On the upper line-distinguishing and upper harmonious chromatic numbers of a graph

Guantao Chen, Gayla S. Domke and Johannes H. Hattingh
Department of Mathematics and Statistics,
Georgia State University, Atlanta, GA 30303, U.S.A.

Renu C. Laskar
Department of Mathematical Sciences,
Clemson University, Clemson, SC 29634, U.S.A.

Abstract

We introduce and study two new parameters, namely the upper harmonious chromatic number, $H(G)$, and the upper line-distinguishing chromatic number, $H'(G)$, of a graph G. $H(G)$ is defined as the maximum cardinality of a minimal harmonious coloring of a graph G, while $H'(G)$ is defined as the maximum cardinality of a minimal line-distinguishing coloring of a graph G. We show that the decision problems corresponding to the computation of the upper line-distinguishing and upper harmonious chromatic numbers are NP-complete for general graphs G. We then determine $H'(P_n)$ and $H(P_n)$. Lastly, we show that H and H' are incomparable, even for trees.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

1 Introduction

Graph theory terminology not presented here may be found in [1]. Let $G = (V, E)$ be a graph with n vertices.

If $A \subseteq V$ and $B \subseteq V$, we will use $q(A, B)$ to denote the number of edges between the sets A and B. A set $S \subseteq V$ is ***independent*** if for distinct $u, v \in S$, $uv \notin E$. A maximal independent set of G is called an ***independent dominating set*** of G. The ***independent dominating number*** of G, denoted $i(G)$, is defined as $\min\{|S| \mid S \text{ is an independent dominating set of } G\}$.

A ***k-coloring*** of G is a partition Π of V into k sets, $V_1, V_2, \ldots, V_k$. A ***proper k-coloring*** is a k-coloring such that each V_i is independent. A k-coloring is a ***complete coloring*** if for every i, j, $1 \leq i < j \leq k$, $q(V_i, V_j) \geq 1$.

The ***chromatic number*** $\chi(G)$ is defined as $\min\{k|G \text{ has a proper } k\text{-coloring}\}$, while the ***achromatic number*** $\psi(G)$ is defined as $\max\{k|G \text{ has a proper complete } k\text{-coloring}\}$.

A ***k-line-distinguishing coloring*** of G is a partition of V into k sets $V_1, \ldots, V_k$ such that $q(\langle V_i \rangle) \leq 1$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \leq 1$ for $1 \leq i < j \leq k$.

If a line-distinguishing coloring is also a proper coloring, then it is called a ***harmonious coloring***. In other words, the partition $\{V_1, V_2, \ldots, V_k\}$ is a harmonious coloring of G if and only if $q(\langle V_i \rangle) = 0$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \leq 1$, $1 \leq i < j \leq k$.

The ***line-distinguishing coloring number*** $h'(G)$ is defined as $\min\{k|G \text{ has a } k\text{-line-distinguishing coloring}\}$, while the ***harmonious coloring number*** $h(G)$ is defined as $\min\{k|G \text{ has a } k\text{-harmonious coloring}\}$.

The achromatic number was first introduced and studied by Harary, Hedetniemi and Prins [4]. The line-distinguishing number, $h'(G)$, was introduced independently by Frank, Harary and Plantholt [5] and Hopcroft and Krishnamoorthy [6] even though the latter authors called it the harmonious coloring number. However, Miller and Pritikin [7] introduced the harmonious coloring number, which is a proper coloring and a line-distinguishing coloring. Harmonious colorings and some of the complexity questions were investigated in [2] and [3].

Consider a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V according to some specified properties P and Q. This means that $\langle V_i \rangle$ has property P for $i = 1, \ldots, k$ and the bipartite graph (V_i, V_j) has property Q for distinct $i, j \in \{1, \ldots, k\}$. The partition is minimal with respect to properties P and Q if any partition Π' obtained from Π by combining color classes V_i and V_j no longer satisfies properties P and Q. The smallest and largest cardinality of minimal partitions with respect to properties P and Q give rise to two parameters associated with a graph. For example, the chromatic and achromatic numbers are, respectively, the minimum and maximum cardinality of a minimal partition where the property P specifies that the induced subgraph of each

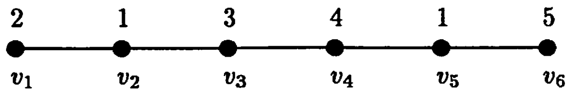

Figure 1: A minimal harmonious coloring of P_6

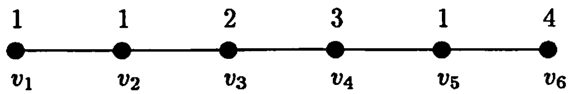

Figure 2: A minimal line-distinguishing coloring of P_6

set in the partition contains no edge.

Let P be the property "contains no edges" and Q be the property "contains at most one edge". If $\Pi = \{V_1, \ldots, V_k\}$ is a partition according to the properties P and Q, then Π is a harmonious coloring of G. If we change property P to "contains at most one edge", then Π becomes a line-distinguishing coloring of G. Before proceeding further, we characterize minimal harmonious and minimal line-distinguishing colorings of a graph. The proofs are easy and therefore omitted.

Lemma 1 *A harmonious coloring $\{V_1, \ldots, V_k\}$ is minimal if and only if for distinct $i, j \in \{1, \ldots, k\}$*

1. *$q(V_i, V_j) = 1$, or*
2. *if $V_i \cup V_j$ is independent, there is an $r \in \{1, \ldots, k\} - \{i, j\}$ such that $q(V_i, V_r) = 1$ and $q(V_r, V_j) = 1$.*

Lemma 2 *A line-distinguishing coloring $\{V_1, \ldots, V_k\}$ is minimal if and only if for distinct $i, j \in \{1, \ldots, k\}$*

1. *$q(\langle V_i \cup V_j \rangle) > 1$, or*
2. *if $q(\langle V_i \cup V_j \rangle) \leq 1$, there is an $r \in \{1, \ldots, k\} - \{i, j\}$ such that $q(V_i, V_r) = 1$ and $q(V_r, V_j) = 1$.*

A harmonious coloring of P_6 is given in Figure 1. Note that this is a minimal harmonious coloring of P_6 since no color class can be combined with $V_1 = \{v_2, v_5\}$ by property 1 of Lemma 1. Also, none of the color classes $V_2, \ldots, V_5$ can be combined with each other by property 2 of Lemma 1 with $V_r = V_1$. Similarly, a 4-line-distinguishing coloring of P_6 is given in Figure 2. This coloring is also minimal since no color class can be combined with $V_1 = \{v_1, v_2, v_5\}$ by property 1 of Lemma 2. Also, none of the color classes $V_2, \ldots, V_4$ can be combined with each other by property 2 of Lemma 2 with $V_r = V_1$.

Let us define the ***upper harmonious chromatic number*** of a graph G, $H(G)$, as the maximum cardinality of a minimal harmonious coloring of G and the ***upper line-distinguishing chromatic number*** of G, $H'(G)$, as the maximum cardinality of a minimal line-distinguishing coloring of G.

In the next section we show that the decision problems corresponding to the computation of the upper line-distinguishing and upper harmonious chromatic numbers are NP-complete for general graphs G. In section 3 we determine $H'(P_n)$ and $H(P_n)$. Lastly, we show that H and H' are incomparable, even for trees.

2 Complexity issues

In this section we show that the decision problem

UPPER LINE-DISTINGUISHING CHROMATIC NUMBER (ULDCN)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is $H'(G) \geq k$?

is NP-complete by describing a polynomial transformation from the following well-known NP-complete problem:

INDEPENDENT DOMINATING SET (IDS)

INSTANCE: A non-trivial connected graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is $i(G) \leq k$?

Theorem 1 ULDCN *is NP-complete.*

Proof. It is clear that **ULDCN** is in NP. To show that **ULDCN** is an NP-complete problem, we will establish a polynomial transformation from **IDS**.

Let the non-trivial connected graph G and the positive integer $k \leq |V(G)|$ be an arbitrary instance of **IDS**. Construct G^* by taking the disjoint union of G, a copy of K_3 with vertex set $\{u, v, w\}$, a copy of K_{k+1}, joining the vertex u to every vertex of G and joining the vertex w to every vertex of the graph K_{k+1}. Let $X = V(G) = \{x_1, \ldots, x_n\}$ and let $Y = V(K_{k+1}) = \{y_1, \ldots, y_{k+1}\}$. Clearly, the construction of the graph G^* can be accomplished in polynomial time.

We will show $i(G) \leq k$ if and only if $H'(G^*) \geq n + 4$, or, equivalently, $i(G) \geq k + 1$ if and only if $H'(G^*) \leq n + 3$.

Suppose $i(G) \geq k + 1$. We will show $H'(G^*) \leq n + 3$. Suppose, to the contrary, $H'(G^*) \geq n + 4$. Then there is a minimal line-distinguishing coloring of G^*, say $V_1, V_2, \ldots, V_m$, such that $m \geq n + 4$. Before proceeding further, we prove four preliminary facts.

Fact 1 Each of the vertices in $\{u, v, w\}$ are colored differently.

Proof. Suppose $v \in V_i$. If $u \in V_i$, then $w \notin V_i$, since otherwise $q(\langle V_i \rangle) \geq 2$, which is a contradiction. But then $w \notin V_i$ is adjacent to at least two vertices in V_i, which is a contradiction. We may therefore assume that $u \in V_j$, where $j \neq i$. Since there is at most one edge between two distinct color classes, $w \notin V_i \cup V_j$, so that $w \in V_\ell$ with i, j and ℓ distinct. The result follows. □

To simplify the next three proofs, we assume $u \in V_1$, $v \in V_2$ and $w \in V_3$.

Fact 2 $|V_1| = |V_2| = |V_3| = 1$.

Proof. Since there is at most one edge between two distinct color classes, $(V_1 \cup V_2) \cap Y = \emptyset$ and $(V_2 \cup V_3) \cap X = \emptyset$. Thus, $|V_2| = 1$.

If $y \in V_3 \cap Y$, then since each color class has at most one edge, $V_3 \cap Y = \{y\}$. However, since $|Y| \geq 2$, there exists $y' \in Y - V_3$ which is adjacent to at least two vertices of V_3, which is a contradiction. Thus, $|V_3| = 1$.

If $x \in V_1 \cap X$, then since each color class has at most one edge, $V_1 \cap X = \{x\}$. However, since G is a connected non-trivial graph, there exists $x' \in X - V_1$ adjacent to x in G. But then x' is adjacent to at least two vertices of V_1, which is a contradiction. Thus, $|V_1| = 1$. □

Fact 3 $|V_i| \leq 2$ for $i = 1, \ldots, m$. Moreover, if $|V_i| = 2$, then $i \geq 4$ and $|X \cap V_i| = |Y \cap V_i| = 1$.

Proof. The result follows if $i \in \{1, 2, 3\}$. Let $i \in \{4, \ldots, m\}$. Then, since u is adjacent to every vertex of X and w is adjacent to every vertex of Y, $|X \cap V_i| \leq 1$ and $|Y \cap V_i| \leq 1$. Thus, $|V_i| \leq 2$ and if $|V_i| = 2$, then $i \geq 4$ and $|X \cap V_i| = |Y \cap V_i| = 1$. □

Fact 4 If $s = |\{V_i \mid |V_i| = 2, i = 4, \ldots, m\}|$, then $s \geq 1$.

Proof. If $s = 0$, then the partition obtained by combining color classes $\{x_1\}$ and $\{y_1\}$ is a line-distinguishing coloring of G^*, which contradicts the minimality of the original partition. □

Assume $V_i = \{x_i, y_i\}$ for $i = 1, \ldots, s$, $V_i = \{x_i\}$ for $i = s+1, \ldots, n$, $V_{n+1} = \{u\}$, $V_{n+2} = \{v\}$, $V_{n+3} = \{w\}$ and $V_{n+3+i} = \{y_{s+i}\}$ for $i = 1, \ldots, k+1-s$. Notice that $m = s + (n-s) + 3 + (k+1-s) = n + 4 + k - s$. Since $m \geq n+4$, it follows that $k \geq s$. For $1 \leq i \neq j \leq s$, y_i is adjacent to y_j, and so x_i is not adjacent to x_j. Thus, $S = \{x_1, \ldots, x_s\}$ is an independent set of G. If S is also dominating, then $i(G) \leq |S| = s \leq k < k+1 \leq i(G)$, which is a contradiction. It follows that there is a vertex in G, say x_{s+1}, which is not dominated by S. Hence, $S \cup \{x_{s+1}\}$ is an independent set of G. But then the partition obtained by combining color classes $\{x_{s+1}\}$ and $\{y_{s+1}\}$ is also a line-distinguishing coloring of G^*, which is a contradiction. Therefore, $H'(G^*) \leq n+3$.

Now suppose $H'(G^*) \leq n+3$. Suppose $S = \{x_1, \ldots, x_s\}$ is an independent dominating set of G. We will show that $s \geq k+1$, thus establishing $i(G) \geq k+1$. If $s \geq k+1$, we are done. Assume therefore $s \leq k$. Construct a coloring of G^* as follows. Let $V_i = \{x_i, y_i\}$ for $i = 1, \ldots, s$, $V_i = \{x_i\}$ for $i = s+1, \ldots, n$, $V_{n+1} = \{u\}$, $V_{n+2} = \{v\}$, $V_{n+3} = \{w\}$ and $V_i' = \{y_i\}$ for $i = s+1, \ldots, k+1$. This is clearly a line-distinguishing coloring of G^*.

We now show that this coloring is a minimal line-distinguishing coloring of G^*. We cannot combine color class V_{n+2} with any other color class C, since in each case there is a vertex, say x, in a color class C' distinct from V_{n+2} and C which is adjacent to at least two vertices in $V_{n+2} \cup C$. Similarly, color classes V_{n+1} and V_{n+3} cannot be combined with another color class.

Let C_1 and C_2 be arbitrary color classes distinct from V_{n+1}, V_{n+2} and V_{n+3}. If $x_i \in C_1$ and $x_j \in C_2$, then we cannot combine color classes C_1 and C_2, since x_i and x_j are both adjacent to u. If $y_i \in C_1$ and $y_j \in C_2$, then color classes C_1 and C_2 cannot be combined since y_i and y_j are both adjacent to w. Thus, $C_1 = \{x_i\}$ for some $i \notin \{1, \ldots, s\}$ and $C_2 = \{y_j\}$ for some $j \notin \{1, \ldots, s\}$ or $C_1 = \{y_j\}$ for some $j \notin \{1, \ldots, s\}$ and $C_2 = \{x_i\}$ for some

$i \notin \{1, \ldots, s\}$. Without loss of generality, assume the former. Since S is a dominating set of G, x_i is adjacent to some x_ℓ with $\ell \in \{1, \ldots, s\}$. But then $x_i \in C_1$ is adjacent to $x_\ell \in V_\ell$ and $y_j \in C_2$ is adjacent to $y_\ell \in V_\ell$. Thus, C_1 and C_2 cannot be combined.

The number of color classes equals $s+(n-s)+3+(k+1-s) = n+k+4-s$. Thus, $n+k+4-s \leq H'(G^*) \leq n+3$, so that $k+1 \leq s$, as required. Thus, $i(G) \geq k+1$. □

Using the same construction and the same proof (although some details may be simplified since each color class is independent) one may show that the decision problem

UPPER HARMONIOUS CHROMATIC NUMBER (UHCN)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is $H(G) \geq k$?

is NP-complete.

3 Paths

In this section we determine the upper line-distinguishing and the upper harmonious chromatic numbers of a path. We start with the following result.

Lemma 3 *If $n \geq 3$, $m \geq \lfloor \frac{2n+1}{3} \rfloor + 1$, and $V_1, \ldots, V_m$ is a coloring of P_n, then there is a color class of cardinality one containing a vertex of degree two.*

Proof. Let ℓ be the number of color classes of cardinality one. Then $n-\ell \geq 2(m-\ell)$, so that $n \geq 2m-\ell \geq 2(\lfloor \frac{2n+1}{3} \rfloor + 1) - \ell$. Let $n = 3q+r$ where k is an integer and $r \in \{0,1,2\}$. Thus, $\ell \geq 2(\lfloor \frac{2n+1}{3} \rfloor) + 2 - n = 2(\lfloor \frac{6k+2r+1}{3} \rfloor) + 2 - 3k - r = \begin{cases} 4k+2-3k = k+2 \geq 3 & \text{if } r = 0 \\ 4k+2+2-3k-1 = k+3 \geq 3 & \text{if } r = 1 \\ 4k+2+2-3k-2 = k+2 \geq 3 & \text{if } r = 2 \end{cases}$. Since every path has at most two end vertices, the result follows immediately. □

Theorem 2 *If $n \geq 1$ is an integer, then $H'(P_n) = \lfloor \frac{2n+1}{3} \rfloor$.*

Proof. The result clearly holds for $n = 1$ and $n = 2$. Assume $n \geq 3$.

We first show that $H'(P_n) \leq \lfloor \frac{2n+1}{3} \rfloor$. Suppose, to the contrary, $H'(P_n) = m \geq \lfloor \frac{2n+1}{3} \rfloor + 1$ and let $V_1, \ldots, V_m$ be a minimal line-distinguishing coloring of P_n.

Without loss of generality assume $V_1 = \{w\}$ with $N(w) = \{x, y\}$ (cf. Lemma 3). Furthermore, assume $x \in V_2$ and $y \in V_3$. Let $A = V_2 \cup V_3$ and let $B = V - V_1 - A$. Since we cannot combine color classes V_1 and $V_i, i = 4, \ldots, m$, some vertex in V_i must be adjacent to some vertex in A. Let $n = 3k + r$ where k is an integer and $r \in \{0, 1, 2\}$. Hence, $q(A, V_1 \cup B) \geq 2 + (m-3) = m - 1 \geq \lfloor \frac{2n+1}{3} \rfloor = \begin{cases} 2k & \text{if } r = 0 \\ 2k+1 & \text{if } r = 1 \\ 2k+1 & \text{if } r = 2 \end{cases}$. It follows that $q(A, B) \geq \begin{cases} 2k-2 & \text{if } r = 0 \\ 2k-1 & \text{if } r = 1 \\ 2k-1 & \text{if } r = 2 \end{cases}$, so that $|B| \geq \begin{cases} 2k-2 & \text{if } r = 0 \\ 2k-1 & \text{if } r = 1 \\ 2k-1 & \text{if } r = 2 \end{cases}$ and $|A| = n - 1 - |B| \leq \begin{cases} k+1 & \text{if } r = 0 \\ k+1 & \text{if } r = 1 \\ k+2 & \text{if } r = 2 \end{cases}$. Thus, $2q(\langle A \rangle) + \left\{ \begin{array}{ll} 2k & \text{if } r = 0 \\ 2k+1 & \text{if } r = 1 \\ 2k+1 & \text{if } r = 2 \end{array} \right\}$
$= 2q(\langle A \rangle) + 2 + \left\{ \begin{array}{ll} 2k-2 & \text{if } r = 0 \\ 2k-1 & \text{if } r = 1 \\ 2k-1 & \text{if } r = 2 \end{array} \right\} \leq 2q(\langle A \rangle) + 2 + q(A, B) = \sum_{v \in A} \deg(v)$
$\leq 2|A| \leq \begin{cases} 2k+2 & \text{if } r = 0 \\ 2k+2 & \text{if } r = 1 \\ 2k+4 & \text{if } r = 2 \end{cases}$. (*)

It now follows that $q(\langle A \rangle) \leq 1$ if $r \in \{0, 2\}$ and $q(\langle A \rangle) = 0$ if $r = 1$. If $q(\langle A \rangle) = 0$, then A is an independent set, and we may combine V_1 and V_2 to obtain a line-distinguishing coloring of P_n, contrary to minimality. Thus, $q(\langle A \rangle) = 1$ and $r \in \{0, 2\}$.

Case 1 $r = 0$.

By (*), $q(A, B) = 2k - 2$, $|A| = k + 1$ and $|B| = 2k - 2$. Let $B = \{v_1, \ldots, v_{2k-2}\}$. Moreover, since $m \geq 2k + 2$, $\{v_i\}$ is a color class and v_i is adjacent to exactly one vertex of A $(i = 1, \ldots, 2k-2)$. If v_i is adjacent to a vertex of V_2 and v_j is adjacent to a vertex of V_3, then we may combine color classes $\{v_i\}$ and $\{v_j\}$ to obtain a line-distinguishing coloring of P_n, contrary to minimality. Thus, without loss of generality, we may assume that v_i $(i = 1, \ldots, 2k-2)$ is adjacent to a vertex of V_2 but not to a vertex of V_3. Furthermore $q(\langle B \rangle) = 3k - 1 - 1 - 2 - (2k-2) = k - 2$, so that $\langle B \rangle \cong \bar{K}_2 \cup (k-2)K_2$. Assume, without loss of generality, that v_1 is an isolated vertex of $\langle B \rangle$. Then v_1 is an end vertex of P_n and is not adjacent to

y. Since we cannot combine color classes $\{v_1\}$ and V_3, some vertex of V_3 is adjacent to a vertex of V_2. We conclude that V_2 is an independent set. We may then combine color classes $\{v_1\}$ and V_2 to obtain a line-distinguishing coloring of P_n, which is a contradiction.

Case 2 $r = 2$.

By (*), $q(A, B) \leq 2k$ and $|A| \geq (2k+3)/2$. Thus, $|A| = k+2$ and $|B| = 2k-1$. Let $B = \{v_1, \ldots, v_{2k-1}\}$.

Case 2.1 $q(A, B) = 2k - 1$.

Since $m \geq 2k+2$, $\{v_i\}$ is a color class and v_i is adjacent to exactly one vertex of A ($i = 1, \ldots, 2k-1$). If v_i is adjacent to a vertex of V_2 and v_j is adjacent to a vertex of V_3, then we may combine color classes $\{v_i\}$ and $\{v_j\}$ to obtain a line-distinguishing coloring of P_n, contrary to minimality. Thus, without loss of generality, we may assume that v_i ($i = 1, \ldots, 2k-1$) is adjacent to a vertex of V_2 but not to a vertex of V_3. Furthermore $q(\langle B \rangle) = 3k+1-1-2-(2k-1) = k-1$, so that $\langle B \rangle \cong K_1 \cup (k-1)K_2$. Assume, without loss of generality, that v_1 is the isolated vertex of $\langle B \rangle$. Then v_1 is an end vertex of P_n and is not adjacent to y. Since we cannot combine color classes $\{v_1\}$ and V_3, some vertex of V_3 is adjacent to a vertex of V_2. We conclude that V_2 is an independent set. We may then combine color classes $\{v_1\}$ and V_2 to obtain a line-distinguishing coloring of P_n, which is a contradiction.

Case 2.2 $q(A, B) = 2k$.

Since $m \geq 2k+2$, $\{v_i\}$ is a color class and v_i is adjacent to at least one vertex of A ($i = 1, \ldots, 2k-1$). But $q(\langle B \rangle) = 3k+1-1-2-2k = k-2$, so that $\langle B \rangle \cong \bar{K}_3 \cup (k-2)K_2$. It follows that the three isolated vertices in $\langle B \rangle$ have degree two, one and one in P_n. Without loss of generality, assume that $\deg(v_1) = 2$ and $\deg(v_2) = \deg(v_3) = 1$. Suppose the neighbor of v_2 is in color class V_2. Since we cannot combine color classes $\{v_2\}$ and $\{v_i\}$ ($i = 3, \ldots, 2k-1$), the neighbor of v_i in A is also in V_2. Since we cannot combine color classes $\{v_2\}$ and V_3, some vertex of V_3 is adjacent to some vertex of V_2. We conclude that V_2 is an independent set. We may then combine color classes $\{v_2\}$ and V_2 to obtain a line-distinguishing coloring of P_n, which is a contradiction.

Thus, $H'(P_n) \leq \lfloor \frac{2n+1}{3} \rfloor$.

Let $n = 3k + r$, where k and r are nonnegative integers and denote the consecutive vertices of the path by $v_1, \ldots, v_n$.

If $r = 0$, then the partition $\{\{v_1, v_2, v_5, \ldots, v_{3k-1}\}, \{v_3\}, \{v_4\}, \{v_6\}, \{v_7\}, \ldots, \{v_{3k-3}\}, \{v_{3k-2}\}, \{v_{3k}\}\}$ is a minimal line-distinguishing coloring of P_n, whence $H'(P_n) \geq 2k = \lfloor \frac{2n+1}{3} \rfloor$.

If $r = 1$, then the partition $\{\{v_2, v_3, \ldots, v_{3k}\}, \{v_1\}, \{v_4\}, \{v_5\}, \ldots, \{v_{3k-2}\}, \{v_{3k-1}\}, \{v_{3k+1}\}\}$ is a minimal line-distinguishing coloring of P_n, whence $H'(P_n) \geq 2k + 1 = \lfloor \frac{2n+1}{3} \rfloor$.

If $r = 2$, then the partition $\{\{v_2, v_3, v_6, \ldots, v_{3k}\}, \{v_1\}, \{v_4\}, \{v_5\}, \ldots, \{v_{3k-2}\}, \{v_{3k-1}\}, \{v_{3k+1}, v_{3k+2}\}\}$ is a minimal line-distinguishing coloring of P_n, whence $H'(P_n) \geq 2k + 1 = \lfloor \frac{2n+1}{3} \rfloor$. The result follows. □

Theorem 3 *If $n \geq 1$ is an integer, then $H(P_n) = \lceil \frac{2n+1}{3} \rceil$.*

Proof. The result clearly holds for $n = 1$ and $n = 2$. Assume $n \geq 3$.

We first show that $H(P_n) \leq \lceil \frac{2n+1}{3} \rceil$. Suppose, to the contrary, $H(P_n) = m \geq \lceil \frac{2n+1}{3} \rceil + 1$ and let $V_1, \ldots, V_m$ be a minimal harmonious coloring of P_n.

Without loss of generality assume $V_1 = \{w\}$ with $N(w) = \{x, y\}$ (cf. Lemma 3). Furthermore, assume $x \in V_2$ and $y \in V_3$. Let $A = V_2 \cup V_3$ and let $B = V - V_1 - A$. Since we cannot combine color classes V_1 and $V_i, i = 4, \ldots, m$, some vertex in V_i must be adjacent to some vertex in A. Hence, $q(A, V_1 \cup B) \geq 2 + (m - 3) = m - 1 \geq \lceil \frac{2n+1}{3} \rceil$. Thus, $q(A, B) \geq \lceil \frac{2n+1}{3} \rceil - 2$, so that $|B| \geq \lceil \frac{2n+1}{3} \rceil - 2$, and $|A| = n - 1 - |B| \leq n - 1 - (\lceil \frac{2n+1}{3} \rceil - 2) = n - \lceil \frac{2n+1}{3} \rceil + 1$. Moreover, $2|A| \geq \sum_{v \in A} \deg(v) = q(A, V_1 \cup B) + 2q(V_2, V_3) \geq q(A, V_1 \cup B) \geq \lceil \frac{2n+1}{3} \rceil$, so that $|A| \geq \frac{1}{2} \lceil \frac{2n+1}{3} \rceil$. Let $n = 3k + r$, where k is a nonnegative integer and $r \in \{0, 1, 2\}$. Then

$$\left\{ \begin{array}{ll} \frac{1}{2}(2k+1) = k + \frac{1}{2} & \text{if } r = 0 \\ \frac{1}{2}(2k+1) = k + \frac{1}{2} & \text{if } r = 1 \\ \frac{1}{2}(2k+2) = k + 1 & \text{if } r = 2 \end{array} \right\} = \frac{1}{2} \lceil \tfrac{2n+1}{3} \rceil \leq |A| \leq n + 1 - \lceil \tfrac{2n+1}{3} \rceil =$$
$$\left\{ \begin{array}{ll} 3k + 1 - (2k+1) = k & \text{if } r = 0 \\ 3k + 1 + 1 - (2k+1) = k + 1 & \text{if } r = 1 \\ 3k + 2 + 1 - (2k+2) = k + 1 & \text{if } r = 2 \end{array} \right. .$$

If $r = 0$, then $k + \frac{1}{2} \leq k$, which is a contradiction. Thus, $r \in \{1, 2\}$.

Case 1 $r = 1$.

Then $|A| = k + 1$ and $|B| = 2k - 1$. Since $2q(V_2, V_3) + 2k + 1 \leq 2q(V_2, V_3) + q(A, V_1 \cup B) \leq 2|A| = 2k + 2$, it follows that $q(V_2, V_3) = 0$. Thus, A is an independent set. Also, $2k + 1 \leq q(A, V_1 \cup B) \leq 2k + 2$, so that $2k - 1 \leq q(A, B) \leq 2k$. Denote the vertices of B by $\{v_1, \ldots, v_{2k-1}\}$.

Case 1.1 $q(A, B) = 2k - 1$. Since $m \geq 2k + 2$, $\{v_i\}$ is a color class and v_i is adjacent to exactly one vertex of A $(i = 1, \ldots, 2k - 1)$. Furthermore $q(\langle B \rangle) = 3k - 2 - (2k - 1) = k - 1$, so that $\langle B \rangle \cong K_1 \cup (k-1)K_2$. Assume, without loss of generality, that v_1 is the isolated vertex of $\langle B \rangle$. Then v_1 is an end vertex of P_n. Without loss of generality assume that the neighbor of v_1 is in V_2. We may then combine color classes $\{v_1\}$ and V_3 to obtain a harmonious coloring of P_n, contrary to minimality.

Case 1.2 $q(A, B) = 2k$.

Since $m \geq 2k + 2$, $\{v_i\}$ is a color class and v_i is adjacent to at least one vertex of A $(i = 1, \ldots, 2k - 1)$. But $q(\langle B \rangle) = 3k - 2 - 2k = k - 2$, so that $\langle B \rangle \cong \bar{K}_3 \cup (k - 2)K_2$. It follows that the three isolated vertices in $\langle B \rangle$ have degree two, one and one in P_n. Without loss of generality, assume that $\deg(v_1) = 2$ and $\deg(v_2) = \deg(v_3) = 1$. Suppose the neighbor of v_2 is in color class V_2. We may then combine color classes $\{v_2\}$ and V_3 to obtain a harmonious coloring of P_n, contrary to minimality.

Case 2 $r = 2$.

Then $|A| = k + 1$ and $|B| = 2k$. Since $2q(V_2, V_3) + 2k + 2 \leq 2q(V_2, V_3) + q(A, V_1 \cup B) \leq 2|A| = 2k+2$, it follows that $q(V_2, V_3) = 0$ and $q(A, V_1 \cup B) = 2k + 2$, so that A is an independent set and $q(A, B) = 2k$. Also, since $m \geq 2k + 3$, each vertex of B forms a color class. Thus, each vertex of B is adjacent to a vertex of A. Denote the vertices of B by $\{v_1, \ldots, v_{2k}\}$. Since $q(\langle B \rangle) = 3k + 1 - 2 - 2k = k - 1$, $\langle B \rangle \cong \bar{K}_2 + (k - 1)K_2$. The isolated vertices in $\langle B \rangle$ are the end vertices of P_n. Without loss of generality, assume $\deg(v_1) = \deg(v_2) = 1$. Assume that the neighbor of v_1 is in V_2. We may then combine color classes $\{v_1\}$ and V_3 to obtain a harmonious coloring of P_n, contrary to minimality.

Thus, $H(P_n) \leq \lceil \frac{2n+1}{3} \rceil$.

Let $n = 3k + r$, where k and r are nonnegative integers and denote the consecutive vertices of the path by $v_1, \ldots, v_n$.

If $r = 0$, then the partition $\{\{v_2, v_5, \ldots, v_{3k-1}\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_6\}, \ldots, \{v_{3k-2}\}, \{v_{3k}\}\}$ is a minimal harmonious coloring of P_n, whence $H(P_n) \geq 2k + 1 = \lceil \frac{2n+1}{3} \rceil$.

If $r = 1$, then the partition $\{\{v_2, v_5, \ldots, v_{3k-1}\}, \{v_1, v_{3k+1}\}, \{v_3\}, \{v_4\}, \ldots, \{v_{3k-3}\}, \{v_{3k-2}\}, \{v_{3k}\}\}$ is a minimal harmonious coloring of P_n, whence $H(P_n) \geq 2k + 1 = \lceil \frac{2n+1}{3} \rceil$.

If $r = 2$, then the partition $\{\{v_2, v_5, \ldots, v_{3k+2}\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_6\}, \ldots, \{v_{3k-2}\}, \{v_{3k}\}, \{v_{3k+1}\}\}$ is a minimal harmonious coloring of P_n, whence

$H(P_n) \geq 2k+2 = \lceil \frac{2n+1}{3} \rceil$. The result follows. □

4 Incomparability

We now show that the parameters H and H' are incomparable. Clearly, $H(K_{1,m}) = m+1$, while $H'(K_{1,m}) = m$. Let $m \geq 2$ be an integer, and construct the tree T by taking two copies of the star $K_{1,m}$ and joining the two central vertices by an edge. It is easily verified that $H(T) = m+2$. The coloring obtained by putting the central vertices in the same color class, and all other vertices in their own color class is, as is easily verified, a minimal line-distinguishing coloring. Thus, $H'(T) = 2m+1 > m+2 = H(T)$.

5 Conclusions

The concept of partitioning the vertices of a graph into sets where there is one property which holds for the vertices in each of the sets and another property which holds between any two sets has been well studied. It seems natural to extend the ideas of k-line-distinguishing and harmonious colorings to include the maximum number of sets in a minimal partition. We have shown that the computation of these upper parameters is NP-complete for general graphs while we have also given specific formulas for paths. It is possible to determine the upper line-distinguishing and upper harmonious chromatic numbers of cycles and classes of trees. A study of these formulas will appear elsewhere.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[2] K. J. Edwards and C.J.H. McDiarmid, "New upper bounds on harmonious colourings," *J. Graph Theory* **18** (1994), 257-267.

[3] K. Edwards and C. McDiarmid, "The complexity of harmonious coloring of trees," *Disc. Appl. Math.* **57** (1995), 133-144.

[4] F. Harary, S.T. Hedetniemi and G. Prins, "An interpolation theorem for graphical homomorphisms," *Portugal. Math.* **26** (1967), 453-462.

[5] O. Frank, F. Harary and M. Plantholt, "The line-distinguishing chromatic number of a graph," *Ars Combin.* **14** (1982) 241-252.

[6] J. Hopcroft and M.S. Krishnamoorthy, "On the harmonious colorings of graphs," *SIAM J. Algebraic Discrete Methods* **4** (1983), 306-311.

[7] Z. Miller and D. Pritikin, "The harmonious coloring number of a graph," *Discrete Math.* **93** (1991), 211-228.