Consecuti ,elings for Graphs

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Abstract

For a graph G, assign an integer value weight to each edge. For a vertex v, the label of v is the sum of weights of the edges incident with it. Further, the weighting is irregular if all the vertex labels are distinct. It is well known that if G has at most than one isolated vertex and no isolated edges then there exist irregular assignments, in fact, using positive edge weights.
In this paper, we consider the following special weighting:

- if G has order $2k+1$, then a consecutive labeling is an assignment where the vertex labels are precisely $-k, -k+1, ..., -1, 0, 1, 2, ..., k-1, k$;
- if G has order $2k$, then a consecutive labeling is an assignment where the vertex labels are precisely $-k+1, ..., -1, 0, 0, 1, 2, ..., k-1$.

Here we show that every graph which has an irregular assignment, also has a consecutive labeling. This concept is extended by considering all consecutive labelings and looking for one that has the smallest maximum, in absolute value, edge weight. This weight is referred to as the consecutive strength. Results parallel to the concept of irregularity strength are presented.

Dedicated to Professor Stephen T. Hedetniemi
on the occasion of his 60^{th} birthday.

1. Introduction.

Having a graph G, we assign an integer value $w(e)$ (the weight of e) to each edge e of G. The label $l(v)$ of a vertex v is the sum of the weights of the edges incident with it. The graph is called irregular if its vertices have different labels. An assignment of labels for vertices of G producing an irregular graph is called an irregular assignment. This concept was introduced in Chartrand et al [2] where it was observed that if a graph G contains a component of order 2, or more than one isolated vertex, then no irregular assignment exists. Throughout this paper we will consider only graphs with no component of order one or two. In [2], Chartrand et al. defined the irregularity strength $s(G)$ of a graph G as the minimum of the largest weight among the edges of G taken over all irregular assignments. The bounds for irregularity strength were studied by Aigner and Triesch [1], and by Jacobson and Lehel [6]. A variation on the irregularity concept was introduced by Jacobson et al [5]. Instead of minimizing the largest weight in an irregular assignment, the sum of the weights was minimized. The minimum sum was called the

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irregularity sum of a graph. A similar concept was investigated by Harary et al [3] and by Harary and Oellermann [4]. For many families of graphs, the minimum irregularity sum is attained when the vertex labels are consecutive integers. The discussion about graphs of this property (called consecutive graphs) is given in [5]. However, the characterization of consecutive graphs remains an open problem even for trees.

In this paper we extend the concept by allowing negative weights (and the weight 0) for edges. Our goal is to produce consecutive and symmetric labelings.

2. Consecutive labelings for graphs.

We say that a graph G with $2k+1$ vertices has a **consecutive labeling** if there exists an irregular, symmetric assignment producing labels $-k, -k+1, \ldots, -1, 0, 1, \ldots, k-1, k$. A graph G of even order $2k+2$ has a **consecutive labeling** if there exists an almost irregular, symmetric assignment producing labels $-k, -k+1, \ldots, -1, 0, 0, 1, \ldots, k-1, k$. For example, a graph given in Figure 1 has a consecutive labeling as shown.

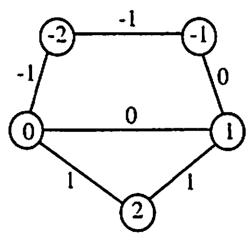

Figure 1

Theorem 1. Every tree has a consecutive labeling.

Proof. In the trivial case of K_1, there are obviously no weights and the label is 0. Let T be a tree of odd order $n = 2k+1$. We will construct a consecutive labeling by induction on n. Select a longest uv-path in the tree T. The tree T is either a path or it has at least three end-vertices. At least two of them, say u an v, must be in the same partite set and, therefore, the uv-path is of even length. By induction, the tree $T-u-v$ has consecutive labeling $-k+1, \ldots, -1, 0, 1, \ldots, k-1$. To produce a consecutive labeling for the tree T we assign the weight 0 to both edges incident with the vertices u and v first. Then, we alternately add weights k and $-k$ to existing weights on subsequent edges of the uv-path. This operation does not change labels on $T-u-v$ and produces labels k and $-k$ on the vertices u and v, respectively. For even values of n, we select an end-vertex u of T and construct, by induction, a consecutive labeling for the tree $T-u$ of order $n-1$. Then we assign the weight 0 to the edge incident with u. ♦

Theorem 2. Every graph with no isolated vertices and no components of order two has a consecutive labeling.

Proof. Notice first that it is enough to produce consecutive labeling for a spanning forest of a given graph G and then assign weight 0 to all remaining edges of G.

Let F be a forest of order n with no isolated vertices and no components of order two. We will construct a consecutive labeling for F by reducing the general case to simpler (smaller) cases.

Assume first that F has two end-vertices u and v such that the uv-path is of even length and $F - u - v$ has no component of order 1 or 2. Assume also that the forest $F - v - u$ has a consecutive labeling. This labeling can be extended to F like in the proof of Theorem 1.

Otherwise, every component of F is either P_3, or $K_{1,3}$, or a path of odd length.

If a forest F has at least two components $K_{1,3}$ and the smaller forest F_I without these components has a consecutive labeling with labels $-M, -M+1, ..., M-1, M$, then this labeling can be extended to two additional copies of $K_{1,3}$, as indicated in Figure 2.

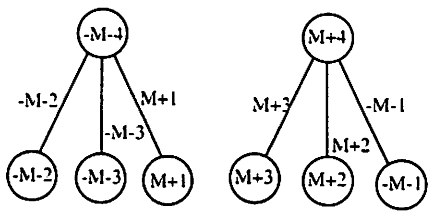

Figure 2

Therefore, we can assume that F has at most one component isomorphic to $K_{1,3}$.

If F contains a path of odd length which is not P_6, then we can reduce its length by 4 as indicated in Figure 3.

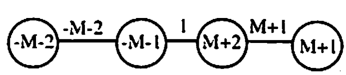

Figure 3

Thus, it remains to construct a consecutive labeling for forests of the form $kP_3 \cup rP_6 \cup sK_{1,3}$, where k and r are nonnegative integers and s is either 0 or 1.

Assume first that $s = 0$. Notice that we can consider P_6 as two copies of P_3 by assigning the weight 0 to the middle edge of P_6. Thus, in this case, we are dealing with $k + 2r = t$ copies of P_3. Let us observe that assigning weights a and b to the edges of P_3 produces the following triple of labels: a, b, $a+b$. Therefore, we need to partition numbers between $-M$ and M into such triples. We distinguish three cases depending on the number t of triples.

a) $t = 2x + 1$, where $x \equiv 0, 1 \bmod 4$.

We partition the numbers 1 through $3x$ into x triples of the form $\{a, b, a+b\}$ using Skolem triple system [7] and use them for assigning weights to x copies of P_3. The next x copies of P_3 are labeled in a similar way by multiplying all weights by -1. The edge weights and the corresponding vertex labels on the remaining copy of P_3 are given in Figure 4.

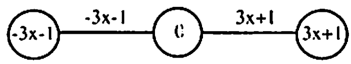

Figure 4

(b) $t = 2x + 1$, where $x \equiv 2, 3 \bmod 4$.

We partition the numbers $1, 2, \ldots, 3x-1, 3x+1$ into x triples of the form $\{a, b, a+b\}$ using O'Keefe triple system [7] and use them for assigning weights to x copies of P_3. The next x copies of P_3 are labeled in a similar way by multiplying all weights by -1. The remaining copy of P_3 has weights $-3x$ and $3x$, which produce labels 0, $3x$, and $-3x$.

(c) $t = 2x + 2$.

On $2x + 1$ copies of P_3 we use triples from either (a) or (b) obtaining labels $-3x-1, -3x, \ldots, -1, 0, 1, \ldots, 3x, 3x+1$. The last copy receives weights as in Figure 5 (the second 0 is produced as a label).

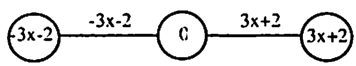

Figure 5

Let us finally consider a situation where one copy of $K_{1,3}$ is present, i.e. the forest F is of the form $kP_3 \cup rP_6 \cup K_{1,3}$. We produce a consecutive labeling $-M, \ldots, M$ on $kP_3 \cup rP_6$ using Skolem or O'Keefe triple systems. There are two triples of the form $\{1, x, x+1\}$, $\{-1, -x, -x-1\}$. We remove these labels from the corresponding copies of P_3 and on these two copies and on one copy of $K_{1,3}$ we introduce labeling as indicated in Figure 6. ♦

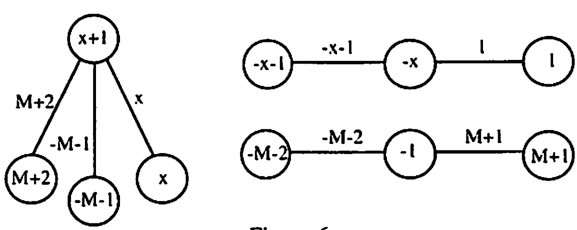

Figure 6

3. Consecutive strength of graphs.

Every graph containing neither K_2 nor K_1 has a consecutive labeling. One can ask about smallest weights producing a consecutive labeling. The **consecutive strength** of a graph G, denoted by $c(G)$, is defined as the minimum of the largest $|w(e)|$ among the edges of G for all assignments producing consecutive labelings. Therefore, if W is the set of all weight assignments for G producing consecutive labelings, then

$$c(G)=\min_{W}\left\{\max_{e\in E(G)}|w(e)|\right\}$$

Theorem 3. For $n \geq 3$, $c(K_n) = 1$.

Proof. Induction on n.

For K_3, the required weight assignment is given in Figure 7.

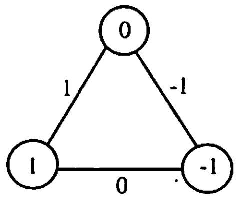

Figure 7

Suppose that K_n has a consecutive labeling with strength 1. Consider the graph K_{n+1}, $n \geq 3$, and distinguish two cases.

Case 1. $n + 1$ is even.
Remove one vertex, say v, from K_{n+1}. The remaining graph K_n has a consecutive labeling. Assign weight 0 to all edges incident with the vertex v.

Case 2. $n+1$ is odd.
Remove one vertex, say v, from K_{n+1}. The remaining graph K_n has consecutive labeling with labels $-\frac{n}{2}+1,-\frac{n}{2}+2,\ldots,-1,0,0,1,\ldots,\frac{n}{2}-2,\frac{n}{2}-1$. Assign weight 1 to all edges between v and vertices with positive labels and to the edge incident with one vertex with label 0. Assign weight -1 to all edges between v and vertices with negative labels and to the edge incident with the other vertex with label 0. The resulting graph K_{n+1} has labels $-\frac{n}{2},-\frac{n}{2}+1,\ldots,-1,0,1,\ldots,\frac{n}{2}-1,\frac{n}{2}$, where the label 0 is produced for v. ♦

Theorem 4. The complete bipartite graph $K_{n,n}$ with $n \geq 2$ has the consecutive strength 1.

Proof. By induction on n. We inductively construct a consecutive labeling of strength 1 satisfying an additional condition, namely, for every n and every partite set of $K_{n,n}$ the number of vertices with positive labels is the same as the number of vertices with negative labels.

If $n = 2$, then the weight assignment for $K_{2,2}$ is given in the Figure 8.

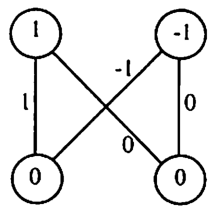

Figure 8

Suppose that $K_{n,n}$ has a consecutive labeling with strength 1 satisfying the extra condition. Consider the graph $K_{n+1,n+1}$, $n \geq 2$, and distinguish two cases.

Case 1. n is even.

Delete two vertices, say u and v, each from one partite set of $K_{n+1,n+1}$. The remaining graph $K_{n,n}$ has consecutive strength 1 and two vertices a and b with label 0 occur in one partite set. Without loss of generality suppose that v, a, and b are in the same partite set. Assign weights to the edges incident with u or v in the following way:

$w(ua) = 1$, $\quad w(ub) = -1$,
$w(ux) = 1$ if $l(x) > 0$, $\quad w(ux) = -1$ if $l(x) < 0$,
$w(vx) = 1$ if $l(x) > 0$, $\quad w(vx) = -1$ if $l(x) < 0$,
$w(uv) = 0$.

Then $l(a) = 1$, $l(b) = -1$, $l(u) = l(v) = 0$, all positive labels are increased by 1 (producing labels from 2 up to n), all negative labels are decreased by 1 (producing labels from -2 down to $-n$).

Case 2. n is odd.

Delete two vertices, say u and v, each from one partite set of $K_{n+1,n+1}$. The remaining graph $K_{n,n}$ has consecutive strength 1 and two vertices a and b with label 0 occur in its different partite sets. Assign weights to the edges incident with u or v in the following way:

$w(ua) = 0$, $\quad w(ub) = -1$,
$w(ux) = 1$ if $l(x) > 0$, $\quad w(ux) = -1$ if $l(x) < 0$,
$w(vx) = 1$ if $l(x) > 0$, $\quad w(vx) = -1$ if $l(x) < 0$,
$w(uv) = 1$.

Then $l(a) = 0$, $l(b) = -1$, $l(u) = 0$, and $l(v) = 1$, all positive labels are increased by 1 (producing labels from 2 up to n), all negative labels are decreased by 1 (producing labels from -2 down to $-n$). ♦

The next two theorems establish lower bounds for the consecutive strength of a graph.

Theorem 5. If G is a graph with n vertices and e edges, then

$$c(G) \geq \left\lceil \frac{3n-2e-4}{6} \right\rceil$$

Proof. Let c denote $c(G)$ and n_i denote the number of vertices of degree i in G. Of course, $\sum_{i=1}^{\Delta} in_i = 2e$. where Δ is the maximum degree of G. Therefore,

$$n_1 + 2n_2 + 3(n_3 + n_4 + \ldots + n_\Delta) \leq 2e,$$

or

$$n_1 + 2n_2 + 3[n - (n_1 + n_2)] \leq 2e,$$

which is equivalent to

$$n_1 + (n_1 + n_2) \geq 3n - 2e.$$

Labels of vertices of degree 1 are between $-c$ and c, so $n_1 \leq 2c + 2$ (two 0's are possible). Labels of $n_1 + n_2$ vertices of degree 1 or 2 are between $-2c$ and $2c$, so $n_1 + n_2 \leq 4c + 2$. Therefore,

$$2c + 2 + 4c + 2 \geq 3n - 2e,$$

which gives the required lower bound. ♦

Corollary 6. If T is a tree with n vertices, then $c(T) \geq \left\lceil \frac{n-2}{6} \right\rceil$.

Theorem 7. Let G be a graph with n vertices and maximum degree Δ. Then,

$$c(G) \geq \frac{n-1}{2\Delta} \quad \text{if } n \text{ is odd,}$$

$$c(G) \geq \frac{n-2}{2\Delta} \quad \text{if } n \text{ is even.}$$

Proof. Let n_i denote the number of vertices of degree i in G. In the case when n is odd,

$$n_1 + n_2 + \ldots + n_r \leq 2c(G)r + 1.$$

In particular, for $r = \Delta$, we have

$$n \leq 2\Delta c(G) + 1 \text{ or}$$

$$c(G) \geq \frac{n-1}{2\Delta}.$$

The case with even value of n is similar. ♦

The next four theorems use the above lower bound and specific constructions to establish the exact values of consecutive strength for paths. cycles, $K_{2,n}$, and wheels.

Theorem 8. For the path P_n with n vertices, $c(P_n) = \left\lceil \frac{n-2}{4} \right\rceil$.

Proof. Theorem 7 implies that $c(P_n) \geq \left\lceil \frac{n-2}{4} \right\rceil$. We will produce a weight assignment for the edges of P_n of strength $\left\lceil \frac{n-2}{4} \right\rceil$. Constructions depend on value of n mod 8.

<u>Case 1</u>. $n = 8k + 1$.

The path P_n has consecutive labeling of strength $2k$ given by the following construction. We list weights for edges of P_n starting from one pendant edge and going toward the central vertex of P_n. The weights on the other half of P_n are obtained by multiplying the weight of the symmetrical edge by -1.

The weight assignment for P_{8k+1}:

$2k, 2k, 2k-1, 2k-1, \ldots, k+1, k+1, k, k, k-1, k-1, \ldots, 2, 2, 1, 1, 0.$

Case 2. $n = 8k + 3$.

The weight assignment for P_{8k+3} of strength $2k+1$:

$2k+1, 2k, 2k, 2k-1, 2k-1, \ldots, k+1, k+1, -k+1, k, k, k-1, k-1, \ldots, 2, 2, 1.$

Case 3. $n = 8k + 5$.

The weight assignment for P_{8k+5} of strength $2k+1$:

$2k+1, 2k+1, 2k, 2k, \ldots, k+1, k+1, -k+1, k, k, k-1, k-1, \ldots, 2, 2, 1.$

Case 4. $n = 8i + 7$.

The weight assignment for P_{8k+7} of strength $2k+2$:

$2k+2, 2k+1, 2k+1, 2k, 2k, \ldots, k+2, k+2, k+1, k, k, k-1, k-1, \ldots, 2, 2, 1, 1, 0.$

For even values of n, we attach an extra vertex to an endpoint of P_{n-1} and assign weight 0 to the new edge. ♦

In the similar way we prove the corresponding result for cycles.

Theorem 9. For a cycle with n vertices, $c(C_n) = \left\lceil \frac{n-2}{4} \right\rceil$.

As an example, a weight assignment of strength 3 producing consecutive labeling for C_{14} is given in Figure 9.

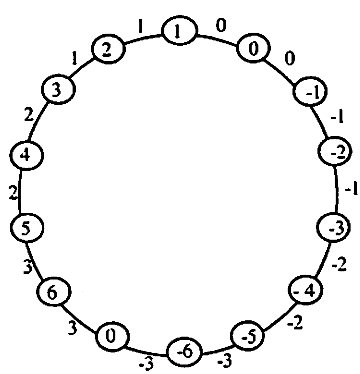

Figure 9.

Theorem 10. For stars $K_{1,n}$, $n \geq 2$, we have $c(K_{1,n}) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof. The star $K_{1,n}$ has $n_1 = n$ vertices of degree 1. Theorem 8 implies that $c(G) \geq \frac{n-2}{2}$ for n odd and $c(G) \geq \frac{n-1}{2}$ for n even. Therefore, $c(G) \geq \left\lceil \frac{n-1}{2} \right\rceil$ for all n. It is easy to construct a weight assignment of the required strength. ♦

Theorem 11. For the complete bipartite graph $K_{2,n}$, $n \geq 2$, we have

$$c(K_{2,n}) = \left\lceil \frac{n-2}{4} \right\rceil.$$

Proof. The graph $K_{2,n}$ has $n_1 = 0$ vertices of degree 1 and $n_2 = n$ vertices of degree 2. Theorem 8 implies that c(G) $\geq \frac{n-2}{4}$ for n even and c(G) $\geq \frac{n-1}{4}$ for n odd. Therefore, $c(G) \geq \left\lceil \frac{n-2}{4} \right\rceil$. Weight assignments of the required strength are given below and they depend on the value of n (mod 4).

<u>Case 1.</u> $n = 4k$, $k \geq 1$ (see Figure 10).

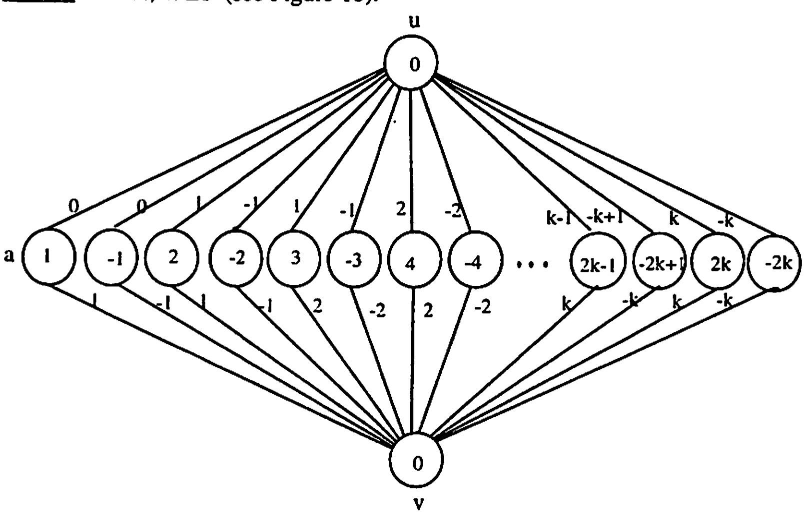

Figure 10.

Case 2. $n = 4k + 1,\ k \geq 1$.

We modify the weight assignment of $K_{2,4k}$ by adding one extra vertex x adjacent to u and to v and by assigning weights $w(ux) = k,\ w(vx) = -k$. We also increase by 1 the weights on $k+1$ consecutive edges incident with u (starting from the edge ua) and decrease by 1 the weights on $k+1$ symmetrical edges incident with v. The new vertex x has label 0, the labels of all other vertices in the larger partite set remain unchanged, the label of u is $2k+1$, and the label of v is $-2k-1$.

Case 3. $n = 4k + 2,\ k \geq 1$.

We modify the weight assignment of $K_{2,4k}$ by adding two extra vertices x and y adjacent to both u and v with weights $w(ux) = w(uy) = k,\ w(vx) = w(vy) = -k$. We also increase the weight on the edge ua by 1 and decrease the weight on the edge av by 1.

Case 4. $n = 4k - 1,\ k \geq 1$.

We modify the weight assignment of $K_{2,4k}$ by deleting the vertex a and by replacing the weight 0 on the next edge incident with u by the weight 1. New labels for vertices u and v are 1 and -1, respectively. ♦

Theorem 12. For a wheel W_n of order $n+1$, $c(W_n) = \left\lceil \frac{n-1}{6} \right\rceil$.

Proof. The wheel W_n has $n_1 = 0,\ n_2 = 0$, and $n_3 = n$. Theorem 8 implies that $c(W_n) \geq \frac{n-2}{6}$ for n odd and $c(W_n) \geq \frac{n-1}{6}$ for n even.

One can construct a weight assignment of the required strength. For example, if $n = 13$ the weight assignment given in Figure 11 has strength 2.

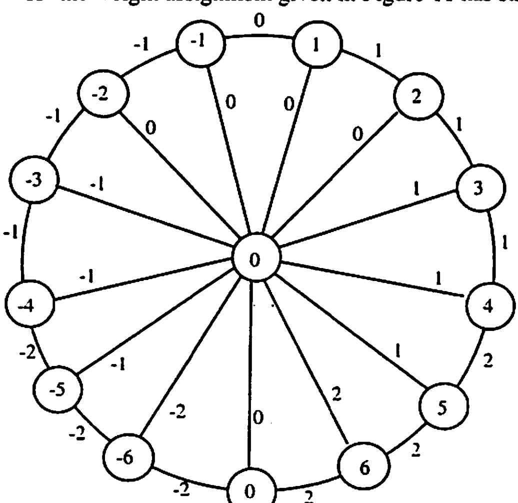

Figure 11.

A slight modification of this pattern gives a weight assignment for W_{2k}, $k \geq 2$. ♦

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