Connected Domination Graphs of Tournaments

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Abstract. The domination graph of a digraph is the graph on the same vertices with an edge between two vertices if every other vertex loses to at least one of the two. This note describes which connected graphs are domination graphs of tournaments.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

Let D be a digraph. Vertex x *beats* vertex y (or y *loses to* x) if (x, y) is an arc in D. Vertices x and y *dominate* (or are a *dominant pair* of) D if x and y beat all other vertices in D. The *domination graph* dom(D) is the graph on the same vertices as D with edges between dominant pairs of D. A *tournament* is a digraph with exactly one arc between each pair of vertices (see Figure 1). See Moon [7], Reid and Beineke [8], and Reid [9] for more on tournaments.

Domination graphs were introduced in conjunction with "competition graphs". The *competition graph* of a digraph D is the graph on the same vertices as D with an edge between two vertices if they beat a common vertex in D. The domination graph of a tournament is the complement of the competition graph of its reversal [2]. See Lundgren [6] and Kim, McKee, McMorris, and Roberts [5] for more about competition graphs.

The *domination digraph* $\mathcal{D}(T)$ of a tournament T is the digraph with the same vertices as T where vertex x beats vertex y in $\mathcal{D}(T)$ if x and y dominate T and x beats y in T. Thus, $\mathcal{D}(T)$ is the orientation of dom(T) induced by T (see Figure 1).

This note extends the authors' previous work [1,2] on domination graphs. A vertex of a graph is *pendant* if it has exactly one neighbor. A *caterpillar* (see Figure 2) is a connected graph whose nonpendant vertices form a (possibly trivial) path. This path is the *spine* of the caterpillar and the number of edges in the spine is the *length* of the caterpillar. A *star* is a graph

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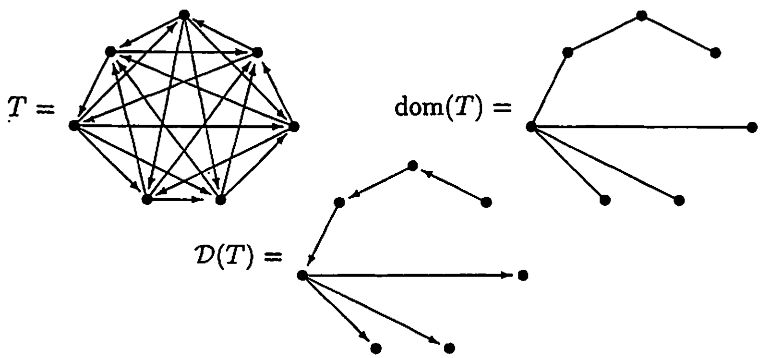

Figure 1: A tournament with its domination graph and domination digraph.

with exactly one vertex adjacent to all others (*i.e.*, a caterpillar of length zero). A *spiked cycle* is a connected graph whose nonpendant vertices form a cycle.

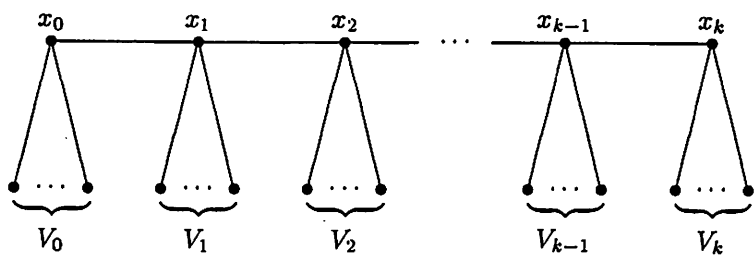

Figure 2: A caterpillar.

PROPOSITION 1. (from [2]) *The domination graph of a tournament is either a spiked odd cycle perhaps with some isolated vertices, or a graph whose components are all caterpillars.*

PROPOSITION 2. (from [2]) *A spiked odd cycle with or without isolated vertices is the domination graph of some tournament.*

PROPOSITION 3. (from [1]) *In the directed domination graph of a tournament, a vertex loses to at most one vertex and beats at most one vertex that beats other vertices.*

PROPOSITION 4. (from [1]) *A path with three or more edges is not the domination graph of any tournament.*

In [1], the authors showed that every caterpillar is the domination graph of an oriented graph (a digraph with at most one arc between each pair of vertices). But Proposition 4 shows that some caterpillars are not the domination graph of any tournament. Theorem 5 describes all connected graphs that can be domination graphs of tournaments (see Figure 3).

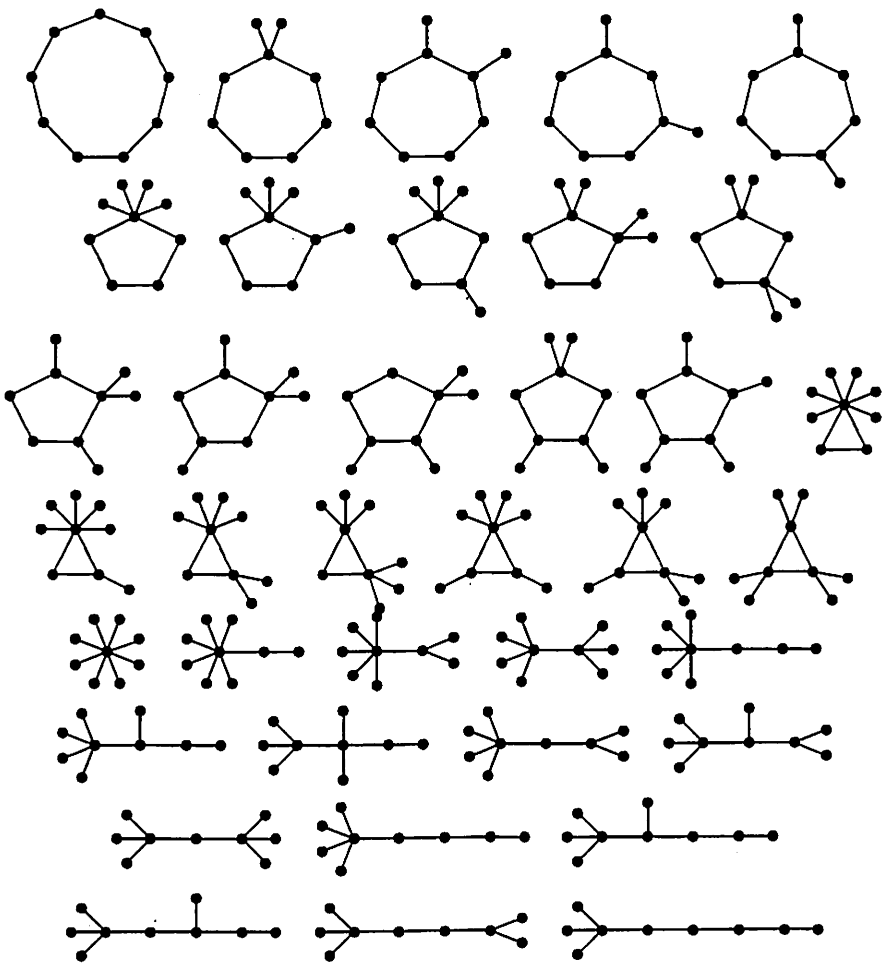

Figure 3: All connected domination graphs of tournaments with 9 vertices

THEOREM 5. *A connected graph is the domination graph of a tournament if and only if it is a spiked odd cycle, a star, or a caterpillar of positive length with three or more pendant vertices adjacent to one end of its spine.*

PROOF. $\Longrightarrow$. Let T be a tournament where $\text{dom}(T) = G$ is connected. Proposition 1 shows that $\text{dom}(T)$ is either a spiked odd cycle or a caterpillar. So all that needs to be shown is that $\text{dom}(T)$ cannot be a caterpillar of positive length with both ends of its spine adjacent to only one or two pendant vertices. Suppose G is such a caterpillar. Consecutively label the vertices of the spine $x_0, x_1, \ldots, x_k$. Let V_i be the set of pendant vertices adjacent to x_i (Figure 2 shows this labeling) with $k \neq 0$ and $|V_0|, |V_k| \in \{1, 2\}$.

Since G is $\text{dom}(T)$ for the tournament T, Proposition 3 shows that the spine of G is a directed path in $\mathcal{D}(T)$. Without loss of generality, assume x_i beats x_{i-1} for $1 \leq i \leq k$. Proposition 3 also implies that with the possible exception of one vertex in V_k, all vertices in V_i lose to x_i in $\mathcal{D}(T)$. If a vertex in V_k does beat x_k, label that vertex x_{k+1}, replace V_k with $V_k - \{x_{k+1}\}$, and let k be what formerly was $k+1$. Thus $\mathcal{D}(T)$ is oriented as in Figure 4 with $|V_0| \in \{1, 2\}$ and $|V_k| \in \{0, 1, 2\}$.

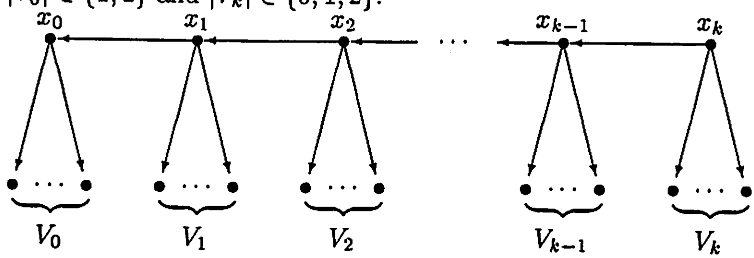

Figure 4: Orientation of a caterpillar.

For $2 \leq i \leq k$, since x_i beats x_{i-1} and (x_{i-2}, x_{i-1}) is a dominant pair, x_{i-2} beats x_i. Then for $3 \leq i \leq k$, since x_{i-3} beats x_{i-1} and (x_{i-1}, x_i) is a dominant pair, x_i beats x_{i-3}. Thus for $4 \leq i \leq k$, since x_i beats x_{i-3} and (x_{i-4}, x_{i-3}) is a dominant pair, x_{i-4} beats x_i. Continuing this, we conclude (see Figure 5):

$$x_i \text{ beats } x_j \iff \text{either } i - j \text{ is odd and positive, or even and negative.}$$

Let $y \in V_i$. Since x_i loses to $\{\ldots, x_{i-4}, x_{i-2}, x_{i+1}, x_{i+3}, \ldots\}$, and (x_i, y) is a dominant pair, y beats $\{\ldots, x_{i-4}, x_{i-2}, x_{i+1}, x_{i+3}, \ldots\}$. Since relabeling assures x_k beats all vertices in V_k, and (x_{i-2}, x_{i-1}) and (x_i, x_{i+1}) are dominant pairs, y loses to x_{i-1} and x_i. Thus for all $y \in V_i$:

$$y \text{ beats } x_j \iff \text{either } i - j \text{ is even and positive, or odd and negative.}$$

Now let $y \in V_i$ and $z \in V_j$ where $i \neq j$. From above, either y beats x_j or z beats x_i. If y beats x_j, since (x_j, z) is a dominant pair, z beats y. If z

beats x_i, since (x_i, y) is a dominant pair, y beats z. So for all $y \in V_i$ and $z \in V_j$ with $i \neq j$:

$$y \text{ beats } z \iff \text{either } i - j \text{ is odd and positive, or even and negative.}$$

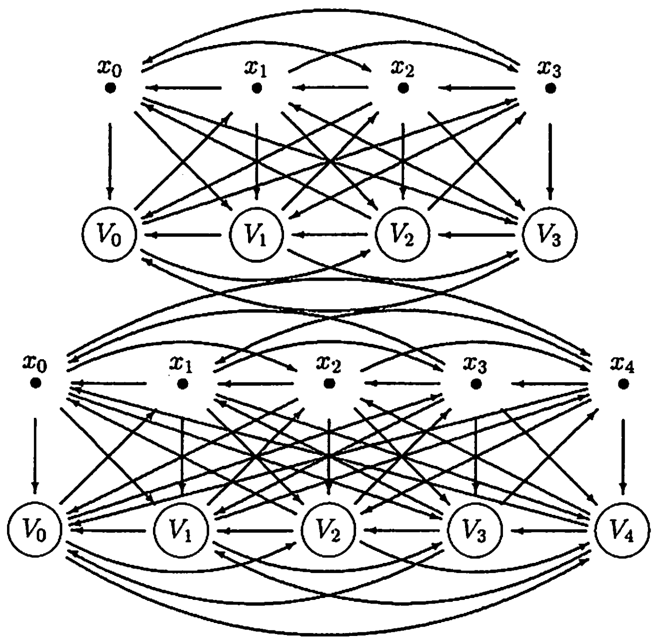

Figure 5: Tournaments whose domination graphs are caterpillars with lengths 3 and 4.

If k is even, x_k beats all x_i for all odd i and all vertices in V_i for all even i, and x_0 beats all x_i for all even $i > 0$ and all vertices in V_i for all odd i. Thus x_0 and x_k dominate T when k is even. If k is odd, x_k beats all x_i for all even i and all vertices in V_i for all odd i. Since $|V_0| \in \{1, 2\}$, some $z \in V_0$ beats all other vertices in V_0. Thus z beats all x_i for all odd i and all vertices in V_i (except itself) for all even i. So z and x_k dominate T when k is odd. Either way, we deduce that dom(T) has an edge not in G, a contradiction. So G is not the domination graph of any tournament.

$\Longleftarrow$. For each graph G allowed by the theorem, we need to give a tournament T with dom(T) = G. If G is a spiked odd cycle, Proposition 2 shows there is such a tournament. If G is a star, then G is the domination graph of any tournament on the same number of vertices where one vertex beats all others.

Now let G be a caterpillar of length $k \neq 0$ with at least three pendant vertices adjacent to one end of its spine. Label G as in Figure 2 with x_0 at an end with three or more pendant vertices. Then $|V_0| \geq 3$. If k is even, let x_{k+1} denote a vertex in V_k (this is possible since x_k is a pendant vertex of the spine, but it is not a pendant vertex of G), redefine V_k as $V_k - \{x_{k+1}\}$, let $V_{k+1} = \emptyset$, and then replace $k+1$ by k.

With or without these redefinitions, k is odd. Let T be a tournament on $\{x_0, x_1, \ldots, x_k\} \cup V_0 \cup V_1 \cup \cdots \cup V_k$ where (see Figure 5):

1. x_i beats x_j if either $i - j$ is odd and positive, or even and negative.
2. For all $y \in V_i$, vertex y beats x_j if either $i - j$ is even and positive, or odd and negative.
3. For all $y \in V_i$ and $z \in V_j$ with $i \neq j$, vertex y beats z if either $i - j$ is odd and positive, or even and negative; otherwise, x_j beats y.
4. The subtournament on V_0 is strongly connected (this is possible since $|V_0| \geq 3$) and the subtournament on V_i for $i > 0$ is arbitrary.

To prove $G = \text{dom}(T)$, we must show that each pair of adjacent vertices of G dominate T, while both vertices in each nonadjacent pair of G lose to some vertex in T:

1. For $1 \leq i \leq k$, vertex x_{i-1} beats $\{\ldots, x_{i-4}, x_{i-2}, x_{i+1}, x_{i+3}, \ldots\} \cup \cdots \cup V_{i-3} \cup V_{i-1} \cup V_i \cup V_{i+2} \cup \cdots$, and x_i beats $\{\ldots, x_{i-3}, x_{i-1}, x_{i+2}, x_{i+4}, \ldots\} \cup \cdots \cup V_{i-2} \cup V_i \cup V_{i+1} \cup V_{i+3} \cup \cdots$. So x_{i-1} and x_i dominate T.
2. For $0 \leq i < j \leq k$ with $j > i+1$, vertices x_i and x_j both lose to x_k if i and j are even, to any vertex in V_0 if i and j are odd, and to x_{i+1} if $i - j$ is odd. So x_i and x_j do not dominate T.
3. For $0 \leq i \leq k$, let $y \in V_i$. Then x_i beats $\{\ldots, x_{i-3}, x_{i-1}, x_{i+2}, x_{i+4}, \ldots\} \cup \cdots \cup V_{i-2} \cup V_i \cup V_{i+1} \cup V_{i+3} \cup \cdots$, and y beats $\{\ldots, x_{i-4}, x_{i-2}, x_{i+1}, x_{i+3}, \ldots\} \cup \cdots \cup V_{i-3} \cup V_{i-1} \cup V_{i+2} \cup V_{i+4} \cup \cdots$. So x_i and y dominate T.
4. Let $y \in V_0$. Since the subtournament on V_0 is strongly connected, some $z \in V_0$ beats y. Then for $1 \leq i \leq k$, vertices x_i and y both lose to x_0 if i is even, and to z if i is odd. So x_i and y do not dominate T.
5. Let $y \in V_i$ for $1 \leq i \leq k$. Then for $0 \leq j \leq k$ with $j \neq i$, vertices y and x_j both lose to x_k if i is odd and j is even, to any vertex in V_0 if i is even and j is odd, to x_i if $j - i$ is even and positive, and to x_{j+1} if $j - i$ is even and negative. So y and x_j do not dominate T.
6. For $0 \leq i \leq j \leq k$, let $y \in V_i$ and $z \in V_j$ with $y \neq z$. Then y and z both lose to x_i if $j - i$ is odd, and to x_j if $j - i$ is even. So y and z do not dominate T.

Thus the dominant pairs of T are exactly the edges of G. ■

What remains in characterizing domination graphs of tournaments is to determine which graphs with two or more components are domination graphs of tournaments. Manuscripts [3] and [4] address this issue. Also, see [4] for some references to some variants of the problem treated here.

Characterizing domination graphs for other types of digraphs remain largely unexplored. *Which graphs are domination graphs of oriented graphs (digraphs with at most one arc between each pair of vertices)? Semicomplete digraphs (digraphs with at least one arc between each pair of vertices)? Symmetric digraphs (or graphs)? Or just digraphs?*

It may also be useful to examine the complement of other graph constructions. The *niche graph* of a digraph D is the graph on the same vertices as D with an edge between two vertices if they either beat a common vertex or lose to a common vertex in D. The complement of the niche graph of a tournament T is $\text{dom}(T) \cap \text{dom}(T^R)$ where T^R is the tournament formed by reversing the arcs of T. Bowser, Cable, and Lundgren (private communication) have characterized these "mixed-pair graphs" for tournaments and hence have characterized niche graphs of tournaments. The *competition-resource graph* of a digraph D is the graph on the same vertices as D with an edge between two vertices if they both beat a common vertex and lose to a common vertex in D. The complement of the competition-resource graph of a tournament T is $\text{dom}(T) \cup \text{dom}(T^R)$. Since the domination graph of a tournament with n vertices has at most n edges (a corollary of Proposition 1), these graphs have at most $2n$ edges. *Can these graphs be characterized?*

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