Lower Bounds for Dominating Cartesian Products

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Abstract

A well-known problem in domination theory is the long standing conjecture of V.G. Vizing from 1963 (see [7]) that the domination number of the Cartesian product of two graphs is at least as large as the product of the domination numbers of the individual graphs. Although limited progress has been made this problem essentially remains open. The usefulness of a maximum 2-packing in one of the graphs in establishing a lower bound has been recognized for some time. In this paper, we shall extend this approach so as to take advantage of 2-packings whose membership can be altered in a certain way. This results in an improved lower bound for graphs which have 2-packings of this type.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60^{th} birthday.

Key words: domination number, Cartesian product, Vizing's conjecture, clique

AMS subject classifications: 05C70, 05C99

1 Introduction and Terminology

In this paper we consider only finite, simple graphs and refer the reader to [5] for terminology and notation, some of which is repeated here for completeness. For vertex subsets X and Y of a graph $G = (V, E)$ we say that X *dominates* Y if $Y \subseteq N[X]$. In particular, when X dominates V we call X a *dominating set* for G. The *domination number* of G is the smallest cardinality, $\gamma(G)$, of a dominating set for G. A subset A of V is called a *2-packing* of G if $N[x] \cap N[y] = \emptyset$ for every pair $x, y \in A$. The *2-packing number* of G is the maximum cardinality, $\rho(G)$, of a 2-packing of G. Since every dominating set for G has a nonempty intersection with each closed neighborhood, it follows that $\rho(G) \leq \gamma(G)$. We use $|G|$ to denote the order of G.

If $G = (V, E)$ and $H = (W, F)$ are graphs, then the *Cartesian product* of G and H is the graph $G \square H$, whose vertex set is the (set) Cartesian product $V \times W$. Two vertices (v_1, w_1) and (v_2, w_2) of $G \square H$ are adjacent if and only if they are equal in one coordinate and adjacent in the other coordinate. Note that we distinguish between the Cartesian product of sets, which is denoted by $\times$, and the Cartesian product of two graphs, which is denoted using the symbol $\square$. It often becomes convenient to consider the subgraph of $G \square H$ induced by the set of vertices $\{u\} \times W$. This subgraph is isomorphic to H and is denoted by H_u.

In [7], the problem of deciding if $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ for all graphs G and H was posed. Probably the most substantial result (and unfortunately not widely known until much later) is that of Barcalkin and German [1] which shows that if a graph G has the property that it is a spanning subgraph of a graph G^* with the same domination number such that the vertex set of G^* can be partitioned into $\gamma(G)$ cliques , then $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ for any graph H. We say such a graph G satisfies the *Barcalkin-German condition*. The reader is referred to [4] for a discussion of other results and a survey of progress on this problem. In [6] the problem of determining when equality can actually be achieved in the case that both graphs are trees was addressed. In [2], Hartnell considered the question of how much the domination number of $G \square H$ exceeds the product $\gamma(G)\gamma(H)$. This was examined under the special circumstances that at least one of the graphs, not necessarily a tree, had the structure that every vertex was either a leaf or had at least one leaf as a neighbor and the other graph had several disjoint maximum 2-packings. This was an attempt to gain some measure of the actual excess (over equality in the conjectured lower bound of Vizing) in dominating the Cartesian product. It has also been observed by Jacobson and Kinch in [6] that $\gamma(G \square H) \geq \max\{\rho(G)\gamma(H), \rho(H)\gamma(G)\}$. In this paper, we try to improve this bound by examining more carefully the situation when one of the graphs has the property that it has several maximum 2-packings, and these 2-packings have all but one member in common.

2 Lower Bounds

We first consider, for a given graph G, a subset S of vertices of G and the subgraph of $G \square H$ induced by $N[S] \times V(H)$ for any other graph H. We determine a lower bound on the number of vertices required to dominate $S \times V(H)$ using vertices from $N[S] \times V(H)$. This result is then employed to give a lower bound on the domination number of $G \square H$. As a very simple illustration, let G be the graph on 6 vertices consisting of a 4-cycle, v_1, v_2, v_3, v_4, and two other vertices, x and y, which are leaves and adjacent to a single vertex, v_1, on the 4-cycle. Since G has a 2-packing which is of the same order as its domination number, it is known that $\gamma(G \square H) \geq \gamma(G)\gamma(H)$, for any graph H. We shall show that, taking $S = \{x, y\}$, at least $\gamma(H) + 1$ vertices are required in $G \square H$ just to dominate $S \times V(H)$. Furthermore, these vertices would have to belong to $N[S] \times V(H)$ and hence, considering $\{v_3\} \times V(H)$, and noting $N[v_3]$ is disjoint from $N[S]$, there must be at least another $\gamma(H)$ vertices in any minimum dominating set of $G \square H$. That is, $\gamma(G \square H) \geq \gamma(G)\gamma(H) + 1$.

Lemma 2.1 *Let G be a connected graph and let $L \subseteq V(G)$ have cardinality m. For any connected graph H consider the Cartesian product $G \square H$, and let D be a subset of $N[L] \times V(H)$ of minimum cardinality such that D dominates $L \times V(H)$. If L is independent, then $|D| \geq \min\{\gamma(H) + m - 1, |H|\}$. If L is not independent, then $|D| \geq \min\{\gamma(H) + m - 2, |H|\}$.*

Proof. First assume that $L = \{u_1, u_2, \ldots, u_m\}$ and that L is independent. Let D be as in the statement of the lemma. Observe that for each i, $1 \leq i \leq m$, it must be the case that $D \cap V(H_{u_i}) \neq \emptyset$, for otherwise $|D| \geq |D \cap (N(u_i) \times V(H))| \geq |H|$. For each i, let $\alpha_i = |D \cap V(H_{u_i})|$. Then it follows that

$$\begin{aligned} |D \cap (N[L] \times V(H))| &\geq \gamma(H) - \min\{\alpha_1, \ldots, \alpha_m\} + m \min\{\alpha_1, \ldots, \alpha_m\} \\ &\geq \gamma(H) + m - 1. \end{aligned}$$

Thus we assume that L is not independent. In addition, suppose $N[L]$ induces a complete subgraph of G. Assume, by way of contradiction, that $|D| \leq \gamma(H) + m - 3$. Let $f : N[L] \times V(H) \to H_{u_1}$ be the projection map defined by $f(a, x) = (u_1, x)$.

As in the case when L is independent, it follows that $D \cap V(H_{u_i}) \neq \emptyset$ for each i. For $1 \leq i \leq m$ let $R_i = D \cap H_{u_i}$ and let $S_i = \{v \in V(H) | (u_i, v) \in$

$R_i\}$. In addition, let $R_0 = D - \cup_{i=1}^m R_i$ and $S_0 = \{v \in V(H) | (w,v) \in R_0$ for some $w \in N[L] - L\}$. For a given i, consider how $V(H_{u_i})$ is dominated by D. Any vertex (u_i, v) of H_{u_i} that does not belong to $N[R_i]$ must be adjacent to a vertex $(u_j, v) \in R_j$, for some j, $j \neq i$, or to a vertex $(w,v) \in R_0$. For each $1 \leq i \leq m$, let

$$C_i = \{x \in V(H) | x \notin \cup_{i=0}^m S_i \text{ but } (u_i, x) \in N(R_i)\}.$$

Note that if $C_i = \emptyset$, then $\cup_{j=0}^m S_j = V(H)$ and so $|D| \geq |H|$. Thus assume $C_i \neq \emptyset$ for every i. Observe that for $j \neq i$, S_j dominates every vertex $y \in C_i$ or otherwise (u_j, y) would not be dominated by D. Therefore, $C_1 = C_2 = \cdots = C_m$.

Let $c \in C_1$. For each $1 \leq i \leq m$, the vertex (u_i, c) has a neighbor (u_i, x_i) in R_i. Note that it is possible for there to be pairs of distinct indices $i \neq j$ for which $S_i \cap S_j \neq \emptyset$, and so it is possible that $x_i = x_j$. Let

$$D' = (D - \{(u_1, x_1), (u_2, x_2), \ldots, (u_{m-1}, x_{m-1})\}) \cup \{(u_1, c)\}.$$

Now $|D'| = |D| - m + 2$, $f(D')$ dominates H_{u_1} and

$$|f(D')| \leq |D'| = |D| - m + 2 \leq \gamma(H) + m - 3 - m + 2 = \gamma(H) - 1,$$

a contradiction. Therefore, $|D| \geq \min\{\gamma(H) + m - 2, |H|\}$.

If L is not independent, but $N[L]$ is not a complete subgraph of G, then add edges to G to make a graph G^* in which $N_{G^*}[L]$ is a complete subgraph. Since $G \square H$ is a spanning subgraph of $G^* \square H$, any subset of $N[L] \times V(H)$ which dominates $L \times V(H)$ in $G \square H$ also dominates $L \times V(H)$ in $G^* \square H$. By the above argument $|D| \geq \min\{\gamma(H) + m - 2, |H|\}$. □

In certain cases the graph G lends itself to applying the above lemma in a number of places and the resulting 'domination excess' can be added.

Corollary 2.2 *For each i such that $1 \leq i \leq n$ let S_i be a star with m_i leaves, and let G be any connected graph formed by arbitrarily adding edges between the centers of these stars. If H is any connected graph of order at least $\gamma(H) + \max\{m_1, m_2, \ldots, m_n\} - 1$, then $\gamma(G \square H) \geq \gamma(G)\gamma(H) + (m_1 - 1) + \cdots + (m_n - 1)$.*

We note that if $t \geq \frac{1}{2}(\max\{m_1, m_2, \ldots, m_n\} - 1)$, then the graph H formed by letting t 4-cycles share a common vertex shows the lower bound in Corollary 2.2 can be attained.

In a more general setting, in the case there are alternate vertices for a maximum 2-packing, we can combine the local result established in Lemma 2.1 with the rest of the Cartesian product and obtain an improved lower bound on the domination number of the Cartesian product.

Theorem 2.3 *Let G be a connected graph and let A be a maximum 2-packing of G. Assume A contains a vertex u_1 such that for some $B \subseteq V(G)$ of cardinality at least one, $(A - \{u_1\}) \cup \{x\}$ is a maximum 2-packing for each $x \in B$. Let $L = B \cup \{u_1\}$ have cardinality m, and assume that if L is not independent, then sufficient edges can be added to G to make a graph G^* in which $N_{G^*}[L]$ is complete and $\gamma(G^*) = \gamma(G)$. Let H be any connected graph. If L is independent, then $\gamma(G \square H) \geq \min\{\rho(G)\gamma(H) + m - 1, (\rho(G) - 1)\gamma(H) + |H|\}$ and if L is not independent then $\gamma(G \square H) \geq \min\{\rho(G)\gamma(H) + m - 2, (\rho(G) - 1)\gamma(H) + |H|\}$*

Proof. Let D be any minimum dominating set for $G \square H$. By Lemma 2.1, $|D \cap (N[L] \times V(H))| \geq \min\{\gamma(H) + m - t, |H|\}$ where $t = 1$ if L is independent and $t = 2$ otherwise. Since D dominates all of $G \square H$ and A is a 2-packing of G, it follows that $|D \cap (N[A - \{u_1\}] \times V(H))| \geq (\rho(G) - 1)\gamma(H)$ and so the result follows. □

In the special case that G is a graph such that $\gamma(G) = \rho(G)$ and G has a maximum 2-packing of the type in Theorem 2.3, then it is possible to conclude that no graph H (of large enough order) exists for which $\gamma(G \square H) = \gamma(G)\gamma(H)$.

Theorem 2.4 *Let G be a connected graph with a 2-packing A of cardinality $\gamma(G)$. Assume A contains a vertex u_1 such that for some $B \subseteq V(G)$ of cardinality at least one, $(A - \{u_1\}) \cup \{x\}$ is a 2-packing of cardinality $\gamma(G)$ for each $x \in B$. Assume $L = B \cup \{u_1\}$ has cardinality m, and let H be any connected graph. If L is independent, then $\gamma(G \square H) \geq \min\{\gamma(G)\gamma(H) + m - 1, (\gamma(G) - 1)\gamma(H) + |H|\}$, and if L is not independent then*

$$\gamma(G \square H) \geq \min\{\gamma(G)\gamma(H) + m - 2, (\gamma(G) - 1)\gamma(H) + |H|\}.$$

Proof. Since $\gamma(G) = \rho(G)$ it is straightforward to see that adding edges to $N[L]$ to form a complete subgraph as in Theorem 2.3 gives a graph G^* with $\gamma(G^*) = \gamma(G)$, and the theorem follows immediately. □

The following corollary gives a start on answering Open Question 5 on page 186 of the survey chapter by Hartnell and Rall, [4]. The proof follows immediately from Theorem 2.4 and the fact that for any tree the domination number and the 2-packing number are equal.

Corollary 2.5 *If a tree T has a vertex adjacent to at least two leaves, then for any connected graph H of order at least two, $\gamma(T \square H) > \gamma(T)\gamma(H)$.*

If G is any graph with domination number one, then $\rho(G) = 1 = \gamma(G)$ and $\{x\}$ is a maximum 2-packing, for every vertex x of G. Thus we have the following corollary which includes the special cases when G is a star or a complete graph.

Corollary 2.6 *If G is a graph of order n and domination number one, then for any connected graph H of order at least two,* $\gamma(G\Box H) \geq \min\{|H|, \gamma(H)+n-2\}$.

Consider the path $P_{3r} : x_1, x_2, x_3, \ldots, x_{3r}$. We note that there exists a maximum 2-packing A of P_{3r} such that $x_3 \in A$. The next corollary follows immediately from Theorem 2.4.

Corollary 2.7 *There does not exist a connected graph H of order at least two such that* $\gamma(P_{3r} \Box H) = \gamma(P_{3r})\gamma(H)$.

Although our main focus has been on employing 2-packings to improve the lower bound of Jacobson and Kinch, combining Lemma 2.1 with other results can be informative as demonstrated in the following theorem. Recall that a vertex x of a graph is called a *simplicial* vertex if $N[x]$ is a clique.

Theorem 2.8 *Let G be a graph with the property that it is a spanning subgraph of a graph G^* with the same domination number where the vertex set of G^* can be partitioned into $\gamma(G)$ cliques. Furthermore, in G^* at least one of these cliques, say C, has the property that it contains m simplicial vertices and $\gamma(G^* - N[C]) = \gamma(G^*) - 1$. Then,*

$$\gamma(G \Box H) \geq \min\{\gamma(G)\gamma(H) + (m-2), (\gamma(G)-1)\gamma(H) + |H|\}$$

for any graph H.

Proof. Consider graphs G and G^* and a clique C of G^* satisfying the hypothesis of the theorem. Let S denote the set of m simplicial vertices of C, and let $T = C - S$. Observe that $\gamma(G^* - N[T]) = \gamma(G^*) - 1$, that $V(G^* - N[T])$ can be partitioned into $\gamma(G^*) - 1$ cliques and that $G^* - N[T]$ satisfies the Barcalkin-German condition. Thus $\gamma((G^* - N[T]) \Box H) \geq \gamma(G^* - N[T])\gamma(H) = (\gamma(G^*)-1)\gamma(H)$. But at least $\min\{|H|, \gamma(H)+(m-2)\}$ vertices of $C \times V(H)$ are required to dominate $S \times V(H)$ by Lemma 2.1 . The result follows. □

We conclude with an illustration. Consider the graph G in Figure 1. Following the notation of the previous theorem, the clique $C = \{a, u, v, w, x\}$ has 4 simplicial vertices u, v, w, x, and $\gamma(G^* - N[C]) = 2 = \gamma(G^*) - 1$. Therefore, for any graph H it follows that

$$\begin{aligned}\gamma(G \Box H) &\geq \min\{\gamma(G)\gamma(H) + (4-2), (\gamma(G)-1)\gamma(H) + |H|\} \\ &= \min\{3\gamma(H) + 2, 2\gamma(H) + |H|\}.\end{aligned}$$

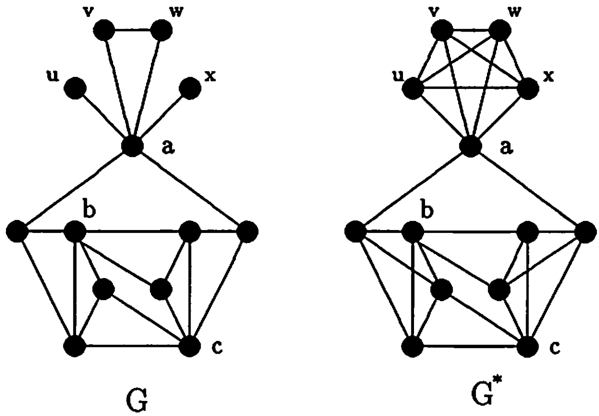

Figure 1: Graphs G and G^*

This lower bound can actually be attained. Let H be the 4-cycle, v_1, v_2, v_3, v_4. The value $\min\{3\gamma(H)+2, 2\gamma(H)+|H|\} = 8$ can be realized by using the dominating set

$$D = (\{a\} \times \{v_1, v_2, v_3, v_4\}) \cup \{(b, v_1), (b, v_3), (c, v_2), (c, v_4)\}.$$

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