Disjoint Cycles in Planar and Triangle-free Graphs

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Abstract

Let p denote the number of vertices in a graph and let q denote the number of edges. Two cycles in a graph are *disjoint* if they have no common vertices. Pósa proved that any graph with $q \geq 3p - 5$ contains two disjoint cycles. This result does not apply to planar graphs, since every planar graph has $q \leq 3p - 6$. In this paper, I show that any planar graph with $q \geq 2p$ contains two disjoint cycles. I also show that this bound is best possible and that there is no minimum number of edges in a planar graph which will ensure the graph contains 3 disjoint cycles. Furthermore, a sufficient condition for any triangle-free graph (and therefore any bipartite graph) to contain k disjoint cycles is given.

Dedicated to Professor Stephen T. Hedetniemi on the occasion of his 60th birthday

1 Introduction

Graph terminology not presented here can be found in [1]. Let G be a graph. Let p denote the number of vertices and let q denote the number of edges in G. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their endpoints. A planar graph drawn in such a way is called a *plane* graph. A plane graph G partitions the rest of the plane into regions and each such region is called a *face*. The *wheel with n spokes*, denoted W_n is the graph obtained by taking the cycle C_n and a vertex x, and joining every vertex of the cycle to x. This planar graph has $p = n + 1$ and $q = 2n = 2p - 2$. The *degree* of a vertex x is the number of vertices adjacent to x. A *leaf* in a graph is a vertex of degree 1. The *degree sum* of a subgraph H is the sum of the degrees of all the vertices in $V(H)$. Let H, K be two vertex disjoint subgraphs of a graph G. Then $E(H, K)$

JCMCC **31** (1999), pp. 177-182

is the set of edges with one endpoint in $V(H)$ and the other endpoint in $V(K)$. Euler's formula relates the number of vertices (p), edges (q), and faces (f) of a connected plane graph.

Theorem 1 *[1] If G is a connected plane graph, then*

$$p - q + f = 2.$$

Since every face in a plane graph is bounded by at least 3 edges, and every edge is in the boundary of exactly 2 faces, we get $2q \geq 3f$. From Euler's formula, is follows that in every planar graph,

$$q \leq 3p - 6.$$

A *subdivision of an edge uv* is obtained by introducting a new vertex w and replacing the edge uv with edges uw, vw. A *subdivision of a graph G* is any graph that can be obtained from G by a sequence of edge subdivisions. In 1930, Kuratowski gave a characterization of planar graphs.

Theorem 2 *[4] A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

It is well known that any graph with $q \geq p$ contains at least one cycle. Two cycles are *disjoint* if they have no vertices in common. There have been several results published of the form: If $q \geq f(p)$, then G contains k disjoint cycles. (See [2], [3], [5], [6], [7].)
The first result, published in 1963, is due to Pósa

Theorem 3 *[7] If $q \geq 3p - 5$ then G contains 2 disjoint cycles.*

This result is sharp, since the graph $K_3 + (p-3)K_1$ has $q = 3p-6$ and does not contain two disjoint cycles. Pósa's theorem does not apply to planar graphs, since as noted earlier, if G is a planar graph, then $q \leq 3p - 6$.
The *girth* of a graph is the length of its smallest cycle. A *triangle* is a cycle of length 3. A graph is called *triangle-free* if its girth is at least 4.
In this paper I give a bound on the number of edges a planar graph needs to ensure it contains 2 disjoint cycles and a bound on the number of edges a triangle-free graph needs to ensure it contains k disjoint cycles.

2 Disjoint cycles in planar graphs

In this section, I will use the following result on claw-free graphs due to Matthews. A graph is *claw-free* if it does not contain the graph $K_{1,3}$ as an induced subgraph.

Theorem 4 *[6] Let G be a claw-free graph with $q \geq p+6$. Then G contains 2 disjoint cycles.*

The next result appears as an exercies in [1]. For completeness, I will give a proof here.

Lemma 5 *Let G be a planar graph with $q \geq 2p - 3$. Then G contains a triangle.*

Proof Suppose a planar graph G has girth at least 4, and $q \geq 2p - 3$. Every face in a plane representation of G is bounded by at least 4 edges, and every edge is in exactly 2 faces, so $2q \geq 4f$. Using Euler's formula, we get

$$2 = p - q + f \leq p - q + q/2.$$

Rearranging, $q \leq 2p - 4$, a contradiction. Thus G contains a triangle. □

The following Lemma will be the base step for an inductive proof of the main theorem in this section.

Lemma 6 *Let G be a planar graph with $p = 6$ and $q \geq 12$. Then G contains 2 disjoint cycles.*

Proof Let G be a planar graph with $V(G) = \{a, b, c, d, e, f\}$ and $q \geq 12$. First suppose that G contains a claw $\{a, b, c, d\}$ with $ad, bd, cd \in E(G)$ and no edges between any pair of a, b, c. Since $q \geq 12$, there must be all possible edges from the set $\{a, b, c\}$ to the set $\{d, e, f\}$ and the set $\{d, e, f\}$ must induce a complete graph. But then G contains $K_{3,3}$ as a subgraph, contradicting the planarity of G. Thus, G must be claw-free. By Theorem 4, we have $q \geq 12 = p + 6$, so G contains 2 disjoint cycles. □

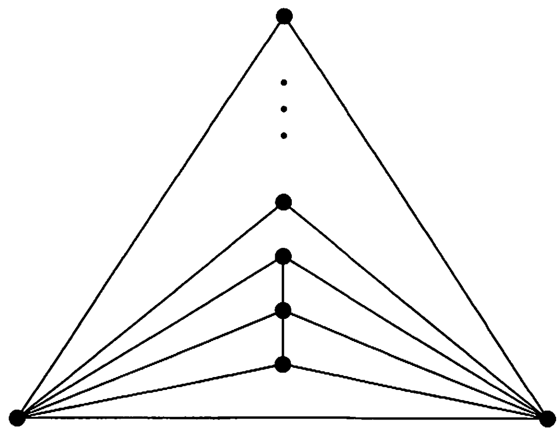
I will now prove the main result of this section.

Theorem 7 *If G is a planar graph with $q \geq 2p$, then G contains 2 disjoint cycles.*

Proof
Let G be the smallest connected planar graph for which the statement does not hold. By Lemmas 5 and 6, G must contain a triangle and have $p \geq 7$. Out of all the triangles in G, let $T = \{x, y, z\}$ be one with the smallest degree sum. Since G does not contain 2 disjoint cycles, the graph induced by $G - T$ is a forest.
Claim 1: $\delta(G) \geq 3$.
Suppose v is a vertex in G of degree 1 or 2. Then the graph $G - v$ is planar and has $|E(G-v)| \geq 2|V(G-v)|$, contradicting the minimality of G. Thus, $\delta(G) \geq 3$.

Claim 2: $G-T$ contains no vertices of degree 0.
Suppose $G-T$ contains a vertex v of degree 0 in $G-T$. Since in G vertex v has degree at least 3, we must have v adjacent to each of x, y, z. The minimality of the degree sum of T implies that $\deg(x) = \deg(y) = \deg(z) = \deg(v) = 3$. But G is connected, so $p = 4$, a contradiction. Thus $G-T$ contains no vertices of degree 0.
Claim 3: Every leaf of $G-T$ is adjacent to exactly two vertices of T and the third vertex must have degree 3.
Let l be a vertex of degree 1 in $G-T$. Suppose l is adjacent to each of x, y, z. Then $\deg(l) = 4$, and the minimality of the degree sum of T implies $\deg(x), \deg(y), \deg(z) \leq 4$. Thus there are at most 6 edges from T to $G-T$. Counting edges $2p \leq q \leq |E(T)|+|E(T, G-T)|+|E(G-T)| \leq 3+6+(p-4)$, giving $p \leq 5$, a contradiction. Thus, any leaf in $G-T$ is adjacent to 2 vertices in x, y, z only. Without loss of generality, leaf l is adjacent to x, y, so $\deg(l) = 3$ and the minimality of the degree sum of T imples that $\deg(z) = 3$.
Claim 4: All leaves in $G-T$ are adjacent to x, y but not to z.
Let l_1 be a leaf in $G-T$, with $l_1 \neq l$. Suppose l_1 is adjacent to y, z in T. By Claim 3, $\deg(x) = 3 = \deg(z)$. Vertex y has at most $p-3$ neighbours in $G-T$, each of x, z has at most one neighbour in $G-T$. Counting edges: $2p \leq q \leq |E(T)|+|E(T, G-T)|+|E(G-T)| \leq 3+(p-1)+(p-4) = 2p-2$, a contradiciton. Thus, every leaf in $G-T$ is adjacent to x, y but is not adjacent to z, and $\deg(z) = 3$.
We will now arrive at our final contradiciton. Since $\deg(z) = 3$, z has a neighbour a in $G-T$. Let C be the component of $G-T$ that contains a. Let l_1, l_2 be leaves of C in difference components of $C - \{a\}$. Then the graph containing x, y, z, l_1, l_2, a, the edges from $\{x, y\}$ to $\{z, l_1, l_2\}$, the edge az and the paths in $G-C$ from a to each of l_1, l_2 is a subdivision of $K_{3,3}$. This contradicts the planarity of G. This is our final contradiciton, so it must be the case that G contains 2 disjoint cycles. □

This result is best possible as shown by the following example: Start with the planar graph $K_{2,p-2}$, $p \geq 6$. Join the two vertices in the partite set of size 2 by an edge. Using 3 vertices in the partite set of size $p-2$, form a P_3. This graph is planar, has $q = 2p-1$ and does not contain 2 disjoint cycles.



I end this section by noting that there are planar graphs with the maximum possible number of edges which do not contain 3 disjoint cycles. For example, the graph obtained by taking the wheel W_{p-2} with $p-2$ spokes. Add one vertex and join it to every vertex except the one of degree $p-2$. This graph has p vertices, and $q = 3p - 6$, but does not contain 3 disjoint cycles. Thus, for a general planar graph, there is no minimum number of edges which will ensure the graph contains 3 disjoint cycles.

3 Disjoint cycles in triangle-free graphs and bipartite graphs

In this section I give a lowerbound on the number of edges needed in a triangle-free graph to ensure the graph contains k disjoint cycles. This bound is best possible.

Theorem 8 *Let G be a triangle-free graph. If $q \geq (2k-1)p-(2k-1)^2+1$ and $p \geq 4k$, then G contains k disjoint cycles.*

Proof

Clearly the statement is true for $k = 1$. We proceed by induction on k. Let $k \geq 2$ and suppose the statement is true for all $k' < k$. Let G be a triangle-free graph with $q \geq (2k-1)p - (2k-1)^2 + 1$ and $p \geq 4k$.
Let C be a shortest cycle in G and let $|C| = g \geq 4$. Consider the graph $G - C$. Note that $G - C$ is also triangle-free. If $G - C$ contains $k - 1$ disjoint cycles, then these cycles together with C form a set of k disjoint cycles in G. If $G - C$ does not contain $k - 1$ disjoint cycles, then by induction, $q(G - C) \leq (2k - 3)(p - g) - (2k - 3)^2$. If a vertex of $G - C$ is adjcacent to more than 2 vertices in C, then there is a cycle of length $< g$.

Thus, each vertex of $G - C$ is adjacent to at most two vertices in C and $|E(G - C, G)| \leq 2(p - g)$. Counting the edges in G, we get

$$\begin{aligned} q &= |E(C)| + |E(G - C, C)| + |E(G - C)| \\ &\leq g + 2(p - g) + (2k - 3)(p - g) - (2k - 3)^2 \\ &= (2k - 1)p - (2k - 1)^2 - (g - 4)(2k - 2) \\ &\leq (2k - 1)p - (2k - 1)^2 \end{aligned}$$

since $g \geq 4$, a contradiction. Thus, G must contain k disjoint cycles. □
A *bipartite graph* is a graph which contains no cycles of odd length. Clearly, a bipartite graph is triangle-free. We get the following corollary of Theorem 8.

Corollary 9 *Let G be a bipartite graph. If $q \geq (2k - 1)p - (2k - 1)^2 + 1$ and $p \geq 4k$, then G contains k disjoint cycles.*

These results are sharp, since the triangle-free (and bipartite) graph $K_{(2k-1),(p-2k+}$ has $q = (2k - 1)p - (2k - 1)^2$, and does not contain k disjoint cycles.

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