Strong Distance in Strong Digraphs

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ABSTRACT

For two vertices u and v in a strong oriented graph D of order $n \geq 3$, the strong distance $\mathrm{sd}(u, v)$ between u and v is the minimum size of a strong subdigraph of D containing u and v. For a vertex v of D, the strong eccentricity $\mathrm{se}(v)$ is the strong distance between v and a vertex farthest from v. The minimum strong eccentricity among the vertices of D is the strong radius $\mathrm{srad}\, D$, and the maximum strong eccentricity is its strong diameter $\mathrm{sdiam}\, D$. It is shown that every pair r, d of integers with $3 \leq r \leq d \leq 2r$ is realizable as the strong radius and strong diameter of some strong oriented graph. Also, for every strong oriented graph D of order $n \geq 3$, it is shown that $\mathrm{sdiam}(D) \leq \lfloor 5(n-1)/3 \rfloor$. Furthermore, for every integer $n \geq 3$, there exists a strong oriented graph D of order n such that $\mathrm{sdiam}(D) = \lfloor 5(n-1)/3 \rfloor$.

Dedicated to Professor Stephen T. Hedetniemi
on the occasion of his 60th birthday

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1 A Class of Minimally Strong Digraphs

A digraph D is *strong* if for every pair u, v of distinct vertices of D, there is both a directed $u - v$ path and a directed $v - u$ path in D. A digraph D is an *oriented graph* if D is obtained by assigning a direction to each edge of a graph G. The graph G is thus the underlying graph of D. In this paper we will be interested in strong oriented graphs. The underlying graph of a strong oriented graph is necessarily 2-edge connected.

A strong digraph D is called *minimally strong* if $D - e$ is not strong for every arc e of D. In 1971, Hedetniemi [4] surveyed results on minimally strong digraphs and proved that a strong digraph D is minimally strong if and only if D contains no pseudocycle (a directed cycle the direction of one of whose arcs has been reversed). Minimally strong digraphs have also been studied by Brualdi and Manber [1], Chen and Zhang [2], and Geller [3].

We will be interested in a certain class of minimally strong oriented graphs. A strong oriented graph D is called a $u - v$ *strong path*, where $u, v \in V(D)$, if there is no proper strong subdigraph of D containing u and v. An oriented graph D is simply called a *strong path* if D is a $u - v$ strong path for some pair u, v of vertices of D. The directed cycle $\vec{C}_n$ $(n \geq 3)$ is certainly a strong path (see $\vec{C}_6$ in Figure 1). The strong oriented graph D of Figure 1 is a strong path (indeed, it is a $u - v$ strong path) and is minimally strong as well. While every strong path is minimally strong, a minimally strong oriented graph need not be a strong path. For example, the strong oriented graph D' of Figure 1 is minimally strong but is not a strong path.

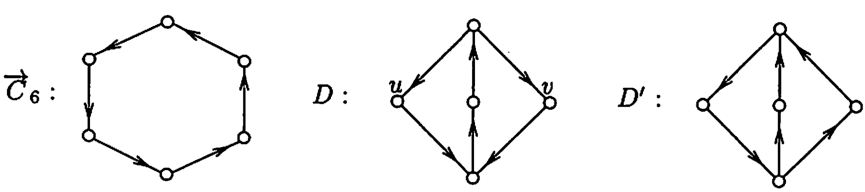

Figure 1: Strong paths and minimally strong oriented graphs

We now make a few elementary observations concerning strong paths. We denote the arc set of an oriented graph D by $E(D)$. Certainly if a $u - v$ strong path D contains a directed $u - v$ path P and a directed $v - u$ path Q, then $H = \langle E(P) \cup E(Q) \rangle$ is a subdigraph of D. However, if H is a proper subdigraph of D, then we will have contradicted the definition of a strong path. This gives us the following.

Lemma 1.1 *If D is a $u-v$ strong path, then $D = \langle E(P) \cup E(Q) \rangle$, where P is some directed $u-v$ path and Q is a directed $v-u$ path.*

As a consequence of Lemma 1.1, if P_1 and P_2 are both directed $u-v$ paths in a $u-v$ strong path D and Q_1 and Q_2 are directed $v-u$ paths, then $E(P_1) \cup E(Q_1) = E(P_2) \cup E(Q_2)$. In an oriented graph D, the *degree* $\deg v$ of a vertex v is the sum of its outdegree and indegree, that is, $\deg v = \text{od}\, v + \text{id}\, v$.

Lemma 1.2 *If D is a $u-v$ strong path, then* $\deg u = \deg v = 2$.

Proof. By Lemma 1.1, $D = \langle E(P) \cup E(Q) \rangle$, where P is some $u-v$ directed path and Q is some $v-u$ path. Then u has outdegree 1 in P and indegree 1 in Q. Hence $\deg u = 2$ in D. Similarly, $\deg v = 2$. ∎

Consequently, every strong path contains at least two vertices of degree 2. Since D is strong, no vertex can have degree 1. Let $P: u = v_1, v_2, \ldots, v_s = v$ be a $u-v$ path in D and $Q: v = v_s, v_{s+1}, \ldots, v_t = u$ be a $v-u$ path. Certainly, $v_1 = v_t$, but there may be indices s' and t' $(1 \leq s' < s < t' < t)$ for which $v_{s'} = v_{t'}$. Each vertex x of D can occur at most twice in the closed walk $v_1, v_2, \cdots, v_t$, namely, once on P and once on Q, so $\deg x \leq 4$. Hence, the degree of every vertex of D is 2, 3, or 4.

We state some other facts about a directed $u-v$ path in a $u-v$ strong path.

Lemma 1.3 *If $P: u = v_1, v_2, \cdots, v_s = v$ is a shortest directed $u-v$ path in a $u-v$ strong path D, then for all integers i and j with $1 \leq i < j \leq s$, (v_i, v_j) is an arc of D if and only if $j = i+1$. Furthermore, if x is a vertex of degree 3 or 4 in D then x lies on P.*

We now have the following.

Corollary 1.4 *Let D be a strong $u-v$ path. Every vertex of D of degree 3 or 4 lies on both a directed $u-v$ path and on a directed $v-u$ path.*

These observations imply that a $u-v$ strong path consists of a single directed $u-v$ path and a single directed $v-u$ path.

Theorem 1.5 *If D is a $u-v$ strong path, then D contains a unique directed $u-v$ path and a unique directed $v-u$ path.*

Proof. It suffices to show that D contains a unique directed $u-v$ path. If D is a directed cycle, then the result follows immediately, so we may assume that D is not a directed cycle. Thus u and v are not adjacent. Let $P: u = v_1, v_2, \cdots, v_s = v$ be a shortest directed $u-v$ path in D and assume, to the contrary, that $P': u = v'_1, v'_2, \cdots, v'_t = v$ is a directed $u-v$

path in D distinct from P. Since u and v have degree 2 in D, it follows that $v_2 = v'_2$ and $v_{s-1} = v'_{t-1}$. Let $a \geq 2$ be the smallest integer for which $v_a = v'_a$ and $v_{a+1} \neq v'_{a+1}$. If the degree of v'_{a+1} exceeds 2, it follows from Lemma 1.3 that v'_{a+1} lies on P and that $v'_{a+1} = v_k = v'_k$ for some $k < a$, which contradicts the assumption that P' is a directed path. Consequently, v'_{a+1} has degree 2.

Let $b > a + 1$ be the smallest integer for which v'_b has degree 3 or 4. Thus v'_b lies on P and $v'_b = v_{b-\ell}$ for some $\ell \geq 0$, for otherwise the directed $u - v'_b$ path induced by P' is shorter than the directed $u - v'_b$ path induced by P, contradicting our choice of P. Since the vertices $v'_{a+1}, v'_{a+2}, \cdots, v'_{b-1}$ do not appear on P and D is a $u - v$ strong path, it follows that every directed $v - u$ path contains the vertices $v'_{a+1}, v'_{a+2}, \cdots, v'_{b-1}$. Let Q be a directed $v - u$ path. Then we can form a directed $v - u$ walk W by following Q to v'_a and then P to v'_b, and finally Q to u. So W does not contain any of $v'_{a+1}, v'_{a+2}, \cdots, v'_{b-1}$. Since W contains a directed $v - u$ path Q', it follows that Q' does not contain any of $v'_{a+1}, v'_{a+2}, \cdots, v'_{b-1}$, which is a contradiction. Consequently, no such directed path P' exists. ∎

2 Strong Distance

The familiar distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. Equivalently, this distance is the minimum size of a connected subgraph of G containing u and v. Using this equivalent formulation of distance, we extend this concept to connected digraphs, in particular to strongly connected (strong) oriented graphs.

Let D be a strong oriented graph of order $n \geq 3$ and size m. We define the *strong distance* $\text{sd}(u, v)$ between u and v as the minimum size of a strong subdigraph of D containing u and v. A $u - v$ *strong geodesic* is a strong subdigraph of D of size $sd(u, v)$ containing u and v. If $u \neq v$, then $3 \leq \text{sd}(u, v) \leq m$. Clearly, $\text{sd}(u, v) = 3$ if and only if u and v belong to a directed 3-cycle in D. In the strong oriented graph of Figure 2, $\text{sd}(v, w) = 3$, $\text{sd}(u, y) = 4$, and $\text{sd}(u, x) = 5$.

Strong distance is a metric on the vertex set of a strong oriented graph D. Certainly $\text{sd}(u, v) = 0$ if and only if $u = v$ and $\text{sd}(u, v) = \text{sd}(v, u)$ for all $u, v \in V(D)$. It remains only to verify the triangle inequality. Let $u, v, w \in V(D)$. Furthermore, let D_1 be a $u - v$ strong geodesic and D_2 a $v - w$ strong geodesic. The subdigraph D_3 defined by $V(D_3) = V(D_1) \cup V(D_2)$ and $E(D_3) = E(D_1) \cup E(D_2)$ is strong and contains u and w. Also, the size of D_3 is at most $\text{sd}(u, v) + \text{sd}(v, w)$. Hence

$$\text{sd}(u, w) \leq \text{sd}(u, v) + \text{sd}(v, w).$$

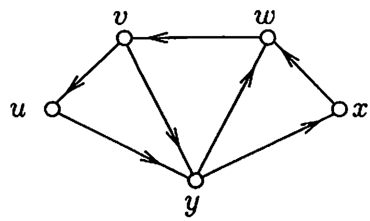

Figure 2: Strong distance in a strong oriented graph

Following common terminology used for distance in connected graphs, we define the *strong eccentricity* $\mathrm{se}(v)$ of a vertex v in a strong oriented graph D by

$$\mathrm{se}(v) = \max\{\mathrm{sd}(v,x) \mid x \in V(D)\}.$$

The *strong radius* $\mathrm{srad}(D)$ of D is

$$\mathrm{srad}(D) = \min\{\mathrm{se}(v) \mid v \in V(D)\};$$

while the *strong diameter* $\mathrm{sdiam}(D)$ of D is

$$\mathrm{sdiam}(D) = \max\{\mathrm{se}(v) \mid v \in V(D)\}.$$

The strong eccentricities of the vertices of the strong oriented graph D of Figure 3 are shown in the figure as well. Hence $\mathrm{srad}(D) = 6$ and $\mathrm{sdiam}(D) = 10$. Observe that, unlike the situation for eccentricities in connected graphs, if k is an integer such that $\mathrm{srad}(D) < k < \mathrm{sdiam}(D)$, then there need not be a vertex v of D such that $\mathrm{se}(v) = k$.

The strong radius and strong diameter of a strong oriented graph satisfy familiar inequalities, which are verified with familiar arguments.

Theorem 2.1 *For every strong oriented graph D,*

$$\mathrm{srad}(D) \leq \mathrm{sdiam}(D) \leq 2\,\mathrm{srad}(D).$$

Proof. The inequality $\mathrm{srad}(D) \leq \mathrm{sdiam}(D)$ follows directly from the definitions. To verify the other inequality, let u and w be vertices such that $\mathrm{sd}(u,w) = \mathrm{sdiam}(D)$ and let v be a vertex such that $\mathrm{se}(v) = \mathrm{srad}(D)$. Then

$$\mathrm{sdiam}(D) = sd(u,w) \leq sd(u,v) + sd(v,w) \leq 2\,\mathrm{se}(v) = 2\,\mathrm{srad}(D). \quad \blacksquare$$

We now show that every pair r, d of integers with $3 \leq r \leq d \leq 2r$ is realizable as the strong radius and strong diameter of some strong oriented graph.

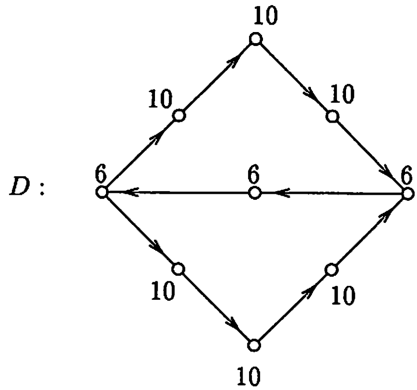

Figure 3: A strong oriented graph

Theorem 2.2 *For every pair r, d of integers with* $3 \le r \le d \le 2r$, *there exists a strong oriented graph D with* $\text{srad}(D) = r$ *and* $\text{sdiam}(D) = d$.

Proof. For $d = r$, let D be the directed cycle $\vec{C}_r$ of order r. Then $\text{srad}(D) = \text{sdiam}(D) = r$, as desired. For $d = 2r$, let D be obtained from two copies D_1 and D_2 of $\vec{C}_r$ by identifying a vertex in D_1 and a vertex in D_2. Then $\text{srad}(D) = r$ and $\text{sdiam}(D) = 2r = d$ and we have the desired digraph.

We now assume that $d = r + \ell$, where $1 \le \ell \le r - 1$. We consider two cases.

Case 1. $\ell = 1$. Then let D be obtained from the directed cycle $\vec{C}_{r+1}$: $v_1, v_2, \cdots, v_{r+1}, v_1$ by adding two arcs (v_1, v_3) and (v_2, v_4). Then $\text{se}(v_2) = \text{se}(v_3) = r + 1$ and $\text{se}(v_i) = r$ for all i with $1 \le i \le r+1$ and $i \ne 2, 3$. Hence $\text{srad}(D) = r$ and $\text{sdiam}(D) = r + 1$.

Case 2. $2 \le \ell \le r - 1$. Let D be obtained from the directed cycle $\vec{C}_r$: $u_1, u_2, \cdots, u_r, u_1$ and the directed path $\vec{P}_{\ell-1}$: $v_1, v_2, \cdots, v_{\ell-1}$ by adding two arcs (u_2, v_1) and $(v_{\ell-1}, u_1)$. Now $\text{se}(u_1) = \text{se}(u_2) = r$ and $\text{se}(v) = r + \ell = d$ for all $v \in V(D) - \{u_1, u_2\}$. Thus $\text{srad}(D) = r$ and $\text{sdiam}(D) = r + \ell = d$. Therefore, D has the desired property. ∎

If H is a $u - v$ strong geodesic in a strong oriented graph D, then certainly H has no proper strong subdigraph containing u and v. Hence every $u - v$ strong geodesic in D is a $u - v$ strong path in D. However, the converse is not true. Figure 4 shows a strong oriented graph D. The subdigraph D_2 is a $u - v$ strong path but not a $u - v$ strong geodesic. However, the subdigraph D_1 is a $u - v$ strong geodesic as $\text{sd}(u, v) = 4$.

Certainly, every strong path contains a directed cycle. Thus if D is a strong path of order n and size m, then $m \ge n$. Since the length of a

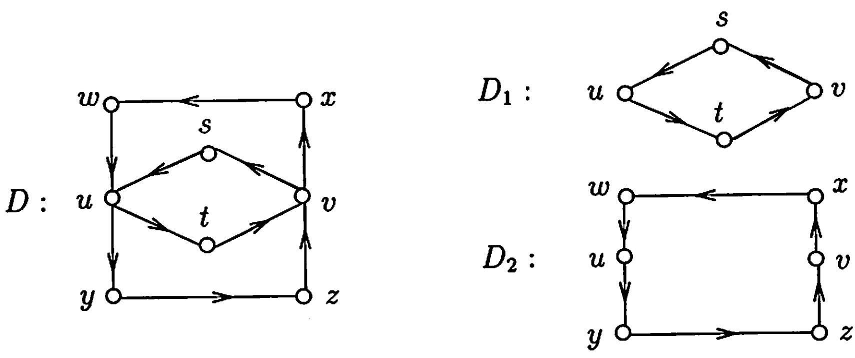

Figure 4: Strong geodesics and strong paths

directed $u - v$ path (and of a directed $v - u$ path) is at most $n - 1$, it follows that $m \leq 2n - 2$ and so sdiam $D \leq 2n - 2$.

3 An Upper Bound for the Strong Diameter of a Strong Oriented Graph

We have already noted that if D is a strong oriented graph of order $n \geq 3$, then $\text{sdiam}(D) \leq 2(n-1)$ and so $\text{sdiam}(D)/(n-1) \leq 2$. As we shall see, there is no strong oriented graph of order n for which $\text{sdiam}(D)/(n-1) = 2$. In this section, we establish a sharp constant upper bound M for $\text{sdiam}(D)/(n-1)$, thereby producing the sharp upper bound $M(n-1)$ for all strong oriented graphs of order $n \geq 3$.

Let H be a strong oriented graph, and u and v be vertices of H such that $\text{sd}(u, v) = \text{sdiam}(H)$. Furthermore, let D be a $u - v$ strong geodesic in H. Thus the size $|E(D)|$ of D is $\text{sdiam}(H)$. As noted earlier, D is also a $u - v$ strong path. Therefore,

$$\frac{\text{sdiam}(H)}{|V(H)|} \leq \frac{\text{sdiam}(H)}{|V(D)|} = \frac{|E(D)|}{|V(D)|}$$

and so

$$\frac{\text{sdiam}(H)}{|V(H)| - 1} \leq \frac{|E(D)|}{|V(D)| - 1}. \tag{1}$$

Hence (1) shows that if we can establish an upper bound for m/n for a strong path of order n and size m, then we will have found an upper bound for the strong diameter of a strong oriented graph in terms of its order.

For a $u - v$ strong path D, let $V_2(D)$ denote the set of vertices of degree 2 in D. Since $\deg u = \deg v = 2$, it follows that $|V_2(D)| \geq 2$. Let

$X_2(D) = V_2(D) - \{u, v\}$. For a $u-v$ strong path D, let D_{uv} and D_{vu} denote the unique directed $u-v$ path and unique directed $v-u$ path, respectively. We now present the first of four lemmas.

Lemma 3.1 *For every strong path D of order n and size m with $X_2(D) \neq \emptyset$, there exists a strong path D' of order n' and size m' with $|X_2(D')| < |X_2(D)|$ such that $m'/n' \geq m/n$.*

Proof. Let D be a $u-v$ strong path such that $X_2(D) \neq \emptyset$. If all of the vertices of D_{uv} and D_{vu} have degree 2, then D is a directed cycle and $m = n$. Hence $m/n = 1$ and surely any strong path D' of order n' and size m' satisfies $m'/n' \geq 1$. Hence we may assume that neither D_{uv} nor D_{vu} contains only vertices of degree 2.

Suppose first that D contains vertices $a, b \in X_2(D)$ such that a is adjacent to b. Let d be the vertex adjacent to a and c be the vertex adjacent from b. Now $c \neq d$ since D is a $u-v$ strong path. We construct a new digraph D' by deleting b (and its two incident arcs) and adding the arc (a, c). Clearly, D' is a $u-v$ strong path of order $n-1$ and size $m-1$. Since $|X_2(D')| = |X_2(D)| - 1$ and $(m-1)/(n-1) \geq m/n$, we have the desired result. Thus we may assume that every vertex of degree 2 is adjacent to and from vertices of degree 3 or 4 if neither of these vertices is u or v.

Let $t \in X_2(D)$. Then t is adjacent from a vertex x and to a vertex y. We now describe a desired $u-v$ strong path whose construction depends on whether x and y are adjacent.

If x and y are not adjacent in D, then D' is constructed by deleting t (and its two incident arcs) and adding the arc (x, y). If x and y are adjacent, then necessarily (y, x) is an arc of D since D is a $u-v$ strong path and D_{uv} and D_{vu} are the unique directed $u-v$ path and directed $v-u$ path in D. If (y, x) is an arc in D, then D' is constructed from D by deleting t (and its two incident arcs), the arc (y, x), and adding two new vertices x_1 and y_1 together with the arcs (x, x_1), (x_1, y_1), (y_1, y), (y_1, x), and (y, x_1). See Figure 5.

Let D' be a digraph of order n' and size m'. We show that $m'/n' \geq m/n$. In the case where x and y are not adjacent, $n' = n-1$ and $m' = m-1$ and $m'/n' \geq m/n$. In the case where (y, x) is an arc of D, $n' = n+1$ and $m' = m+2$ and, once again, $m'/n' \geq m/n$ since $m \leq 2n-2$.

It remains only to show that D' is a $u-v$ strong path. Certainly D' is strong and contains both u and v. Thus D' contains a $u-v$ strong path D''. We show that $D'' = D'$ in both cases.

Case 1. x and y are not adjacent in D (so that (x, y) is an arc in D'). Necessarily, D'' contains the arc (x, y) as well, for otherwise neither D_{uv} nor D_{vu} contains the vertex t, contradicting the fact that D is a strong path. Therefore, if $D'' \neq D'$, then D' contains some arc (w, z) that is not

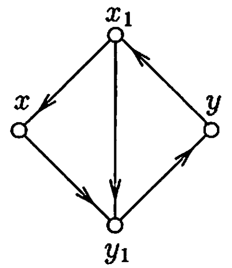

Figure 5: Constructing D'

present in D''. Consequently, there is both a directed $u-v$ path and a directed $v-u$ path in D' that do not contain the arc (w,z). Hence there is a directed $u-v$ path and a directed $v-u$ path in D that do not contain (w,z), contradicting the fact that D is a strong path.

Case 2. *x and y are adjacent.* The argument here is similar to Case 1 and is therefore omitted. ∎

As a consequence of Lemma 3.1, the maximum value of the ratio m/n for a strong path of order n and size m occurs among those strong paths D for which $X_2(D) = \emptyset$. Hence, with the exception of the two vertices of degree 2, every vertex of D has degree 3 or 4. The next two lemmas show that there are some restrictions on the location of vertices of degree 4 in a $u-v$ strong path in which u and v are the only vertices of degree 2.

Lemma 3.2 *In any $u-v$ strong path D with $X_2(D) = \emptyset$, neither u nor v is adjacent to or from a vertex of degree* 4.

Proof. Assume first that u is adjacent to a vertex of degree 4. Let D_{uv} : $u = v_1, v_2, \cdots, v_n = v$. Since $\operatorname{od} u = 1$, it follows that $\deg v_2 = 4$. Then v_2 is adjacent to v_3 and a vertex v_k with $k > 3$. However, this contradicts Lemma 1.3.

Assume next that u is adjacent from a vertex of degree 4. Let D_{vu} : $v = u_1, u_2, \cdots, u_n = u$. Since $\operatorname{id} u = 1$, it follows that $\deg u_{n-1} = 4$. Then u_{n-1} is adjacent from u_{n-2} and a vertex u_r with $r < n-2$. This again contradicts Lemma 1.3. ∎

Lemma 3.3 *If D is a strong path with $X_2(D) = \emptyset$, then no two vertices of degree* 4 *are adjacent in D.*

Proof. Let D be a $u-v$ strong path and assume, to the contrary, that D contains two vertices x and y of degree 4 such that (x,y) is an arc of D. Since D is a strong path, (x,y) lies on at least one of D_{uv} or D_{vu}, say D_{uv}. Hence we may assume that D_{uv} is the directed $u-v$ path $u = v_1, v_2, \cdots, v_i = x, v_{i+1} = y, \cdots, v_n = v$. By Corollary 1.4, x and y lie on D_{vu} as well. Let D_{vu} be the directed $v-u$ path $v = v_n, v_{r_2}, v_{r_3}, \cdots, v_{r_{n-1}}, v_1 = u$. Thus $i = r_s$ and $i+1 = r_t$ for distinct integers s and t. Necessarily, $r_{s+1} \neq i+1$, $r_{s-1} \neq i-1$, $r_{t+1} \neq i+2$, and $r_{t-1} \neq i$. By Lemma 3.2, no neighbor of x and y is u or v.

We now consider two cases, according to whether $s < t$ or $s > t$.

Case 1 $s < t$. Then $v = v_n, v_{r_2}, v_{r_3}, \cdots, v_{r_{s-1}}, v_i, v_{i+1}, v_{r_{t+1}}, \cdots, v_1 = u$ is a directed $v-u$ path whose length is less than that of D_{vu}, which is impossible.

Case 2 $s > t$. Lemma 1.3 implies that $r_{t+1} < r_s = i$. But then the path

$$v = v_n, v_{r_2}, \ldots, v_{r_t}, v_{r_{t+1}}, v_{r_{t+1}+1}, v_{r_{t+1}+2}, \ldots, v_{i-1}, v_i = v_{r_s}, v_{r_{s+1}}, \ldots, v_1 = u$$

is a directed $v-u$ path that, unlike D_{vu}, contains the arc (v_{i-1}, v_i). This contradicts the uniqueness of D_{vu}. ∎

We observed in Lemma 3.2 that if D is a $u-v$ strong path for which $X_2(D) = \emptyset$, then u and v are adjacent to and from vertices of degree 3. In the next lemma, we provide additional information about the vertices of degree 3 in D.

Lemma 3.4 *Let $D_{uv} : v_1, v_2, \cdots, v_n$ be the unique directed $u-v$ path in a $u-v$ strong path D with $X_2(D) = \emptyset$, and let v_i be a vertex of degree* 3.

(*a*) *If* $\text{id}\, v_i = 2$, *then* $\deg v_{i+1} = 3$ *and* $\text{od}\, v_{i+1} = 2$.

(*b*) *If* $\text{od}\, v_i = 2$, *then* $\deg v_{i-1} = 3$ *and* $\text{id}\, v_{i-1} = 2$.

Proof. Let $D_{vu} : v = v_n, v_{r_2}, v_{r_3}, \cdots, v_{r_{n-1}}, v_1 = u$, where $i = r_s$. Assume first that $\text{id}\, v_i = 2$. Then $i + 1 = r_{s+1}$. Since v_{i+1} immediately follows v_i both on D_{uv} and D_{vu} and $\deg v_{i+1} \neq 2$, it follows that $\deg v_{i+1} = 3$ and $\text{od}\, v_{i+1} = 2$. This establishes (a). The proof of (b) is similar. ∎

We are now prepared to present the desired upper bound for the strong diameter of a strong oriented graph.

Theorem 3.5 *If D is a strong path of order $n \geq 3$ and size m, then*

$$\frac{m}{n-1} \leq \frac{5}{3}.$$

Proof. Let D be a $u-v$ strong path. By Lemma 3.1, we may assume that $X_2(D) = \emptyset$. Let $D_{uv} : u = v_1, v_2, v_3, \cdots, v_{n-1}, v_n = v$ be the unique directed $u-v$ path in D. Then v_1 and v_n are the only vertices of degree 2 in D. By Lemma 3.2, vertices v_2 and v_{n-1} have degree 3. By Lemmas 3.3

and 3.4, at most one-third of the vertices v_i $(3 \le i \le n-2)$ have degree 4. Thus

$$2m \le 2 \cdot 2 + 2 \cdot 3 + \frac{1}{3}(n-4) \cdot 4 + \frac{2}{3}(n-4) \cdot 3 = \frac{10n-10}{3}$$

and so $\frac{m}{n-1} \le \frac{5}{3}$. ∎

Combining Theorem 3.5 and inequality (1), we have the following.

Corollary 3.6 *If D is a strong oriented graph of order $n \ge 3$, then*

$$\text{sdiam}(D) \le \frac{5}{3}(n-1).$$

The upper bound for sdiam(D) given in Corollary 3.6 is sharp; indeed, it is attainable for every integer $n \ge 3$, as is show in Figure 6.

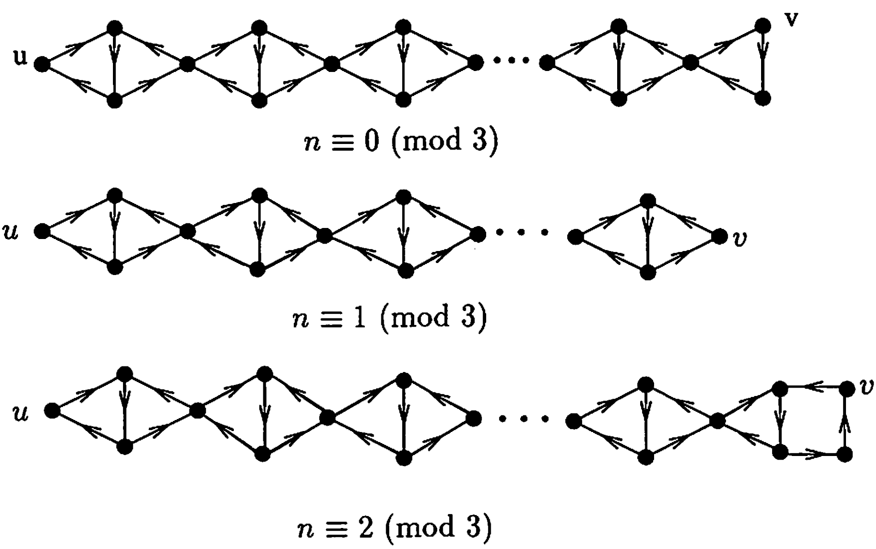

Figure 6: Strong oriented graphs D of order $n \ge 3$ with sdiam(D) = $\lfloor 5(n-1)/3 \rfloor$

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