2$.*

Proof: Immediate since $c(X) \leq |X|$. □

Sharpness of the preceding corollary is shown by P_{2k+1}, where $k \geq 2$.

Corollary 12 *Let G be a connected graph having at least one connected minimum vertex cover. Then $\gamma_c[L(G)] = \alpha_0(G) - 1$.*

Proof: Follows from Theorem 5 and Lemma 9. □

The corona of any cycle with K_1 demonstrates sharpness of the preceding bound. Next we examine a bound involving the total domination numbers of a graph G and its line graph. Let X be a minimum total dominating set of $L(G)$. Dutton and Brigham [3] show that any component of $\langle X \rangle$, designated $K_m^{(s)}$, is constructed from a K_m, $m \geq 2$, by adding s pendant edges, $0 \leq s \leq m$, with at most one incident to any vertex of the K_m. Then the edges in G corresponding to the vertices of $K_m^{(s)}$ induce, except possibly when $m = 3$, a $K_{1,m}$ with s of its edges subdivided. If $m = 3$, these edges could form a triangle, but only if $s = 0$. It is now possible to proceed to the next result.

Theorem 13 *Suppose neither G nor $L(G)$ have isolated vertices and X is a minimum total dominating set of $L(G)$. Then $\gamma_t(G) \leq \gamma_t[L(G)] + c(X)$.*

Proof: Let $K_m^{(s)}$ be a component of $\langle X \rangle$. The corresponding edges in G are incident to at most $s + m + 1$ vertices. The vertices arising from all the components form a total dominating set of G of size at most $|X| + c(X) = \gamma_t[L(G)] + c(X)$. □

The next corollary follows since $c(X) \leq |X|/2$.

Corollary 14 *Suppose neither G nor $L(G)$ have isolated vertices. Then $\gamma_t(G) \leq \frac{3}{2}\gamma_t[L(G)]$.*

Sharpness of the preceding theorem and corollary is given by the graph H_k constructed from a path

$$P_{4k+3} = \langle a_1, b_1, c_1, d_1, a_2, \ldots, a_k, b_k, c_k, d_k, a_{k+1}, b_{k+1}, c_{k+1} \rangle$$

by adding pendant edges a_ix_i, b_iy_i, and c_iz_i for $1 \leq i \leq k+1$. Figure 2 shows H_2 and its line graph. For H_k, $X = \{a_ib_i, b_ic_i : 1 \leq i \leq k+1\}$ forms

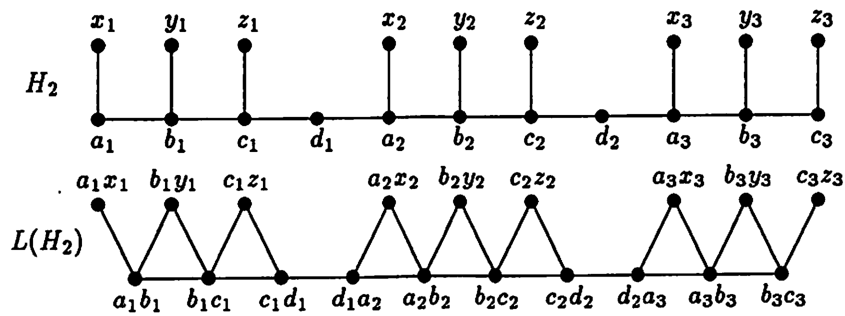

Figure 2: The graph H_2 and its line graph

a minimum total dominating set of $L(G)$ of size $2(k+1)$ and $c(X) = k+1$. Furthermore, $\{a_i, b_i, c_i : 1 \leq i \leq k+1\}$ forms a minimum total dominating set of G of size $3(k+1)$.

The next result gives a lower bound on $\gamma_2[L(G)]$. If uv is a vertex of $L(G)$, we will say that u and v are *in* uv.

Theorem 15 *For a graph G having no isolated vertices, $\gamma(G) \leq \gamma_2[L(G)]$.*

Proof: Let X' be a minimum 2-dominating set of $L(G)$ and Y' be a maximum sized independent set of vertices of $L(G) - X'$ of the form ab with b not in any vertex of X'. If $Y' = \emptyset$, set $t = 0$. Otherwise $Y' = \{a_ib_i : 1 \leq i \leq t\}$. Let $X_1' = N(Y') \cap X'$. Since X' is a 2-dominating set and b_i is not in any vertex of X', a_i must be in at least two vertices of X_1'. Hence, because Y' is an independent set of vertices and no $K_{1,3}$ can occur, $|X_1'| \geq |Y'| = t$. Define $Z \subseteq V(G)$ to be a set containing $a_1, a_2, \ldots, a_t$ and exactly one of v or w for each vertex $vw \in X' - X_1'$. Observe that $|Z| \leq t + |X' - X_1'| = t + \gamma_2[L(G)] - |X_1'| \leq \gamma_2[L(G)]$. The result will be proven by showing Z is a dominating set of G.

Suppose $w \in V(G)$ is not dominated by Z. Then $w \notin Z$ and, for any $u \in N(w)$, $u \notin Z$. Thus $wu \notin X' - X_1'$. This means wu must have been a candidate for inclusion in Y' and was excluded because it is adjacent to some a_ib_i which is in Y'. If $w = b_i$, a_i is a neighbor of w in G and hence dominates it, a contradiction. If $u = b_i$, wu is not dominated in $L(G)$, another contradiction. It follows that $u = a_i$ which is in Z. This final contradiction establishes $\gamma(G) \leq |Z| \leq \gamma_2[L(G)]$. □

Sharpness is achieved by the corona of the cycle C_n with K_1.

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