# Linear algorithms for $w$-medians of graphs*

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**Abstract**

Suppose $G = (V, E)$ is a graph in which every vertex $v$ has a non-negative real number $w(v)$ as its weight. The $w$-distance sum of $v$ is $D_{G,w}(v) = \sum_{u \in V} d(v,u)w(u)$. The $w$-median $M_w(G)$ of $G$ is the set of all vertices $v$ with minimum $w$-distance sum $D_{G,w}(v)$. This paper gives linear-time algorithms for computing the $w$-medians of interval graphs and block graphs.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

## 1 Introduction

The concept of center and median arise from facility location problems, which deal with the job of choosing a site subject to certain criterion. These distance related concepts have been extensively studied, see the book by Buckley and Harary [4]. In particular, algorithms have been developed for

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them, see [2, 5, 8, 10, 11, 12, 16, 17, 23, 24]. The purpose of this paper is to study $w$-medians of graphs from an algorithmic point of view.

All graphs in this paper are simple, i.e., undirected, loopless and without multiple edges. In a graph $G = (V, E)$, the *distance* $d_G(u, v)$ between two vertices $u$ and $v$ is the minimum number of edges in a $u$-$v$ path; $d_G(u, v) = \infty$ if there is no $u$-$v$ path. Suppose every vertex $v$ has a non-negative real number $w(v)$ as its weight. The *$w$-distance sum* of a vertex $v$ in $G$ is

$$D_{G,w}(v) = \sum_{u \in V} d_G(v, u) w(u).$$

The *$w$-median* $M_w(G)$ of $G$ is the set

$$M_w(G) = \{v \in V : D_{G,w}(v) \leq D_{G,w}(u) \text{ for all } u \in V\}.$$

For the case in which $w(v) = 1$ for all vertices $v$, the $w$-distance sum $D_{G,w}(v)$ is called the *distance sum* $D_G(v)$ and the $w$-median $M_w(G)$ is the *median* $M(G)$.

Slater [22] showed that for every (not necessarily connected) graph $H$ there exists a graph $G$ such that $H$ is the subgraph of $G$ induced by the median $M(G)$. Lee and Chang [14] generalized this result to $w$-medians. Zelinka [27] showed that the median of a tree is a clique of size one or two. This also follows from a more general result obtained by Truszczyński [25] that says the median of a connected graph lies in a block of the graph. Slater [22] showed that the median of a 2-tree is a clique of size at most three. Nieminen [18] and Yushmanov [26] proved that the median of a Ptolemaic graph is a clique. Lee and Chang [13] showed that the $w$-median of a strongly chordal graph is a clique if the weight function $w$ is positive. Note that trees are block graphs, block graphs are Ptolemaic, and Ptolemaic graphs are strongly chordal.

In general, the $w$-median of a graph $G = (V, E)$ can be computed by finding the distances between all pairs of vertices. A standard breadth-first-search, which costs $O(|V||E|)$ time, does the job. This paper employs the idea of the proof in [13] to obtain an algorithm for finding the $w$-median of a strongly chordal graph. This algorithm is then adapted to linear-time algorithms for the $w$-median problem in interval graphs and block graphs, which are both strongly chordal.

## 2 Strongly chordal graphs

A graph is *chordal* (or *triangulated*) if every cycle with more than three vertices has a *chord*, i.e., an edge joining two non-contiguous vertices of the cycle. A *p-sun* is a chordal graph with a vertex set $\{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p\}$

such that $\{y_1, y_2, ..., y_p\}$ is an independent set, $(x_1, x_2, ..., x_p, x_1)$ is a cycle and each vertex $y_i$ has exactly two neighbors $x_{i-1}$ and $x_i$, where $x_0 = x_p$. A graph $G$ is *strongly chordal* if it is chordal and contains no $p$-sun for $p \geq 3$. A vertex $v$ is *simple* if for any two vertices $x, y \in N_G[v]$ either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$, where $N_G(x) = \{y : xy \in E(G)\}$ and $N_G[x] = \{x\} \cup N_G(x)$. Note that if $v$ is a simple vertex, then $G - v$ is a *distance-invariant* induced subgraph of $G$, i.e., $d_{G-v}(x,y) = d_G(x,y)$ for all vertices $x$ and $y$ in $G - v$. A *maximal neighbor* of a simple vertex $v$ is a vertex $x \in N_G[v]$ such that $N_G[y] \subseteq N_G[x]$ for all $y \in N_G[v]$. Farber [6] proved that every strongly chordal graph that is not a complete graph has two non-adjacent simple vertices. Furthermore, a graph $G = (V, E)$ is strongly chordal if and only if it has a *simple ordering*, i.e., an ordering $(v_1, v_2, \cdots, v_n)$ of $V$ such that $v_i$ is a simple vertex of the graph $G - \{v_1, v_2, \cdots, v_{i-1}\}$. The main result of [13] is:

**Theorem 1** [13] *The $w$-median of a connected strongly chordal graph is a clique if $w$ is a positive weight function.*

We first sketch the proof of Theorem 1. In order to prove Theorem 1, [13] introduced the following more general concept. Every vertex $v$ in $G = (V, E)$ has a non-negative *weight* $w(v)$ and a non-negative *cost* $c(v)$. The *cost $w$-distance sum* of $v$ (*with respect to $w$ and $c$*) is

$$D_{G,w,c}(v) = \sum_{u \in V} d_G(v,u)w(u) - c(v).$$

The *cost $w$-median* $M_{w,c}(G)$ of $G$ is

$$M_{w,c}(G) = \{v \in V : D_{G,w,c}(v) \leq D_{G,w,c}(u) \text{ for all } u \in V\}.$$

It is easy to see that $M_{w,c}(G) = M_w(G)$ when $c(v) = 0$ for all vertices $v \in V$. However, unless we choose $c$ properly, it is not the case that one can modify Theorem 1 to get a cost $w$-median result for an arbitrary cost function $c$. The proof of Theorem 1 requires an inductive approach starting with a connected strongly chordal graph $G = (V, E)$ with a positive weight function $w$ and a cost function $c \equiv 0$. For induction, it uses the following two terms.

First, for any vertex $x \in V$, there exists a set $S_x = \{x \equiv x_0, x_1, \cdots, x_{n(x)}\} \subseteq N_G[x]$ such that (C1) and (C2) hold.

(C1) $N_G[x_0] \subseteq N_G[x_1] \subseteq \cdots \subseteq N_G[x_{n(x)}]$.

(C2) If $c(x) > 0$, then $n(x) \geq 1$ and $c(x) < \sum_{i=1}^{n(x)} w(x_i)$ and $c(x) < \sum_{i=1}^{j} w(x_i) + c(x_j)$ for $1 \leq j \leq n(x) - 1$.

Initially, each $S_x = \{x \equiv x_0\}$ and so conditions (C1) and (C2) hold. Secondly, it uses a poset (partially ordered set) $P$ whose elements are precisely the vertices of $G$ and $y < z$ in $P$ if $y = x_i$ and $z = x_j$ for some $x \in V$ with $0 \le i < j \le n(x)$. Note that $P$ is not necessarily a poset if $\{S_x : x \in V\}$ is not chosen properly. However, initially each $|S_x| = 1$ and so $P$ is simply a poset in which each pair of distinct elements is incomparable.

Theorem 1 is clear when $G$ is a complete graph. Suppose $G$ is not a complete graph. Choose a pair of non-adjacent simple vertices $u$ and $v$ (see [6]). Without loss of generality, we may assume that $u$ and $v$ are chosen so that they are as small as possible in the poset $P$. Suppose $u$ is not a minimal element in $P$. Then $u = x_i$ for some $x \in V$ and $x_i \in S_x$ with $i \ge 1$. Since $N_G[x_0] \subseteq N_G[x_i]$ and $x_i$ is a simple vertex not adjacent to $v$, $x_0$ is also a simple vertex not adjacent to $v$. But then $x_0$ is smaller than $u = x_i$ in $P$, which contradicts our choice of $u$. So, $u$ is a minimal element in $P$. Similarly, $v$ is minimal in $P$. Without loss of generality, we may assume that $w(v) + c(v) \le w(u) + c(u)$.

Now choose a maximal neighbor $m$ of $v$ in $G$. Without loss of generality, we may assume that $m$ is chosen so that it is as large as possible in the poset $P$. Suppose $m$ is not a maximal element in $P$. Then $m = x_i$ for some $x \in V$ and $x_i \in S_x$ with $i < n(x)$. Since $N_G[x_i] \subseteq N_G[x_{n(x)}]$ and $x_i$ is a maximal neighbor of $v$, $x_{n(x)}$ is also a maximal neighbor of $v$. But then $x_{n(x)}$ is larger than $m \equiv x_i$ in $P$, which contradicts our choice of $m$. So, $m$ is maximal in $P$.

Keeping all these results in mind, we now consider the distance-invariant subgraph $G - v$ of $G$, denoted by $G' = (V', E')$, which is also connected strongly chordal. We define the new weight function $w'$ and the new cost function $c'$ on $V'$ as:

$$w'(x) = w(x) + w(v) \text{ if } x = m \text{ and } w'(x) = w(x) \text{ otherwise,}$$

$$c'(y) = c(y) + w(v) \text{ if } y \in N_G(v) - \{m\} \text{ and } c'(y) = c(y) \text{ otherwise.}$$

It remains true that $w'$ is positive and $c'$ is non-negative. We also update $\{S_x : x \in V\}$ and $P$ as follows. Since $m$ is a maximal element of the poset $P$, for any vertex $x \in V$, either $m \notin S_x$ or $m = x_{n(x)}$. Let

$$S'_x = S_x \cup \{m\} \text{ if } x \in N_G(v) \text{ with } m \notin S_x \text{ and } S'_x = S_x \text{ otherwise.}$$

Now the poset $P'$ contains elements of $V'$. Some new relationships are also added to $P'$ when $S'_x = S_x \cup \{m\}$ for some $x \in V'$. However, since $m$ is a maximal element in $P$, $P'$ remains a poset even when new relationships are added to it.

Theorem 1 then follows from induction and the following lemmas.

**Lemma 2** [13] $M_{w,c}(G) = M_{w',c'}(G')$.

**Lemma 3** [13] *$\{S'_x : x \in V'\}$ satisfies conditions (C1) and (C2) of $G'$.*

**Lemma 4** *If $G$ is a complete graph, then $M_{w,c}(G) = \{u \in V : w(u) + c(u) \geq w(v) + c(v)$ for all $v \in V\}$.*

**Proof.** Lemma 4 follows from that fact that for any vertices $v$ and $u$ of $G$,

$$\begin{aligned} D_{w,c}(v) - D_{w,c}(u) &= \Big(\sum_{x \neq v} w(x) - c(v)\Big) - \Big(\sum_{x \neq u} w(x) - c(u)\Big) \\ &= (w(u) + c(u)) - (w(v) + c(v)). \end{aligned}$$

Q.E.D.

To implement the idea of the proof for Theorem 1, we need not to keep the sets $S_x$ and the poset $P$. Instead, a "mark" is given to each vertex $v$ that ensures the maximal neighbor of a simple vertex $v$ is the last deleted vertex in $N_G(v)$. More precisely, we have the following algorithm.

**Algorithm MS.** Compute the $w$-median of a connected strongly chordal graph.
**Input:** A connected strongly chordal graph $G = (V, E)$ in which every vertex $v$ has a positive weight $w(v)$.
**Output:** The $w$-median $M_w(G)$.
**Method:**

**begin**
  $c(v) \leftarrow 0$ for all $v \in V$;
  mark$(v) \leftarrow 0$ for all $v \in V$;
  **while** $V$ is not a clique **do**
    find two non-adjacent simple vertices $u$ and $v$ in $G$ with the smallest
      mark$(u)$ + mark$(v)$, W.L.O.G., assume $w(v) + c(v) \leq w(u) + c(u)$;
    choose a maximal neighbor $m$ of $v$ with the largest mark$(m)$;
    $w(m) \leftarrow w(m) + w(v)$;
    $c(y) \leftarrow c(y) + w(v)$ for all $y \in N(v) - \{m\}$;
    mark$(m) \leftarrow \max\{$mark$(y) : y \in N(v)\} + 1$;
    $G \leftarrow G - v$;
  **enddo**;
  $M_w(G) \leftarrow \{u \in V : w(u) + c(u) \geq w(v) + c(v)$ for all $v \in V\}$;
  **output**$(M_w(G))$
**end.**

Since finding two simple vertices during any iteration is costly for a general strongly chordal graph, the time complexity of this algorithm is greater than $O(|V||E|)$. However, in the next two sections, we shall modify the algorithm to obtain linear-time algorithms for the problem in interval graphs and block graphs.

# 3 Interval graphs

A graph $G = (V, E)$ is an *interval graph* if there exists a family $\{I_v : v \in V\}$ of intervals such that two distinct vertices $u$ and $v$ are adjacent in $G$ if and only if $I_u \cap I_v \neq \emptyset$; such a family $\{I_v : v \in V\}$ is referred to as an *interval representation* of $G$.

Gilmore and Hoffman [7] showed that a graph is an interval graph if and only if its maximal cliques can be linearly ordered into $C_1, C_2, \cdots, C_m$, such that for every vertex $v$, the maximal cliques containing $v$ occur contiguously. Suppose for every vertex $v$, $i_v$ (respectively, $j_v$) is the minimum (respectively, maximum) index $i$ such that $v \in C_i$. $\{[i_v, j_v] : v \in V\}$ is then an interval representation of $G$, which we call a *canonical representation.* Booth and Lueker [3] gave an $O(|V| + |E|)$-time algorithm for an arbitrary graph $G = (V, E)$ that tests whether $G$ is an interval graph. In the case in which $G$ is an interval graph, the algorithm also gives an ordering $C_1, C_2, ..., C_m$ of its maximal cliques, and so, a canonical interval representation.

Roberts [21], Ramalingam and Pandu Rangan [20], and Olariu [19] gave another characterization in which a graph $G$ is an interval graph if and only if an *interval ordering* $(v_1, v_2, \cdots, v_n)$ of $V$ exists such that $i < j < k$, and $v_i v_k \in E$ imply $v_j v_k \in E$, or equivalently, $i \leq j \leq k$ and $v_i \in N_G[v_k]$ imply $v_j \in N_G[v_k]$.

**Lemma 5** *Any interval ordering $(v_1, v_2, \cdots, v_n)$ of a graph $G$ is a simple ordering. Consequently, an interval graph is a strongly chordal graph.*

**Proof.** We only need to show that $v_1$ is a simple vertex of $G$. Suppose $v_i, v_j \in N_G[v_1]$. Assume $i \leq j$. We shall prove that $N_G[v_i] \subseteq N_G[v_j]$. Suppose $v_k \in N_G[v_i]$. For the case in which $j \leq k$, since $i \leq j \leq k$ and $v_i \in N_G[v_k]$, $v_j \in N_G[v_k]$ and so $v_k \in N_G[v_j]$. For the case in which $k < j$, since $1 \leq k \leq j$ and $v_1 \in N_G[v_j]$, $v_k \in N_G[v_j]$. Thus, $v_1$ is a simple vertex of $G$. Q.E.D.

Suppose $\{I_v = [a_v, b_v] : v \in V\}$ is an interval representation of an interval graph $G$. Sort the right end-point $b_v$'s of the intervals $I_v$'s into $b_{v_{r(1)}} \leq b_{v_{r(2)}} \leq \cdots \leq b_{v_{r(n)}}$. It is straightforward to check that $R = (v_{r(1)}, v_{r(2)}, \cdots, v_{r(n)})$ is an interval ordering of $G$. Similarly, if we sort the left end-point $a_v$'s into $a_{v_{l(1)}} \geq a_{v_{l(2)}} \geq \cdots \geq a_{v_{l(n)}}$, then $L = (v_{l(1)}, v_{l(2)}, \cdots, v_{l(n)})$ is an interval ordering of $G$. Note that if the interval representation is canonical, then we can use bucket sorts to sort the interval end-points allowing $R$ and $L$ to be computed in linear time. Suppose $G_{ij}$ is the graph $G - \{v_{r(1)}, v_{r(2)}, \cdots, v_{r(i-1)}, v_{l(1)}, v_{l(2)}, \cdots, v_{l(j-1)}\}$. If $v_{r(i)}$ and $v_{l(j)}$ are in $G_{ij}$, then they are simple vertices of $G_{ij}$. This, and the following lemma, can be used for efficient implementation of Algorithm MS for interval graphs.

**Lemma 6** *If $v_{r(i)}$ and $v_{l(j)}$ are in $G_{ij}$ and $v_{r(i)} \in N_{G_{ij}}[v_{l(j)}]$, then $G_{ij}$ is a complete graph.*

**Proof.** Since $I_{v_{r(i)}}$ ($I_{v_{l(j)}}$) has the smallest (largest) right (left) endpoints among all vertices in $G_{ij}$ and $v_{r(i)} \cap v_{l(j)} \neq \emptyset$, $a_{v_{r(i)}} \leq a_{v_{l(j)}} \leq b_{v_{r(i)}} \leq b_{v_{l(j)}}$. If $x = v_{r(i')} = v_{l(j')}$ is a vertex in $G_{ij}$, then $a_x \leq a_{v_{l(j)}} \leq b_{v_{l(j)}} \leq b_x$ and so $I_x$ contains $a_{v_{l(j)}}$. Thus $G_{ij}$ is a complete graph. Q. E. D.

We are now able to modify Algorithm MS to get a linear-time algorithm for the $w$-median of an interval graph.

**Algorithm MI.** Compute the $w$-median of a connected interval graph.
**Input:** A connected interval graph $G = (V, E)$ in which every vertex $v$ has a positive weight $w(v)$. Two interval orderings $R$ and $L$ as above.
**Output:** The $w$-median $M_w(G)$ of $G$.
**Method:**

**begin**
  $i \leftarrow 1$;
  $j \leftarrow 1$;
  $c(v) \leftarrow 0$ for all $v \in V$;
  **while** $v_{r(i)} \notin N_{G_{ij}}[v_{l(j)}]$ **do**
    **if** $w(v_{r(i)}) + c(v_{r(i)}) \leq w(v_{l(j)}) + c(v_{l(j)})$ **then**
      choose a maximal neighbor $m$ of $v_{r(i)}$;
      $w(m) \leftarrow w(m) + w(v_{r(j)})$;
      $c(y) \leftarrow c(y) + w(v_{r(j)})$ for all $y \in N(v_{r(j)}) - \{m\}$;
      $G \leftarrow G - v_{r(j)}$;
    **else**
      choose a maximal neighbor $m$ of $v_{l(j)}$;
      $w(m) \leftarrow w(m) + w(v_{l(j)})$;
      $c(y) \leftarrow c(y) + w(v_{l(j)})$ for all $y \in N(v_{l(j)}) - \{m\}$;
      $G \leftarrow G - v_{l(j)}$;
    **endif**;
    **while** $v_{r(i)}$ not in $G$ **do** $i \leftarrow i + 1$;
    **while** $v_{l(j)}$ not in $G$ **do** $j \leftarrow j + 1$;
  **enddo**;
  $M_w(G) \leftarrow \{u \in V : w(u) + c(u) \geq w(v) + c(v)$ for all $v \in V\}$;
  **output**$(M_w(G))$
**end.**

# 4 Block graphs

The concept of a block graph was introduced by Harary [9], who defined the *block graph* $B(G)$ of a graph $G$ as the intersection graph of blocks of $G$.

He then proved that a graph is the block graph of some graph if and only if all of its blocks are complete graphs. So, we may define a *block graph* as a graph whose blocks are complete graphs.

A graph with one or more cut-vertices contains at least two blocks, each of which contains exactly one cut-vertex; we call them *end blocks* (see [1]). If a graph has vertices $u$ and $v$ that are not in the same block, then any path from $u$ to $v$ must pass through a unique sequence of blocks $B_1, B_2, \ldots, B_n$, where $B_i$ and $B_{i+1}$, $i = 1, 2, \ldots, n-1$, have a common cut-vertex that is a vertex of the path. Moreover, for any graph $G$ containing $m$ blocks $B_1, B_2, \ldots, B_m$ and $n$ cut-vertices $c_1, c_2, \ldots, c_n$, consider the graph $G^* = (V^*, E^*)$, which we call the *block-cut-vertex structure* of $G$, where

$$\begin{aligned} V^* &= \{B_1, B_2, \ldots, B_m, c_1, c_2, \ldots, c_n\} \text{ and} \\ E^* &= \{(B_i, c_j) : 1 \le i \le m, 1 \le j \le n, c_j \in B_i\}. \end{aligned}$$

Then $G^*$ is a forest whose leaves are exactly the end blocks of $G$ and whose isolated vertices are exactly those blocks without cut-vertices in $G$. The *block-cut-vertex structure* $G^*$ of a graph $G$ can be constructed in linear time by using a depth-first search.

An *end vertex* of a block graph is a vertex in some end block but is not a cut-vertex. It is easy to show that an end vertex of a block graph is a simple vertex with the cut-vertex in the end block containing it being its maximal neighbor. Consequently, block graphs are strongly chordal. Note that if the block-cut-vertex structure is found then the two non-adjacent end vertices can be found in a constant time. Therefore, we now modify Algorithm MS to get a linear-time algorithm for the $w$-median problem in block graphs.

**Algorithm MB.** Compute the $w$-median of a connected block graph.
**Input:** A connected block graph $G = (V, E)$ in which every vertex $v$ has a positive weight $w(v)$ and its block-cut-vertex structure $T$.
**Output:** The $w$-median $M_w(G)$ of $G$.
**Method:**

**begin**
    $c(v) \leftarrow 0$ for all $v \in V(G)$;
    **while** $|V(T)| > 1$ **do**
        find two end vertices $u$ and $v$ from different end blocks $B_i$ and
            $B_j$ of $T$, W.L.O.G., assume $w(v) + c(v) \le w(u) + c(u)$;
        $w(m) \leftarrow w(m) + w(v)$   where $m$ is the cut vertex of $B_i$;
        $c(y) \leftarrow c(y) + w(v)$ for all $y \in B_i - \{m\}$;
        $G \leftarrow G - v$;
        $B_i \leftarrow B_i - \{v\}$;
        **if** $B_i = \{m\}$ **then**

$T \leftarrow T - B_i$;
**if** $m$ is a leaf of $T$ **then** $T \leftarrow T - m$ **endif**;
**endif**;
**enddo**;
$M_w(G) \leftarrow \{u \in V(G) : w(u) + c(u) \geq w(v) + c(v)$ for all $v \in V(G)\}$;
**output** $M_w(G)$
**end.**

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