A characterisation of graphs with minimum degree 2 and domination number exceeding a third their size

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set if every vertex not in S is adjacent to a vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Sanchis [8] showed that a connected graph G of size q and minimum degree at least 2 has domination number at most $(q+2)/3$. In this paper, connected graphs G of size q with minimum degree at least 2 satisfying $\gamma(G) > q/3$ are characterised.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday

1 Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighbourhood of v is $N(v) =$

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$\{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $\langle S \rangle$. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$). A cycle of length n is an *n-cycle.* A graph of order n that is a path or a cycle is denoted by P_n or C_n, respectively.

A set $S \subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. (That is, $N[S] = V$.) The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [1] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi and Slater [2, 3].

The decision problem to determine the domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the domination number of a graph. Various authors have investigated upper bounds on the domination number of a connected graph in terms of the minimum degree and order of the graph. The earliest such result is due to Ore [5].

Theorem 1 (Ore) *If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma(G) \leq n/2$.*

A large family of graphs attaining the bound in Theorem 1 can be established using the following transformation of a graph. The *corona* of a graph G, denoted by G^+, is the graph obtained from G by adding an adjacent end-vertex to each vertex of G. Payan and Xuong [6] characterised those graphs with no isolated vertex and with domination number exactly half their order.

Theorem 2 (Payan, Xuong) *If G is a connected graph of order n, then $\gamma(G) = n/2$ if and only if $G \cong C_4$ or $G \cong H^+$ for some connected graph H.*

McCraig and Shepherd [4] investigated upper bounds on the domination number of a connected graph with minimum degree at least 2.

Theorem 3 (McCraig, Shepherd) *If G is a connected graph of order n with $\delta(G) \geq 2$, and if G is not one of seven exceptional graphs (one of order 4 and six of order 7), then $\gamma(G) \leq 2n/5$.*

McCraig and Shepherd [4] also characterised those graphs G of order n which are edge-minimal with respect to satisfying G connected, $\delta(G) \geq 2$, and $\gamma(G) \geq 2n/5$. Reed [7] investigated upper bounds on the domination number of a connected graph with minimum degree at least 3.

Theorem 4 (Reed) *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.*

Sanchis [8] investigated upper bounds on the domination number of a connected graph in terms of the minimum degree and size of the graph.

Theorem 5 (Sanchis) *If G is a connected graph of size q with $\delta(G) \geq 2$, then $\gamma(G) \leq (q+2)/3$ with equality if and only if G is a cycle of length n where $n \equiv 1 \pmod 3$.*

In this paper, we characterise connected graphs G of size q with minimum degree at least 2 satisfying $\gamma(G) > q/3$.

2 Main result

We will refer to a graph G as an $\frac{q}{3}$**-graph** if G is a connected graph of size q with minimum degree at least 2 satisfying $\gamma(G) > q/3$. We shall characterise $\frac{q}{3}$-graphs. For this purpose, we introduce a family $\mathcal{G}$ of $\frac{q}{3}$-graphs and a collection $\mathcal{H}$ of five $\frac{q}{3}$-graphs.

We define a *unit* to be either a 4-cycle with a path of length 1 attached to a vertex of the 4-cycle, which we call a *type-1 unit*, or a 5-cycle, which we call a *type-2 unit*. If v is a vertex of a graph, then by *attaching a type-1 unit* to v we mean adding a 4-cycle and joining v with an edge to one vertex of the cycle (see Figure 1(a)). By *attaching a type-2 unit* to v we mean adding a (disjoint) 5-cycle to the graph and identifying one of its vertices with v (see Figure 1(b)). We now introduce a family $\mathcal{G}$ of $\frac{q}{3}$-graphs.

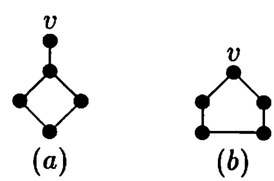

Figure 1: (a) type-1 unit and (b) type-2 unit.

Let F be a forest that consists of $k \geq 1$ nontrivial components $F_1, \dots, F_k$. For $i \in \{1, \dots, k\}$, we let S_i be a distinguished set of vertices of F_i that satisfies the following two conditions: (i) every end-vertex of F_i belongs to S_i (but not every vertex of S_i is necessarily an end-vertex of F_i); (ii) if $V(F_i) \neq S_i$, then $F_i - S_i$ is a forest whose vertex set can be partitioned into $\ell \geq 1$ sets each of which induce a path P_3 on three vertices, the central vertex of which has degree 2 in F_i. We refer to the partition in (ii) as the *path-partition* of $V(F_i) - S_i$. Let $S_F = \cup_{i=1}^{k} S_i$.

If $k \geq 2$, then we construct a tree T from the forest F by adding $k-1$ edges $e_1, \ldots, e_{k-1}$ to F where both ends of e_i belong to S_F for $i = 1, \ldots, k-1$. Let $E^* = \{e_1, \ldots, e_{k-1}\}$ and let S_F^* denote the vertices incident with some edge of E^*. (Thus, $S_F^* \subset S_F$.) Let $S_F' = S_F$ if $k = 1$ and let $S_F' = S_F - S_F^*$ if $k \geq 2$. If $k = 1$, then we let $T = F$.

We now construct a graph G from T as follows. Notice that each component of the subgraph $\langle E^* \rangle$ induced by E^* is a nontrivial tree. Each component of $\langle E^* \rangle$ of order ℓ we replace with a $(3\ell-1)$-cycle in which the ℓ vertices in the component are the ℓ vertices on the $(3\ell-1)$-cycle in positions $1, 3, 6, \ldots, 3(\ell-1)$. (In particular, each component of $\langle E^* \rangle$ that is a path P_2 is replaced with a 5-cycle in which the two vertices of the path are non-adjacent vertices on the cycle.) Finally, we attach a type-1 unit or a type-2 unit to each vertex of S_F'. Let G denote the resulting graph. We refer to the forest F as the **underlying forest** of G and the tree T as the **underlying tree** of G. The collection of all such graphs G we denote by $\mathcal{G}$.

If $F \cong K_2$, for example, then $T = F$ and G is one of the three graphs shown in Figure 2 (where u and v denote the two vertices of F).

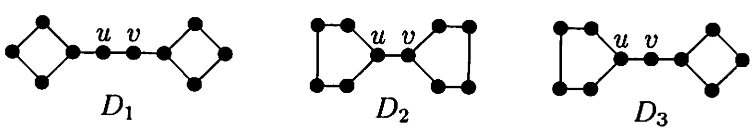

Figure 2: Three graphs in the family $\mathcal{G}$ constructed from $F = K_2$.

As a further example of our construction, consider the graph G in the family $\mathcal{G}$ that is shown in Figure 3.

The underlying forest F of the graph G of Figure 3 is shown in Figure 4 where the vertices of S_F are darkened. In this example, the forest F consists of two components, namely a component F_1 containing the vertex named u and a component F_2 containing the vertex named v. The underlying tree T of G is constructed from F by adding the edge uv. The graph G is constructed from T by replacing

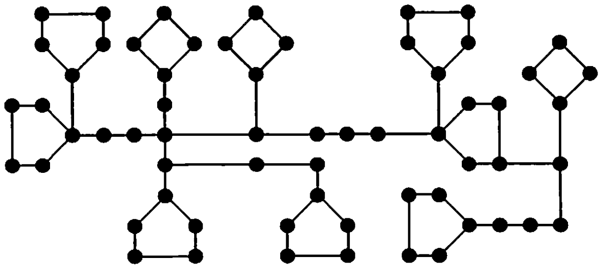

Figure 3: A graph G in the family $\mathcal{G}$.

the edge uv with a 5-cycle in which u and v are non-adjacent vertices on the 5-cycle, and by attaching a type-1 unit or a type-2 unit to each vertex of $S'_F = S_F - \{u, v\}$.

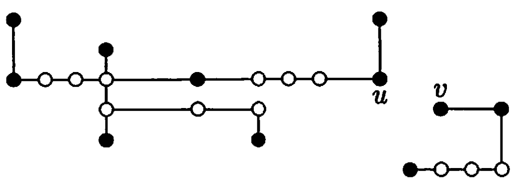

Figure 4: The underlying forest F of the graph G of Figure 3.

The final two examples of our construction are shown in Figure 5 and Figure 6. These examples serve to illustrate two graphs G in the family $\mathcal{G}$ with different underlying trees T but with the same underlying forest F.

Next we define a collection $\mathcal{H}$ of five $\frac{2}{3}$-graphs.

Let G be a nonempty graph. We define an *elementary* 3*-subdivision* of G as a graph obtained from G by subdividing some edge three times. A 3*-subdivision* of G is a graph obtained from G by a succession of elementary 3-subdivisions (including the possibility of none). We denote the family of all 3-subdivisions of G by G^*; that is,

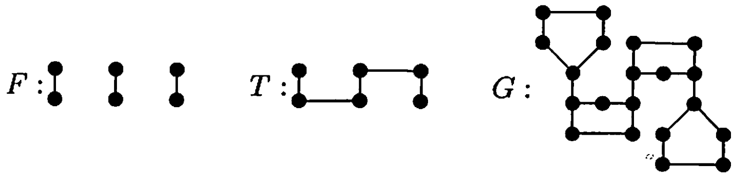

Figure 5: A graph G in $\mathcal{G}$ with underlying tree T and underlying forest F.

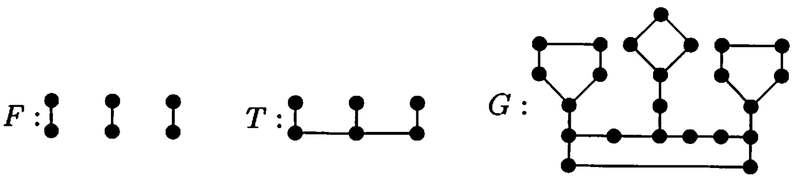

Figure 6: A graph G in $\mathcal{G}$ with underlying tree T and underlying forest F.

$G^* = \{H \mid H \text{ is a 3-subdivision of } G\}$. Let

$$\mathcal{G}^* = \bigcup_{G \in \mathcal{G}} G^* \text{ and } \mathcal{H}^* = \bigcup_{H \in \mathcal{H}} H^*.$$

For $i = 0, 1, 2$, let $\mathcal{C}_i = \{C_n \mid n \equiv i \,(mod\, 3)\}$. Notice that $H_1^* = \mathcal{C}_1$ and $H_2^* = \mathcal{C}_2$. We shall prove:

Theorem 6 *If G is a $\frac{2}{3}$-graph, then $G \in \mathcal{G}^* \cup \mathcal{H}^*$.*

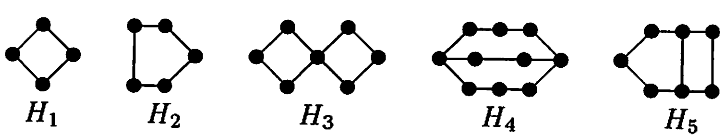

Figure 7: Graphs in the collection $\mathcal{H}$.

As a consequence of Theorem 6, we have the following result.

Theorem 7 *If G is a connected graph of size q with minimum degree at least 2, then $\gamma(G) \le q/3$ unless either $G \in \mathcal{C}_1$, in which case $\gamma(G) = (q+2)/3$, or $G \in \mathcal{G}^* \cup (\mathcal{H}^* - \mathcal{C}_1)$, in which case $\gamma(G) = (q+1)/3$.*

3 Preliminary Results

The following lemma will prove to be useful.

Lemma 8 *Let G be a connected nontrivial graph and let G' be obtained from G by an elementary 3-subdivision. Then $\gamma(G') = \gamma(G) + 1$.*

Proof. Suppose $e = uv$ is the edge of G that is subdivided three times to produce G'. Let u, a, b, c, v be the resulting u-v path of length 4. Let D be a γ-set of G. If $u, v \in D$, then $D \cup \{b\}$ is a dominating set of G'. If $u \in D$ and $v \notin D$ (say), then $D \cup \{c\}$ is a dominating set of G'. If $u, v \notin D$, then $D \cup \{b\}$ is a dominating set of G'. In any event, D can be extended to a dominating set of G' by adding one vertex. Hence $\gamma(G') \le \gamma(G) + 1$. Now let D' be a γ-set of G'. If $b \in D'$, then we may assume $a, c \notin D'$ (if $a \in D'$, then we replace a with u in D'), whence $D' - \{b\}$ is a dominating set of G. If $b \notin D'$, then we may assume $a \in D'$ and $v \in D'$, whence $D' - \{a\}$ is a dominating set of G. In any event, we can construct a dominating set of G of cardinality $|D'| - 1$, and so $\gamma(G') \ge \gamma(G) + 1$. Consequently, $\gamma(G') = \gamma(G) + 1$. □

An immediate corollary of Lemma 8 now follows.

Corollary 9 *Let G be a connected nontrivial graph and let G' be obtained from G by an elementary 3-subdivision. If G has size q and G' has size q', then G is a $\frac{q}{3}$-graph if and only if G' is a $\frac{q'}{3}$-graph.*

The domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is easy to compute.

Fact 1 *For $n \geq 3$, $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$.*

For $n_1, n_2 \geq 3$ and $k \geq 1$, we define a *dumb-bell* $D(n_1, n_2, k)$ to be the graph obtained from $C_{n_1} \cup C_{n_2}$ by joining a vertex of C_{n_1} to a vertex of C_{n_2} and subdividing this edge $k-1$ times. Thus the dumb-bell $D(n_1, n_2, k)$ has order $n = n_1 + n_2 + k - 1$ and size $q = n + 1$. The following result is straightforward to verify.

Fact 2 *Suppose $G \cong D(n_1, n_2, k)$ is a dumb-bell of size q where $3 \leq n_i \leq 5$ and $1 \leq k \leq 3$. Then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \in \{D_1, D_2, D_3\}$ where D_1, D_2, D_3 are the three graphs of Figure 2.*

An immediate consequence of Lemma 8 and Fact 2 now follows.

Fact 3 *If G is a dumb-bell of size q, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \in D_1^* \cup D_2^* \cup D_3^*$ where D_1, D_2, D_3 are the three graphs of Figure 2.*

A *daisy* with $m \geq 2$ *petals* is a connected graph with one vertex of degree $2m$ and all other vertices of degree 2. That is, a daisy with $m \geq 2$ petals is constructed from m disjoint cycles by identifying a set of m vertices, one from each cycle, into one vertex.

Fact 4 *If G is a daisy of size q that contains no cycle of length greater than 5, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \cong H_3$.*

Proof. Let v denote the vertex of degree $2m$ in G, and let F_1, $F_2, \cdots, F_m$ denote the m cycles passing through v, where $F_i \cong C_{n_i+1}$ for $i = 1, 2, \ldots, m$. By assumption, $2 \leq n_i \leq 4$ for all $i = 1, 2, \ldots, m$. Let $I = \{i \mid 1 \leq i \leq m,\ n_i \geq 3\}$. Then G has order $n = 1 + \sum_{i=1}^{m} n_i \geq$

$1+\sum_{i\in I} n_i \geq 1+3|I|$ and size $q = \sum_{i=1}^{m}(n_i+1) = n+m-1 \geq 3|I|+m$. Hence $(q+1)/3 \geq (3|I|+m+1)/3 \geq |I|+1 = |I|+|\{v\}| \geq \gamma(G)$. Furthermore, if $(q+1)/3 = \gamma(G)$, then we must have $m = 2$ and $n_i = 3$ for all $i = 1, 2, \ldots, m$, i.e., $G \cong H_3$. Clearly, if $G \cong H_3$, then $\gamma(G) = (q+1)/3$. □

An immediate consequence of Lemma 8 and Fact 4 now follows.

Fact 5 *If G is a daisy of size q, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \in H_3^*$.*

We define a *pumpkin* to be a graph of maximum degree at least 3 obtained from a forest F every component of which is a path (possibly trivial) by adding two new (possibly adjacent) vertices u and v (of degrees at least 3), joining u and v to every isolated vertex of F, and for each nontrivial path in F joining u to one end-vertex and v to the other end-vertex on the path. We call F the *underlying forest* of the pumpkin.

Fact 6 *If G is a pumpkin of size q and if every path in the underlying forest of G has order at most 3, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \in \{H_4, H_5\}$.*

Proof. Let u and v be the two vertices in G of degrees at least 3. For $i = 1, 2, 3, 4$, let n_i denote the number of u-v paths of length i. Suppose firstly that u and v are adjacent vertices, i.e., $n_1 = 1$. Since u (v) has degree at least 3, $n_2 + n_3 + n_4 \geq 2$. Suppose $n_4 = 0$. If $n_3 = 0$, then $\gamma(G) = 1 < 5/3 \leq q/3$. On the other hand, if $n_3 \geq 1$, then $\gamma(G) = |\{u, v\}| = 2 \leq q/3$. Suppose then that $n_4 \geq 1$. If $n_2 + n_3 = 0$, then $n_4 \geq 2$ and $\gamma(G) = n_4 + 1 \leq (4n_4+1)/3 \leq q/3$. Hence $n_2+n_3 \geq 1$. If $n_3 = 0$, then $\gamma(G) = n_4+1 < (4n_4+3)/3 \leq q/3$. On the other hand, if $n_3 \geq 1$, then $\gamma(G) = |\{u, v\}| + n_4 = 2 + n_4 \leq (4n_4+5)/3 \leq (q+1)/3$ with equality if and only if $n_2 = 0$ and $n_3 = n_4 = 1$, i.e., if and only if $G \cong H_5$. Hence if $n_1 = 1$, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \cong H_5$.

Suppose, next, that u and v are not adjacent, i.e., $n_1 = 0$. Then $n_2 + n_3 + n_4 \geq 3$. If $n_4 = 0$, then $\gamma(G) = 2 \leq 6/3 \leq q/3$. Suppose

$n_4 \geq 1$. If $n_2+n_3 = 0$, then $n_4 \geq 3$ and $\gamma(G) = n_4+1 \leq 4n_4/3 = q/3$. Hence we may assume $n_2 + n_3 \geq 1$. If $n_3 = 0$, then $\gamma(G) = n_4 + 1 \leq (4n_4 + 2)/3 \leq q/3$. So we may assume $n_3 \geq 1$. Then $\gamma(G) = |\{u, v\}| + n_4 = 2 + n_4$. If $n_4 = 1$, then $q \geq 9$, and so $\gamma(G) = 3 \leq q/3$. On the other hand, if $n_4 \geq 2$, then $q + 1 \geq 4n_4 + 4 \geq 3n_4 + 6$ with equality if and only if $n_2 = 0$, $n_3 = 1$, and $n_4 = 2$, i.e., if and only if $G \cong H_4$. Hence if $n_4 \geq 2$, then $\gamma(G) = 2 + n_4 \leq (q + 1)/3$ with equality if and only if $G \cong H_4$. Thus if $n_1 = 0$, then $\gamma(G) \leq (q+1)/3$ with equality if and only if $G \cong H_4$. □

An immediate consequence of Lemma 8 and Fact 6 now follows.

Fact 7 *If G is a pumpkin of size q, then $\gamma(G) \leq (q + 1)/3$ with equality if and only if $G \in H_4^* \cup H_5^*$.*

The following two observations about graphs in the families $\mathcal{G} \cup \mathcal{H}$ will be useful.

Observation 1 *Let $G \in \mathcal{G} \cup \mathcal{H}$ have size q, and let v be a vertex of G. Then*

(a) *G is a connected graph and $\delta(G) = 2$,*
(b) *$\gamma(G) = (q + 2)/3$ if $G \cong H_1$ and $\gamma(G) = (q + 1)/3$ otherwise,*
(c) *there is γ-set of G that contains v.*

In particular, notice that each graph in $\mathcal{G} \cup \mathcal{H}$ is a $\frac{q}{3}$-graph.

Observation 2 *Suppose G is obtained from the disjoint union $G_1 \cup G_2$ of two nontrivial connected graphs G_1 and G_2 by joining a vertex v_1 of G_1 to a vertex v_2 of G_2. Suppose v_1 belongs to a γ-set of G_1.*

(a) If $G_2 \in \mathcal{H} - \{H_2\}$, then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2) - 1$.

(b) If $G_2 \in \mathcal{G}$, and either v_2 belongs to a $(3\ell - 1)$-cycle $(\ell \geq 2)$ of G_2 and is adjacent to a vertex of degree at least 3 in G_2 or v_2 belongs to a 4-cycle of a type-1 unit of G_2 or v_2 is the central vertex of a P_3 in the path-partition of $F - S_F$, where F is the underlying forest of G_2, then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2) - 1$.

4 Proof of Theorem 6

We proceed by induction on the size $q \geq 3$ of a $\frac{q}{3}$-graph. Suppose $G = (V, E)$ is a $\frac{q}{3}$-graph of order n. If $q = n$, then $G \cong C_n$, and so, by Fact 1, either $G \in \mathcal{C}_1$, in which case $\gamma(G) = (q+2)/3$ or $G \in \mathcal{C}_2$, in which case $\gamma(G) = (q+1)/3$. Hence if $q = n$, then the result follows. In particular, the base cases when $q = 3$ or $q = 4$ are true. So in what follows we assume that $q > n$. Assume the result is true for all connected graphs G' of size q', where $q' < q$, that satisfy $\delta(G') \geq 2$ and $\gamma(G') > q'/3$. Hence we have the following result.

Lemma 10 *If G' is a connected graph of size $q' < q$ with $\delta(G') \geq 2$, then either $G' \in \mathcal{C}_1$, in which case $\gamma(G') = (q' + 2)/3$, or $G' \in \mathcal{G}^* \cup (\mathcal{H}^* - \mathcal{C}_1)$, in which case $\gamma(G') = (q' + 1)/3$, or $\gamma(G') \leq q'/3$.*

By assumption G is not a cycle. Thus G contains at least one vertex of degree at least 3. Let $S = \{v \in V \mid deg\, v \geq 3\}$. If $|S| = 1$, then G is a daisy, and so, by Fact 5, $G \in H_3^*$. So we may assume that $|S| \geq 2$. For each $v \in S$, we define the *2-graph of* v to be the component of $G - (S - \{v\})$ that contains v. The 2-graph of v consists of edge-disjoint cycles through v, which we call *2-graph cycles*, and paths emanating from v, which we call *2-graph paths.*

Lemma 11 *If G contains a path on five vertices each internal vertex of which has degree 2 in G, then $G \in \mathcal{G}^* \cup \mathcal{H}^*$.*

Proof. Let u and v be the two end-vertices of a path on five vertices each internal vertex of which has degree 2. Let G' be the graph of size $q' = q - 3$ obtained from G by removing the three internal vertices of this path and adding the edge uv. By Lemma 8, $\gamma(G') = \gamma(G) - 1 \geq (q' + 1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}^* \cup \mathcal{H}^*$. However, G is obtainable from G' by an elementary 3-subdivision, and so G also belongs to $\mathcal{G}^* \cup \mathcal{H}^*$. □

By Lemma 11, we may assume that G contains no path on five vertices each internal vertex of which has degree 2 in G, for otherwise $G \in \mathcal{G}^* \cup \mathcal{H}^*$. Hence we may assume that

every 2-graph path in G has length at most 2, and every 2-graph cycle in G has length at most 5.

Hence, by Lemma 10 we have the following result.

Lemma 12 *Suppose G' is a connected subgraph of G of size $q' < q$ with $\delta(G') \geq 2$. If the degrees of all but one of the vertices in G' are the same as their degrees in G, then either $G' \cong H_1$, in which case $\gamma(G') = (q'+2)/3$, or $G' \in \mathcal{G} \cup (\mathcal{H} - H_1)$, in which case $\gamma(G') = (q'+1)/3$, or $\gamma(G') \leq q'/3$.*

The following lemma will prove to be useful.

Lemma 13 *Suppose G is obtained from two (disjoint) graphs G_1 and G_2 by identifying a vertex of G_1 and a vertex of G_2 into one vertex v where v has degree at least 1 in G_1 and degree at least 2 in G_2. Suppose G_1 is a type-1 unit or a type-2 unit or can be obtained from a type-2 unit by attaching a path of length 3 to a vertex of the 5-cycle. Then $G \in \mathcal{G}$.*

Proof. Since G is connected, G_1 and G_2 are both connected. Furthermore, since G has minimum degree at least 2, every vertex of G_1 different from v has degree at least 2 in G_1 while every vertex of G_2 has degree at least 2 in G_2. Suppose G_i has size q_i for $i = 1, 2$. Notice that $\gamma(G_1) = (q_1+1)/3$ and v belongs to a γ-set of G_1. Hence, if G_2 is a cycle, then either $G_2 \cong C_3$, in which case $\gamma(G) = \gamma(G_1) = (q-2)/3$, or $G_2 \in \{C_4, C_5\}$, in which case $\gamma(G) = \gamma(G_1) + 1 = (q_1+1)/3 + 1 \leq q/3$. Both cases produce a contradiction. Hence G_2 cannot be a cycle.

Let G_2' be the graph of size q_2' obtained from $G_2 - v$ by adding as few edges as possible between neighbours of v in G_2 until we produce a connected graph with minimum degree at least 2 (possibly, $G_2' = G_2 - v$). Then $q_2' \leq q_2 - 1$, and so $q \geq q_1 + q_2' + 1$.

We show that $G_2' \in \mathcal{G}^*$. If G_2' is a cycle, then, since G_2 is not a cycle, $q_2' \leq q_2 - 2$. In particular, if $G_2' \in H_2^*$, then $\gamma(G) \leq (q_1 +$

$1)/3 + (q_2' + 1)/3 \leq q/3$, a contradiction. If $G_1' \in \mathcal{H}^* - \{H_2^*\}$, then, since v belongs to a γ-set of G_1, it follows from Observation 2(a) that $\gamma(G) \leq (q_1 + 1)/3 + (q_2' + 2)/3 - 1 \leq (q - 1)/3$, a contradiction. If $\gamma(G_2') \leq q_2'/3$, then $\gamma(G) \leq (q_1 + 1)/3 + q_2'/3 \leq q/3$, a contradiction. Hence $G_2' \notin \mathcal{H}^*$ and $\gamma(G_2') > q_2'/3$. Consequently, by Lemma 10, $G_2' \in \mathcal{G}^*$ and $\gamma(G_2') = (q_2' + 1)/3$. Let F be the underlying forest of G_2'.

If $q_2' \leq q_2 - 2$, then $\gamma(G) \leq (q_1 + 1)/3 + (q_2 - 1)/3 = q/3$, a contradiction. Hence $q_2' = q_2 - 1$. This implies that each neighbour of v in G_2 belongs to a different component of $G_2 - v$. Thus each edge of G_2' that is not in G_2 belongs to F. Hence each neighbour of v in G_2 must belong to F. If some neighbour of v in G_2 is the central vertex of a P_3 in the path-partition of $F - S_F$, then, by Observation 2(b), it follows that $\gamma(G) \leq \gamma(G_1) + \gamma(G_2') - 1 = (q - 2)/3$, a contradiction. Hence each neighbour of v in G_2 either belongs to the set S_F or is an end-vertex of a P_3 in the path-partition of $F - S_F$. But then $G \in \mathcal{G}^*$. (If G_1 is a type-1 or a type-2 unit, then the underlying tree of G is obtained from the underlying tree of G_2' by removing edges joining vertices that are neighbours of v in G_2, adding the vertex v and adding the edges joining v to the vertices that are its neighbours in G_2. If G_1 can be obtained from a type-2 unit by attaching a path of length 3 to a vertex x of the 5-cycle, then the underlying tree of G is as described earlier but with the addition of the v-x path of length 3 which is attached to v. In the latter case, the neighbour of v on the v-x path is a central vertex of a P_3 in the path-partition of $F - S_F$.) However, since every 2-graph path in G has length at most 2 and every 2-graph cycle in G has length at most 5, $G \in \mathcal{G}$. □

Lemma 14 *If S is not an independent set, then $G \in \mathcal{G} \cup \mathcal{H}$.*

Proof. Let $e = uv$ be an edge, where $u, v \in S$. Suppose $G - e$ is a connected graph (of size $q - 1$). Then by the induction hypothesis, $\gamma(G - e) \leq (q + 1)/3$. If $\gamma(G - e) \leq q/3$, then $\gamma(G) \leq \gamma(G - e) \leq q/3$, a contradiction. Hence $\gamma(G - e) = (q + 1)/3$, and so $G - e \in C_1$. Thus G is obtained from a cycle C_n, $n \equiv 1\,(mod\,3)$, by adding the edge e. Hence, by Fact 6, $G \in \{H_4, H_5\}$.

Suppose, next, that e is a bridge of G. Let G_1 and G_2 be the two components of $G - e$, where $u \in V(G_1)$. For $i = 1, 2$, let G_i have order n_i and size q_i. Then $q = q_1 + q_2 + 1$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. If G is a dumb-bell, then $G \cong D(n_1, n_2, 1)$, and so, by Fact 2, $G \cong D_2 \in \mathcal{G}$ (where D_2 is the graph shown in Figure 2). Hence we may assume that G_2 is not a cycle. Thus, by Lemma 10, $\gamma(G_2) \leq (q_2 + 1)/3$.

Suppose $\gamma(G_1) \leq (q_1 + 1)/3$. If $\gamma(G_2) \leq q_2/3$ or $\gamma(G_1) \leq q_1/3$, then $\gamma(G) \leq q/3$, a contradiction. Hence $\gamma(G_i) = (q_i + 1)/3$ for $i = 1, 2$, and so, by Lemma 12, $G_i \in \mathcal{G} \cup \mathcal{H}$. By Observation 1(c), we can choose a γ-set of G_1 to contain u and a γ-set of G_2 to contain v. Hence, if $G_2 \in \mathcal{H}$, then, since $G_2 \not\cong H_2$, Observation 2(a) implies that $\gamma(G) \leq (q - 2)/3$, a contradiction. Thus, $G_2 \in \mathcal{G}$. Furthermore, by Observation 2(b), v belongs to a type-2 unit with both its neighbours having degree 2 in G_2 or v is a vertex in the underlying forest F of G_2 and either belongs to the set S_F or is an end-vertex of a P_3 in the path-partition of $F - S_F$, for otherwise $\gamma(G) \leq (q - 2)/3$. If $G_1 \in \mathcal{H} - \{H_2\}$, then $\gamma(G) \leq (q - 2)/3$, a contradiction. Hence $G_1 \in \mathcal{G} \cup \{H_2\}$. If $G_1 \cong H_2$, then $G \in \mathcal{G}$. On the other hand, if $G_1 \in \mathcal{G}$, then by Observation 2(b), u belongs to a type-2 unit with both its neighbours having degree 2 in G_1 or u is a vertex in the underlying forest F of G_1 and either belongs to the set S_F or is an end-vertex of a P_3 in the path-partition of $F - S_F$, for otherwise $\gamma(G) \leq (q-2)/3$. It follows then that $G \in \mathcal{G}$. Hence we may assume that $\gamma(G_1) = (q_1 + 2)/3$, for otherwise $G \in \mathcal{G}$. Thus $G_1 \cong H_1$, and so $q_1 = n_1 = 4$. Let G_1' be the graph obtained from G_1 by adding v and the edge e. Then G_1' is a type-1 unit. Applying Lemma 13 (with "G_1" replaced by "G_1'"), $G \in \mathcal{G}$. □

By Lemma 14, we may assume that S **is an independent set**, for otherwise $G \in \mathcal{G} \cup \mathcal{H}$.

Lemma 15 *If G contains a 2-graph cycle, then $G \in \mathcal{G}$.*

Proof. Let $v \in S$ and suppose that C_v is a 2-graph cycle of v of length q_1. By Lemma 11, we may assume that $3 \leq q_1 \leq 5$.

Case 1: *$deg\, v \geq 4$.*

Let $G_2 = G - (V(C_v) - \{v\})$. Then G_2 is a connected graph with minimum degree at least 2 and of size $q_2 = q - q_1$. Since $|S| \geq 2$, G_2 is not a cycle. Hence, by Lemma 12, $\gamma(G_2) \leq (q_2 + 1)/3 = (q - q_1 + 1)/3$. Suppose $q_1 = 3$. Then $q = q_2 + 3$. If $G_2 \in \mathcal{G} \cup \mathcal{H}$, then, by Observation 1(c), there is a γ-set of G_2 containing v, whence $\gamma(G) \leq \gamma(G_2) \leq (q_2 + 1)/3 = (q - 2)/3$, a contradiction. Hence $G_2 \notin \mathcal{G} \cup \mathcal{H}$, i.e., $\gamma(G_2) \leq q_2/3$ by Lemma 12. However, any γ-set of G_2 can be extended to a dominating set of G by adding one vertex, and so $\gamma(G) \leq 1 + q_2/3 = q/3$, once again producing a contradiction. Hence $q_1 \neq 3$. Suppose $q_1 = 4$. Then $q = q_2 + 4$. Any γ-set of G_2 can be extended to a dominating set of G by adding one vertex, and so $\gamma(G) \leq 1 + (q_2 + 1)/3 = q/3$, a contradiction. Hence $q_1 \neq 4$. Thus, $q_1 = 5$, i.e., C_v is a type-2 unit. Applying Lemma 13 (with $G_1 = C_v$), $G \in \mathcal{G}$.

Case 2: *$deg\, v = 3$.*

Let $v, v_1, \ldots, v_k, w$ be the path from v to the vertex w of $S - \{v\}$ every internal vertex of which belongs to $V - S$. Since S is independent, $k \geq 1$. Since every 2-graph path of G has length at most 2, $k \leq 2$. Let F_1 and F_2 be the two components of $G - v_k w$, where $w \in V(F_2)$. The graph F_2 is connected of size $q_2 = q - q_1 - k - 1$ with minimum degree at least 2. If F_2 is a cycle, then G is a dumbbell, and so, it follows from Fact 2 that $G \in \{D_1, D_3\} \subset \mathcal{G}$ (where D_1 and D_3 are the graphs shown in Figure 2). Hence we may assume that F_2 is not a cycle. Thus, by Lemma 12, $\gamma(F_2) \leq (q_2 + 1)/3 = (q - q_1 - k)/3$.

If $\gamma(F_1) \leq (q_1 + k)/3$, then $\gamma(G) \leq \gamma(F_1) + \gamma(F_2) \leq q/3$, a contradiction. Hence $(q_1, k) \notin \{(3,1), (4,2), (5,1)\}$. Suppose $q_1 = 3$ and $k = 2$. If $\gamma(F_2) \leq q_2/3$, then $\gamma(G) \leq 2 + q_2/3 = q/3$, a contradiction. Hence, by Lemma 12, $F_2 \in \mathcal{G} \cup (\mathcal{H} - \{H_1, H_2\})$. Thus, by Observation 1(c), w belongs to a γ-set of F_2. It follows that $\gamma(G) \leq 1 + \gamma(F_2) = (q - 2)/3$, a contradiction. Hence $(q_1, k) \neq (3, 2)$. Thus $(q_1, k) \in \{(4,1), (5,2)\}$.

Suppose $q_1 = 4$ and $k = 1$. Notice that v_k belongs to a γ-set of F_1. If $\gamma(F_2) \leq q_2/3$, then $\gamma(G) \leq q/3$, a contradiction. If $F_2 \in \mathcal{H} - \{H_1, H_2\}$, then, by Observation 2(a), $\gamma(G) \leq 1 + (q_2 + 1)/3 =$

$(q-2)/3$, a contradiction. Hence $F_2 \in \mathcal{G}$. By Observation 2(b), w belongs to a type-2 unit in F_2 with both its neighbours having degree 2 in F_2 or w is a vertex in the underlying forest F of F_2 and either belongs to the set S_F or is an end-vertex of a P_3 in the path-partition of $F - S_F$, for otherwise $\gamma(G) \leq (q-2)/3$. It follows that $G \in \mathcal{G}$.

Suppose $q_1 = 5$ and $k = 2$. Let G_1 be obtained from F_1 by adding w and the edge $v_k w$ and let $G_2 = F_2$. Then G_1 can be obtained from a type-2 unit by attaching a path of length 3 to a vertex of the 5-cycle. Hence, applying Lemma 13, $G \in \mathcal{G}$. □

By Lemma 15, we may assume that

there is no 2-graph cycle

in G. Hence, if $|S| = 2$, then G is a pumpkin, and so, by Fact 6, $G \in H_4 \cup H_5$ (for otherwise $\gamma(G) \leq q/3$). Hence we may assume that $|S| \geq 3$.

Lemma 16 *If $v \in S$ and v, a, b is a 2-graph path of v of length 2, then $G - \{a, b\}$ is disconnected.*

Proof. Let $w \in S$ be the neighbour of b different from a. Since S is independent, vw is not an edge. Let $G' = G - \{a, b\}$. Then G' has size $q' = q - 3$ and has minimum degree at least 2. Suppose G' is connected. Since $|S| \geq 3$, G' is not a cycle, and so, by Lemma 10, $\gamma(G') \leq (q'+1)/3$. Any γ-set of G' can be extended to a dominating set of G by adding either a or b. Hence, if $\gamma(G') \leq q'/3$, then $\gamma(G) \leq q'/3 + 1 = q/3$, a contradiction. Thus $\gamma(G') = (q'+1)/3$, and so, by Lemma 10, $G' \in \mathcal{G} \cup (\mathcal{H} - \{H_1, H_2\})$. If $G' \in \mathcal{G}$, then, since G has no 2-graph cycles, G' has exactly two 2-graph cycles, one containing v and the other containing w. However, we can then choose a γ-set of G' to contain both v and w. Hence $\gamma(G) \leq \gamma(G') = (q-2)/3$, a contradiction. On the other hand, if $G' \in \{H_3, H_4, H_5\}$, then we can choose a γ-set of G' to contain any two nonadjacent vertices of G'. In particular, we can choose a γ-set of G' to contain both v and w, once again producing a contradiction. Hence G' must have been disconnected. □

An immediate consequence of Lemma 16 now follows.

Lemma 17 *There is no 5-cycle or 6-cycle in G containing exactly two vertices of S.*

Proof. Suppose G contains a 5-cycle $C\colon v, a, b, w, c, v$ containing exactly two vertices v and w of S. Then C contains the 2-graph path v, a, b of length 2. Since G is connected, so too is $G - \{a, b\}$, contradicting the result of Lemma 16. Hence, there is no 5-cycle in G containing exactly two vertices of S. Similarly, there is no 6-cycle in G containing exactly two vertices of S. □

Lemma 18 *There is no 4-cycle in G containing exactly two vertices of S.*

Proof. Suppose G contains a 4-cycle $C\colon v, a, w, b, v$ containing exactly two vertices v and w of S. Then C contains two 2-graph paths of length 1. Let G' be obtained from $G - \{a, b\}$ by adding the edge vw. Then G' is a connected graph of minimum degree at least 2 with size $q' = q - 3$. Since $|S| \geq 3$, G' is not a cycle, and so, by Lemma 10, $\gamma(G') \leq (q' + 1)/3$.

Suppose $\gamma(G') \leq q'/3$. Let D' be a γ-set of G'. If $v, w \in D'$, then D' is a dominating set of G, whence $\gamma(G) \leq (q-3)/3$, a contradiction. Hence v or w, say v, does not belong to D', whence $D' \cup \{v\}$ is a dominating set of G and so $\gamma(G) \leq q/3$, a contradiction. Hence $\gamma(G') = (q' + 1)/3$ and so, by Lemma 10, $G' \in \mathcal{G} \cup (\mathcal{H} - \{H_1, H_2\})$.

If $G' \in \mathcal{G}$, then G' has at least two 2-graph cycles, at least one of which does not contain the edge vw. But then G has at least one 2-graph cycle, producing a contradiction. On the other hand, if $G' \in \{H_3, H_4, H_5\}$, then we can choose a γ-set of G' to contain any two nonadjacent vertices of G'. In particular, we can choose γ-set of G' to contain v and a neighbour of w different from v. Hence $\gamma(G) \leq \gamma(G') = (q-2)/3$, a contradiction. Hence, there is no 4-cycle in G containing exactly two vertices of S. □

Among all vertices in S, let v be chosen so that $G - v$ contains a component of maximum order. Let S' denote the subset of vertices of $S - \{v\}$ that are adjacent to a vertex on some 2-graph path of v. By Lemma 17 and Lemma 18, the graph G' of order q' obtained from G by removing v and all vertices on a 2-graph path of v has minimum degree at least 2.

Lemma 19 *The graph G' is connected.*

Proof. Suppose G' is disconnected. Let w be a vertex of S' that belongs to a component of G' of minimum order. Then the component of $G - w$ that contains v has order exceeding that of any component of $G - v$. This contradicts our choice of v. □

By Lemma 10, $\gamma(G') \leq (q' + 2)/3$. Let D' be a γ-set of G'. Since G' is connected, Lemma 16 implies that v has no 2-graph path of length 2. Hence every 2-graph path of v has length 1. Hence D' can be extended to a dominating set of G by adding v. Thus, $\gamma(G) \leq (q' + 2)/3 + 1 \leq (q - 1)/3$, a contradiction. This completes the proof of Theorem 7. □

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