On a conjecture concerning irredundant and perfect neighbourhood sets in graphs

E.J. Cockayne
University of Victoria
Victoria, BC, Canada

C.M. Mynhardt
University of South Africa
Pretoria, South Africa

ABSTRACT. It has been conjectured that the smallest cardinality $\theta(G)$ of a perfect neighbourhood set of a graph is bounded above by $\mathrm{ir}(G)$, the smallest order of a maximal irredundant set. We prove results concerning the construction of perfect neighbourhood sets from irredundant sets which could help to resolve the conjecture and which establish that $\theta(G) \leq \mathrm{ir}(G)$ in certain cases. In particular the inequality is proved for claw-free graphs and for any graph which has an ir-set S whose induced subgraph has at most six non-isolated vertices.

Dedicated to Professor Stephen T. Hedetniemi on the occasion of his 60th birthday

1 Introduction

For a simple graph G and $S \subseteq V = V(G)$, vertex u of G is called *S-perfect* if $|N[u] \cap S| = 1$ ($N(u)$, $N[u]$ will denote the open (closed) neighbourhoods of vertex u). The set S is called a *perfect neighbourhood* set (abbreviated, S is a PN-set) if for all $v \in V$, v or some neighbour of v is S-perfect. We observe that S is a PN-set if and only if the set of S-perfect vertices is a dominating set of G. The parameters $\theta(G)$ and $\Theta(G)$ are the smallest and largest cardinalities of PN-sets of G.

For $s \in S \subseteq V$, the *S-private neighbourhood of s*, is the set

$$\mathrm{pn}(s, S) = N[s] - N[S - \{s\}] ,$$

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where for $A \subseteq V$, $N[A]$ denotes the union of closed neighbourhoods of elements of A. Elements of $\mathrm{pn}\,(s, S)$ are called *S–private neighbours of s*. An S–private neighbour of s is either s itself, in which case s is an isolated vertex of $G[S]$, or is a neighbour of s in $V - S$ which is not adjacent to any vertex of $S - \{s\}$. This latter type will be called an *external S–*private neighbour (abbreviated S–epn) of s. Observe that $s_1 \neq s_2$ implies $\mathrm{pn}\,(s_1, S) \cap \mathrm{pn}\,(s_2, S) = \emptyset$. The set S is *irredundant* if for all $s \in S$, $\mathrm{pn}\,(s, S) \neq \emptyset$ and the parameters $\mathrm{ir}\,(G)$, $\mathrm{IR}\,(G)$ are the smallest and largest cardinalities of maximal irredundant sets of G.

Perfect neighbourhood sets and irredundant sets are clearly related. Observe that $|N[x] \cap S| = 1$ if and only if $x \in \mathrm{pn}\,(s, S)$ for some $s \in S$. Thus the set of S–perfect vertices is precisely $\bigcup_{s\in S} \mathrm{pn}\,(s, S)$. Our first result follows.

Proposition 1. [3] *If S is a PN–set, then S is irredundant.*

Proof: Suppose S is not irredundant. Then there exists $s \in S$ with $\mathrm{pn}\,(s, S) = \emptyset$. Neither s nor any neighbour of s is S–perfect, hence S is not a PN–set.

Perfect neighbourhood sets were introduced and researched by Fricke, Haynes, Hedetniemi, Hedetniemi and Henning [5]. These authors defined PN–sets while studying functions $f\colon V \to \{0, 1\}$ subject to various edge and neighbourhood conditions, examples of which are given in [5, 7]. If the function f satisfies the condition that for all $u \in V$, there exists $v \in N[u]$ with $\sum_{x\in N[v]} f(x) = 1$, then $S = \{s \in V \,|\, f(s) = 1\}$ is a PN–set.

The following relations involving perfect neighbourhood parameters and $\gamma(G)$ and $\Gamma(G)$, the smallest and largest cardinalities of minimal dominating sets of G, were obtained in [5].

Theorem 2. [5] *For any graph G,* (a) $\theta(G) \leq \gamma(G)$ (b) $\Theta(G) = \Gamma(G)$.

Irredundant sets have been well–studied in the literature. The reader is referred to [1, 6, 8] for bibliographies containing approximately one hundred references. It is well–known that for any graph G, $\mathrm{ir}\,(G) \leq \gamma(G)$. In view of this inequality and Theorem 2(a), the authors of [5] were led to compare $\mathrm{ir}\,(G)$ and $\theta(G)$ and conjectured that for any graph G, $\theta(G) \leq \mathrm{ir}\,(G)$.

In [3] Cockayne, Hedetniemi, Hedetniemi and Mynhardt proved that for any maximal irredundant set S of a tree T, there exists an independent PN–set U with $|U| \leq |S|$. This clearly implies that $\theta(T) \leq \mathrm{ir}\,(T)$ for any tree T.

In this paper we continue work on the conjecture $\theta(G) \leq \mathrm{ir}\,(G)$. We prove a variety of results concerning the construction of perfect neighbourhood sets from maximal irredundant sets under certain special conditions. It is hoped that these results and the ideas involved in their proofs will be useful in an eventual resolution of the conjecture. These results are applied to prove that if G is claw–free or G has an ir–set (i.e. a maximal irredundant set of minimum cardinality) S for which $G[S]$ has at most six non–isolated vertices, then $\theta(G) \leq \mathrm{ir}(G)$.

2 PN–sets from maximal irredundant sets

As noted in [3], a vertex subset S of G induces a partition of the vertex set. Specifically

$$V = Z_S \cup F_S \cup Y_S \cup E_S \cup Q_S \cup R_S \qquad \text{(disjoint union)} \qquad (1)$$

where

$$\begin{aligned} Z_S &= \{v \in V \,|\, v \text{ is isolated in } G[S]\}, \\ Y_S &= S - Z_S, \\ F_S &= \{v \in V - S \,|\, v \text{ is an } S\text{–epn of a vertex of } Z_S\}, \\ E_S &= \{v \in V - S \,|\, v \text{ is an } S\text{–epn of a vertex of } Y_S\}, \\ R_S &= \{v \in V - S \,|\, N[v] \cap S = \emptyset\} \end{aligned}$$

and

$$Q_S = \{v \in V - S \,|\, |N(v) \cap S| \geq 2\}.$$

We will often deal with irredundant sets S such that $G[S]$ is isolate–free, in which case $Z_S = F_S = \emptyset$.

Several proofs will require the following characterisation of maximal irredundant sets given by Cockayne, Grobler, Hedetniemi and McRae [2].

Theorem 3. *The irredundant set S of a graph G is maximal if and only if for each $r \in N[R_S]$, there exists $s \in S$ such that $pn(s, S) \subseteq N[r]$.*

If $r \in N[R_S]$ and s is a vertex whose existence is asserted by Theorem 3, then we say that r *annihilates* s.

For subsets A, B of V, A *dominates* B (denoted $A \succ B$) if $B \subseteq N[A]$. Further, we abbreviate $A \succ \{b\}$ to $A \succ b$. We say that vertex v is *adjacent to* A if v is adjacent to some vertex of A and that A *is adjacent to* B if ab is an edge of G for some $a \in A$, $b \in B$. The next result which will be used extensively to prove that sets have the PN property, was given in [3].

Proposition 4. *The irredundant set S is a PN–set if and only if $E_S \cup F_S \cup Z_S \succ V$.*

Proof: The set of S–perfect vertices is $E_S \cup F_S \cup Z_S$.

Our first new theorem will show that isolated vertices of the subgraph induced by maximal irredundant sets may be ignored in the quest to settle the conjecture.

Theorem 5. *Suppose that for any graph H and any maximal irredundant set T of H with $Z_T = \emptyset$, there exists a PN–set U_H of H with $|U_H| \leq |T|$. Then for any graph G, $\theta(G) \leq ir(G)$.*

Proof: Let S be an ir–set of G. If $Z_S = \emptyset$, then the result is true by hypothesis. If $Y_S = \emptyset$, then S is independent and hence dominating. It follows that $\text{ir}(G) = \gamma(G)$ and $\theta(G) \leq \text{ir}(G)$ by Theorem 2(a).

So we may assume that $Y_S \neq \emptyset$ and $Z_S \neq \emptyset$. Define $Q'_S = \{v \in Q_S \mid v$ is adjacent to $Z_S\}$ and $H = G[V - (Z_S \cup F_S \cup Q'_S)]$. The set Y_S is irredundant in H and the set of vertices undominated by Y_S in H, is precisely R_S. Let $v \in N_G[R_S] \cap V(H)$. Since S is maximal irredundant in G, by Theorem 3 applied to S and G, v annihilates some $s \in S$ and since v is not adjacent in G to Z_S, this annihilated vertex $s \in Y_S$. Moreover, $\text{pn}(s, S, G) = \text{pn}(s, Y_S, H)$ and so in the subgraph H, v annihilates $s \in Y_S$. From Theorem 3 applied to Y_S and H, we conclude that Y_S is maximal irredundant in H. Since $H[Y_S]$ has no isolates, we may apply the hypothesis to H with $T = Y_S$, to assert the existence of a PN–set W of H satisfying $|W| \leq |Y_S|$. Define

$$\begin{aligned} Z(W) &= \{z \mid z \text{ is isolated in } H[W]\} \\ Y(W) &= W - Z(W) \\ E(W) &= \{x \mid x \text{ is a } W\text{–epn of some } y \in Y(W) \text{ in } H\} \\ F(W) &= \{x \mid x \text{ is a } W\text{–epn of some } z \in Z(W) \text{ in } H\}\ . \end{aligned}$$

Since W is a PN–set of H, by Proposition 4, we have

$$Z(W) \cup E(W) \cup F(W) \succ V(H)\ . \tag{2}$$

Let $U = W \cup Z_S$. Clearly $|U| \leq |S|$. Observe that

$$\left.\begin{aligned} Z(W) \cup Z_S &= Z_U \\ Y(W) &= Y_U \\ E(W) &= E_U \\ F(W) &\subseteq F_U\ . \end{aligned}\right\} \tag{3}$$

By (2) and (3)

$$Z_U \cup E_U \cup F_U \supseteq Z(W) \cup E(W) \cup F(W) \succ V(H)$$

and

$$Z_U \supseteq Z_S \succ V(G) - V(H)\ .$$

Thus $Z_U \cup E_U \cup F_U \succ V(G)$ and (by Proposition 4) U is a PN–set of G. Therefore $\theta(G) \leq |U| \leq |S| = \text{ir}(G)$.

The remaining results of this section show that from maximal irredundant sets S satisfying a variety of special conditions, PN–sets of no greater cardinality may be constructed. In view of Theorem 5, we are only interested in sets S for which $Z_S = \emptyset$ (i.e. $Y_S = S$).

Proposition 6. *Suppose that S is maximal irredundant in G, W is a subset of E_S containing exactly one S–epn of each $s \in Y_S (= S)$ and for each $r \in R_S$, $|N(r) \cap W| = 1$. Then W is a PN–set of G of order $|S|$.*

Proof: It is easily verified that $Z_W \cup E_W \cup F_W \succ V$.

In order to state the conditions of the next result, we will need the following definitions concerning vertex subsets of the induced subgraph $G[S]$ (again we emphasize that $G[S]$ is assumed to be isolate–free). For $S' \subseteq S$ define

$$f(S') = \bigcup_{s \in S'} \mathrm{pn}\,(s, S) \ ,$$

where we will abbreviate $f(\{s\})$ to $f(s)$ ($s \in S$). Further, for $B \subseteq S$, define

$$\begin{aligned} B_1 &= \{b \,|\, b \text{ is isolated in } G[B]\} \\ B_2 &= B - B_1 \\ B_3 &= \{b \in S - B \,|\, |N(b) \cap B| = 1\} \\ B_4 &= \{b \in S - (B \cup B_3) \,|\, b \text{ is adjacent to } B_1 \cup B_3\} \end{aligned}$$

and

$$B_5 = S - (B \cup B_3 \cup B_4) \ .$$

Note that $\bigcup_{i=1}^{5} B_i = S$ (disjoint union). Further, let D_B be the set of all $q \in Q_S$ satisfying

D1. $|N(q) \cap B| = 1$,

D2. q is not adjacent to $f(B_4 \cup B_5)$

and

D3. q is not adjacent to any $r \in R_S$ for which $N(r) \cap f(B) = \emptyset$.

Theorem 7. *Let S be maximal irredundant in G with $G[S]$ isolate–free and $B \subseteq S$ which satisfies*

(i) *Each $b \in B_5$ is adjacent to D_B.*

(ii) *For each $q \in Q_S$, $q \in N[D_B \cup B_1 \cup B_3 \cup f(B)]$.*

Then G has a PN–set U with $|U| \le |S|$.

Proof: Let $B_4 \cup B_5 = \{b_1, \ldots, b_t\}$. Construct U by the following procedure.

ALGORITHM 1

Step 1 (Initialize)

$$R \leftarrow \{r \in R_S \big| N(r) \cap f(B) = \emptyset\}$$
$$X \leftarrow \{s \in S - B \big| f(s) \text{ is adjacent to } R\}$$
$$k \leftarrow 0$$
$$U \leftarrow B$$

Step 2 (Construct subset $\{r_1, \dots r_k\}$ of R_S)

WHILE ($\exists r \in R$ with r adjacent to $f(X)$)

$$k \leftarrow k + 1$$
$$r_k \leftarrow r$$
$$X \leftarrow X - \{x \in X \big| r \text{ is adjacent to } f(x)\}$$
$$R \leftarrow R - \{r' \in R \big| r' \text{ is adjacent to } N(r) \cap E_S\}$$

Step 3 (Augment U, initialize for Step 4)

$$U \leftarrow U \cup \{r_1, \dots, r_k\}$$
$$E \leftarrow f(B) \cup \left(\bigcup_{i=1}^{k} (N(r_i) \cap E_S) \right)$$
$$j \leftarrow 0$$

Step 4 (Add a subset of E_S to U)

FOR $\quad i = 1, \dots, t \quad$ DO

IF ($\exists c \in f(b_i)$ which is not adjacent to $E \cup U$)

THEN $j \leftarrow j + 1$

$c_j \leftarrow c$

$U \leftarrow U \cup \{c_j\}$

OD.

END.

Let $T = \{r_1, \dots, r_k\}$ and $C = \{c_1, \dots, c_j\}$. We will first show that the set $U = B \cup T \cup C$ (disjoint union) constructed by the procedure satisfies $|U| \le |S|$ and then use Proposition 4 to prove that U is a PN–set.

By Theorem 3, each $r \in R$ annihilates some $s \in S$. Since no $r_i \in T$ is adjacent to $f(B)$, each r_i annihilates some $s \in B_3 \cup B_4 \cup B_5$. Further, by Step 2, the sets $N(r_i) \cap E_S$, $i = 1, \dots, k$, are disjoint; hence there exist distinct vertices $s_1, \ldots, s_k \in B_3 \cup B_4 \cup B_5$ such that r_i annihilates s_i for $i = 1, \ldots, k$. Suppose that $\{s_1, \ldots, s_m\} \subseteq B_4 \cup B_5$ while $\{s_{m+1}, \ldots, s_k\} \subseteq B_3$. Whenever an element s of $\{s_1, \ldots, s_m\}$ is encountered as a b_i in the DO loop of Step 4, the IF condition is not satisfied (since the annihilator of s is already in U). Hence $m + |C| \leq |B_4 \cup B_5|$ and

$$\begin{aligned} |T \cup C| &= |T| + |C| \\ &\leq |T| + |B_4 \cup B_5| - m \\ &= |B_4 \cup B_5| + k - m \\ &\leq |B_3 \cup B_4 \cup B_5| \,. \end{aligned}$$

Therefore

$$\begin{aligned} |U| &= |B \cup T \cup C| \\ &\leq |B| + |B_3 \cup B_4 \cup B_5| = |S| \,. \end{aligned}$$

By Proposition 4, it remains to show that $Z_U \cup E_U \cup F_U$ dominates each $v \in V$. The following assertions are all implied by the definitions of the various sets, private neighbour properties and the construction.

Firstly, C is independent and hence

$$B_1 \cup C \subseteq Z_U \,. \tag{4}$$

Secondly, $f(B) \cup D_B \cup B_3$ is not adjacent to $T \cup C$. We deduce that

$$f(B) \cup D_B \cup B_3 \subseteq E_U \cup F_U \,. \tag{5}$$

Lastly (as noted above) the sets $N(r_i) \cap E_S$, $i = 1, \ldots, k$, are disjoint and furthermore not adjacent to $B \cup C$. Thus for each $i = 1, \ldots, k$,

$$N(r_i) \cap E_S \subseteq E_U \cup F_U \,. \tag{6}$$

There are now four cases to consider, depending on the membership of v in the blocks of the partition $V = S \cup Q_S \cup E_S \cup R_S$.

Case 1 $v \in Q_S$.

By condition (ii), (4) and (5),

$$Z_U \cup E_U \cup F_U \supseteq D_B \cup B_1 \cup B_3 \cup f(B) \succ v \,.$$

Case 2 $v \in S$.

If $v \in B \cup B_3$, then $f(B) \cup B_3 \succ v$; each $v \in B_4$ is dominated by $B_1 \cup B_3$ and if $v \in B_5$, then (condition (i)) v is adjacent to D_B. From (4) and (5) we have

$$Z_U \cup E_U \cup F_U \supseteq f(B) \cup D_B \cup B_1 \cup B_3 \succ v \,.$$

Case 3 $v \in R_S$.

By the construction and Theorem 3, v is adjacent to $f(B)$ or to $N(r_i) \cap E_S$ for some $i = 1, \ldots, k$. By (6), $E_U \cup F_U \succ v$.

Case 4 $v \in E_S$.

If $v \in f(B) \cup f(B_3)$, then (by (5)) $E_U \cup F_U \supseteq B_3 \cup f(B) \succ v$. Otherwise $v \in f(b_i)$ for some $i \in \{1, \ldots, t\}$ and one of the following possibilities holds:

(a) $E_U \cup F_U \supseteq E \succ v$ (by (5) and (6)).

(b) $Z_U \supseteq C \succ v$ (by (4)).

(c) There exists $c \in (f(b_i) - \{v\}) \cap C$. Since $c \in Z_U$, it follows that $b_i \in F_U$, hence $F_U \succ v$.

Thus for all $v \in V$, $Z_U \cup E_U \cup F_U \succ v$ and by Proposition 4, U is a PN–set as asserted. □

The next two results are special cases of Theorem 7. We use the notation developed for that result.

Corollary 8. *Let S be an ir–set of G where $G[S]$ is isolate–free and suppose that S contains an independent set B such that $|B_1 \cup B_3| \geq |S| - 1$. Then $\theta(G) \leq ir(G)$.*

Proof: Since B is independent, $B = B_1$ and $B_2 = \emptyset$. If $B_1 \cup B_3 = S$, then $B_4 = B_5 = \emptyset$ and conditions (i) and (ii) of Theorem 7 are satisfied. If $B_1 \cup B_3 = S - \{s\}$, then s is adjacent to $B_1 \cup B_3$ (otherwise s is an isolate of $G[S]$). Hence $s \in B_4$ and $B_5 = \emptyset$. Since each $q \in Q_S$ is adjacent to at least two vertices of S, q is adjacent to $B_1 \cup B_3$. Again the conditions of Theorem 7 are satisfied. □

Corollary 9. *If S is an ir–set of G where $G[S]$ is isolate–free and $\Delta(G[S]) \geq |S| - 2$, then $\theta(G) \leq ir(G)$.*

Proof: Let s have degree $\Delta(G[S])$ in $G[S]$. Then $B = \{s\}$ satisfies the hypothesis of Corollary 8. □

The case $\Delta(G[S]) = |S| - 3$ will now be discussed.

Theorem 10. *Let S be an ir–set of G such that $G[S]$ is isolate–free. Further, let $r^* \in R_S$ and $s_1 \in S$ satisfy*

(i) *s_1 has degree $\Delta(G[S]) = |S| - 3$ in $G[S]$,*

(ii) *$N(r^*) \cap f(s_1) = \emptyset$ and*

(iii) *$N(r^*) \cap f(s) \neq \emptyset$ for at least two vertices $s \in S - \{s_1\}$.*

Then $\theta(G) \leq ir(G)$.

Proof: Let $S = \{s_1, \ldots, s_m\}$ and use the notation of Theorem 7 with $B = \{s_1\}$. Then $B_2 = \emptyset$ and (say) $B_3 = \{s_2, \ldots, s_{m-2}\}$. If $s \in \{s_{m-1}, s_m\}$ is not adjacent to B_3, then $B' = \{s_1, s\}$ satisfies the conditions of Corollary 8 and the result follows. Hence we may assume that $B_4 = \{s_{m-1}, s_m\}$ and $B_5 = \emptyset$. Firstly, let $s_1 \in U$ and then augment U with an independent subset $T = \{r_1, \ldots, r_k\}$ of R_S formed precisely as in Steps 1 and 2 of ALGORITHM 1 with the added provision that $r_1 = r^*$. Note that by definition of r^*, $|X|$ is reduced by at least two after r^* is added to T and by at least one after each other r_j is added to T. Thus if X_1 is the initial set X (i.e. X_1 contains all $s \in S - B$ for which there exists $r \in R_S$ adjacent to $f(s)$ but not to $f(s_1)$), then $|T| \leq |X_1| - 1 \leq m - 2$. Further, at each insertion, r_i annihilates some s in the current set X (Theorem 3). Hence $|T| \leq p$, the number of vertices of X_1 which are annihilated by some vertex of R_S. Define A_S to be the set of all $q \in Q_S$ such that

(i) $N(q) \cap S = \{s_{m-1}, s_m\}$ and

(ii) q is not adjacent to $f(s_1) \cup T \cup (N(T) \cap E_S)$.

The remaining construction requires two cases:

Case 1 $A_S \neq \emptyset$.

Choose any $q^* \in A_S$ and add it to U. Then the final set $U = \{s_1, q^*\} \cup T$ and $|U| \leq 2 + (|X_1| - 1) \leq m$.

Case 2 $A_S = \emptyset$.

For $i = m-1$ and m, if there exists $u_i \in f(s_i)$ such that u_i is not adjacent to $f(s_1) \cup T \cup (N(T) \cap E_S)$, then choose any one such u_i and add it to U. Observe that if such a u_i is added, then s_i is not annihilated by any vertex of T. Thus if $j (= 1$ or $2)$ vertices are added to U in this case,

$$\begin{aligned} |U| &= 1 + |T| + j \\ &\leq 1 + p + j \\ &\leq 1 + (m - 1 - j) + j = m \,. \end{aligned}$$

It now remains to show that $Z_U \cup E_U \cup F_U \succ v$ for all $v \in V$. In each of the above cases

$$B \cup T \subseteq Z_U$$

and

$$B_3 \cup f(B) \cup (N(T) \cap E_S) \subseteq F_U \,.$$

Hence it is easily seen that

$$Z_U \cup F_U \succ (Q_S - A_S) \cup S \cup R_S \cup f(B \cup B_3) \,.$$

Finally, consider $v \in A_S \cup f(B_4)$. If Case 1 occurred, then $q^* \in Z_U$ and $F_U \supseteq B_4 \succ v$. Otherwise Case 2 occurs, $A_S = \emptyset$, $v \in f(\{s_{m-1}, s_m\})$ and one of the following possibilities occurs:

(i) v is adjacent to $f(s_1) \cup T \cup (N(T) \cap E_S) \subseteq F_U \cup Z_U$, or

(ii) $v \in f(s_i)$ where $i = m-1$ or m and there exists $w \in f(s_i) \cap U$ (possibly $w = v$). In this case $E_U \cup F_U \supseteq \{s_i\} \succ v$.

□

Theorem 11. *Let S be an ir–set of G such that $G[S]$ is isolate–free and there exists $r^* \in R_S$ such that for each $s \in S$, $N(r^*) \cap f(s) \neq \emptyset$. Then $\theta(G) \leq ir(G)$.*

Proof: Let $|S| = m$ and $\{s_1, \ldots, s_t\}$ be the vertices of S which are not annihilated by r^*. Initially put r^* into U and define $E = N(r^*) \cap E_S$. Now add to U an independent set C of E_S which is constructed by the DO loop of Step 4 of ALGORITHM 1. Let $C = \{c_1, \ldots, c_j\}$ and relabel $\{s_1, \ldots, s_t\}$ so that for each $i = 1, \ldots, j$, $c_i \in f(s_i)$. We next define W_S to be the subset of Q_S containing all $q \in Q_S$ such that $q \notin N(\{r^*\} \cup E \cup C \cup \{s_1, \ldots, s_j\})$. Note that each $q \in W_S$ is adjacent to at least two vertices of $S' = S - \{s_1, \ldots s_j\}$. The last part of this construction is to add an independent subset of W_S to U by the following procedure.

Step 1 (Initialize)

$L \leftarrow W_S$

$P \leftarrow S'$

Step 2 WHILE ($\exists\, q \in L$ such that $N(q) \cap S \subseteq P$)

$U \leftarrow U \cup \{q\}$

$L \leftarrow L - \{x \in L \,|\, x \in N[q]\}$

$P \leftarrow P - \{s \in P \,|\, s \in N(q)\}$.

END.

Suppose that D is the subset of W_S added into U by this procedure. With each insertion of q into D, $|P|$ is reduced by at least two. Hence $|D| \leq \lfloor (m-j)/2 \rfloor$ and so $|U| \leq 1 + j + \lfloor (m-j)/2 \rfloor$. Since r^* annihilates at least one s, $j \leq m-1$ and we conclude $|U| \leq m$.

In order to see that $Z_U \cup E_U \cup F_U \succ v$ for each $v \in V$, we observe that $Z_U = U = \{r^*\} \cup C \cup D$ (i.e. U is independent) and

$$E = N(r^*) \cap E_S \subseteq F_U \ ,$$
$$N(D) \cap S \subseteq F_U \ ,$$
$$\{s_1, \ldots, s_j\} \subseteq F_U \ .$$

Case 1 $v \in R_S$.
In this case v annihilates some $s \in S$ and by hypothesis there exists $x \in f(s) \cap N(r^*)$. Thus v is adjacent to $x \in F_U$.
Case 2 $v \in S$.
By hypothesis there exists $x \in N(r^*) \cap f(v)$ and v is adjacent to $x \in F_U$.
Case 3 $v \in Q_S$.
If $v \neq W_S$, then v is adjacent to $\{r^*\} \cup C \subseteq Z_U$ or to $E \cup \{s_1, \ldots, s_j\} \subseteq F_U$. Otherwise $v \in N[D]$ and is dominated by $D \subseteq F_U$.
Case 4 $v \in E_S$.
One of the following possibilities holds:

$$v \in N[E] \subseteq F_U \ ,$$
$$v \in N[C] \text{ which is dominated by } C \subseteq Z_U \ ,$$
$$\text{or } v \in f(s_i) \text{ for some } i \in \{1, \ldots, j\}, \text{ in which case } F_U \succ v \ .$$

□

Corollary 12. *Let S be an ir–set with $G[S]$ isolate–free and $\delta(G[S]) \geq |S| - 3$. Then $\theta(G) \leq ir(G)$.*

Proof: If $\Delta(G[S]) \geq |S| - 2$, then the result holds by Corollary 9 and so we may assume that $G[S]$ is $(|S| - 3)$–regular. Choose $W \subseteq E_S$ so that $|W \cap f(s)| = 1$ for each $s \in S$. If $|N(r) \cap W| = 1$ for each $r \in R$, then the result is true by Proposition 6. Hence we may assume the existence of $r^* \in R$ such that $|N(r^*) \cap W| \geq 2$. Theorem 11 implies the result unless there exists $s_1 \in S$ such that $N(r^*) \cap f(s_1) = \emptyset$. But now r^* and s_1 satisfy the hypothesis of Theorem 10. □

Theorem 13. *Let G have an ir–set S such that $G[S]$ has at most six non–isolates. Then $\theta(G) \leq ir(G)$.*

Proof: The argument used to establish Theorem 5 shows that it is sufficient to prove $\theta(G) \leq ir(G)$ for any graph G which has an ir–set S where $H = G[S]$ is isolate–free and $m = |S| \leq 6$. For $m \leq 5$ it is very easy to show the existence of a subset B of S satisfying Corollary 8, so we now assume that $m = 6$. If $\Delta = \Delta(H) \neq 3$, then again it is easy to prove the result with Corollary 8 or 9. Therefore, suppose that $\Delta = 3$. The case $\delta(H) = 3$ is handled by Corollary 12 and the two remaining situations will be proved with Corollary 8.

Let $V(H) = \{1, \ldots, 6\}$ and suppose vertex 1 has degree two in H and is adjacent to 2 and 3. If (say) 4 is not adjacent to $\{2, 3\}$, then without loss of generality $45 \in E(H)$ and $B = \{1, 4\}$. Therefore each vertex of $\{4, 5, 6\}$ is adjacent to $\{2, 3\}$ and without loss of generality $\{24, 25, 36\} \subseteq E(H)$ and $23 \notin E(H)$. If neither 34 nor 35 is in $E(H)$, then $B = \{2, 3\}$. Hence assume $34 \in E(H)$ and deduce that $35 \notin E(H)$. We can set $B = \{3, 5\}$

unless both 45, 56 are in $E(H)$. In the latter case H is completely specified and $B = \{1, 5\}$.

Hence we may assume that no vertex has degree two. Suppose that vertex 1 has 2 as its only neighbour. The case of disconnected H is easily handled, hence assume that 2 has degree three and that $\{23, 24\} \subseteq E(H)$. To avoid $B = \{1, 6\}$ (resp. $B = \{1, 5\}$) vertex 6 (resp. 5) has degree one. In all cases B is easily found and the proof is complete. □

Our final result establishes the inequality for claw–free graphs G (i.e. graphs with no subgraph isomorphic to $K_{1,3}$). We require a preliminary result of Favaron [4].

Lemma 14. *If S is an irredundant set of a claw–free graph G, then each component of $G[S]$ is a complete subgraph.*

Theorem 15. *If G is a claw–free graph, then $\theta(G) \leq ir(G)$.*

Proof: Let S be an ir–set of G. If G has no induced subgraph isomorphic to $K_{1,3}$, then each induced subgraph H of G is $K_{1,3}$–free as well and hence (by the proof of Theorem 5) we may assume that $G[S]$ is isolate–free. By Lemma 14, each component of $G[S]$ is complete and the set B which contains exactly one vertex from each of these components, satisfies the hypothesis of Corollary 8. □

Note added in proof: The conjecture that $\theta(G) \leq ir(G)$ for all graphs has recently been disproved by O. Favaron and J. Puech (manuscript), who found a counterexample containing nearly 2 million vertices.

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