MAD Partitioning for Grid Graphs

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ABSTRACT. For a graph facility or multi-facility location problem, each vertex is typically considered to be the location for one customer or one facility. Typically, the number of facilities is predetermined, and one must optimally locate these facilities so as to minimize some function of the distances between customers and facilities (and, perhaps, of the distances among the facilities). For example, p facility locations (such as, for hospitals or fire stations) might be chosen so as to minimize the maximum or the average distance from a customer to the nearest facility. The problem investigated in this paper considers all of the facilities to be distinct, and we seek to minimize the average customer-to-facility distance, primarily for grid graphs.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

1. INTRODUCTION

Within the realm of graph theory, one of the things of particular interest to Hedet is the family of grid graphs. Indeed, his fascination with grids once resulted in giving serious consideration to the publication of a journal devoted exclusively to problems related to grid graphs. Obviously, to be included would be papers concerned with his favorite chessboard problems (queens-, rooks-, bishops-, knights-, and kings-domination), as well as problems concerned with facility location, independence, colorings, packings, tilings, communications, percolation, etc.

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a)

	X		X		X
X		X		X	
	X		X		X
X		X		X	
	X		X		X

b)

		X	X		X
X	X		X		
X	X			X	
		X		X	X
	X		X	X	

c)

X	X	X			
X	X	X			
X			X		X
			X	X	X
			X	X	X

Figure 1. Three 5-by-6 patterns.

a)

	X		X		X
X		X		X	
	X		X		X
X		X		X	
	X		X		X
X		X		X	

b)

X	X	X			
X	X	X			
X	X	X			
			X	X	X
			X	X	X
			X	X	X

c)

X			X	X	
X		X	X		
	X	X			X
	X			X	X
	X		X	X	
X		X			X

Figure 2. Three 6-by-6 patterns.

a)

	X		X		X	
X		X		X		X
	X		X		X	
X		X		X		X
	X		X		X	

b)

X	X	X				
X	X	X	X			
	X		X		X	
			X	X	X	X
				X	X	X

c)

X				X	X	
		X	X		X	X
	X	X	X			
	X		X	X		X
X	X				X	

Figure 3. Three 5-by-7 patterns.

In this paper, we consider a planar facility location problem using the Manhattan metric. Think of each square of an r-by-s chessboard as the possible location for a facility or for a customer, or, equivalently, as the location for one member of a black team or a white team. Note that Figures (1a), (2a) and (3a) give the normal chessboard black/white alternating square patterns. Having located m black team members and rs-m white team members (or m facilities and rs-m customers) on the rs locations, assume we randomly choose a black team player and a white team player. What is the average distance they must travel to get together, assuming a Manhattan metric where one can only move horizontally or vertically, and the distance from square (i,j) to square (h,k) is $|i-h| + |j-k|$? Surprisingly, the answer is the same for all three patterns in each figure, where each X marks the location for a black team member. And if we can choose where to locate the players, how should we do it so as to minimize this average black/white distance? The patterns in Figures 1, 2 and 3 actually produce the minimum possible average black/white distance for arrangements of 15 blacks and 15 whites, 18 blacks and 18 whites, and 17 blacks and 18 whites, respectively.

a)

	57		53		69
57		43		49	
	45		41		57
57		43		49	
	57		53		69

b)

		53	54		68
58	49		45		
55	46			47	
		44		50	59
	58		54	59	

c)

67	56	51			
58	47	42			
55			40		56
			43	48	59
			52	57	68

Figure 4. The b-w distances.

For each black/facility location, add all of the distances from the white/customer locations to it. These distance totals have been calculated for the patterns in Figure 1 and are shown in Figure 4. In all three examples in Figure 4, the sum of all the black-white distances is the same Total Distance TD=799, and because there are 15*15=225 different black:white pairs, the Average Distance is AD=799/225=3.5511. The Minimum Average Distance for the 6-by-6 board with an 18:18 distribution is MAD=1260/324=3.8889, and for the 5-by-7 board with a 17:18 distribution we have MAD=1190/306= 3.8889.

2. FACILITY LOCATION: BACKGROUND

We restrict our attention to unweighted vertex facility location problems for a graph G=(V,E) with vertex set V and edge set E. Thus, each vertex will be the site for one facility or for exactly one customer, and each edge has length one so the distance between two vertices is the minimum number of edges in a path connecting them.

If one is to locate an emergency response facility, such as a hospital or police station, a typical optimization criterion is to minimize the maximum distance (travel time) from the facility to a customer. In 1869 Jordan [4] defined the *eccentricity* e(v) of a vertex v to be the maximum distance from v to any vertex $w \in V(G)$, $e(v) = MAX\{d(v,w): w \in V(G)\}$, and a center vertex is a vertex for which e(v) is minimized. The *center* of G is $C(G) = \{v \in V(G): e(v) \le e(w) \text{ for all } w \in V(G)\}$. For the tree T in Figure 5(a), the eccentricity of each vertex is indicated, and the only center vertex is g with e(g)=6.

For a service facility, one typically wants to select as a facility vertex one for which the average (or, equivalently, the total) distance to the non-facility vertices is minimized. In 1959 Harary [2] defined the *status* s(v) of a vertex v to be the sum of the distances from v to the other vertices, $s(v) = \Sigma\{d(v,w): w \in V(G)\}$. In 1964 Hakimi called s(v) the distance of v, and a median vertex is a vertex for which s(v) is minimized. The *median* of G is $M(G) = \{v \in V(G): s(v) \le s(w) \text{ for all } w \in V(G)\}$. For the tree T in Figure 5(b), the status of each vertex is indicated, and the only median vertex is x with s(x) = 77.

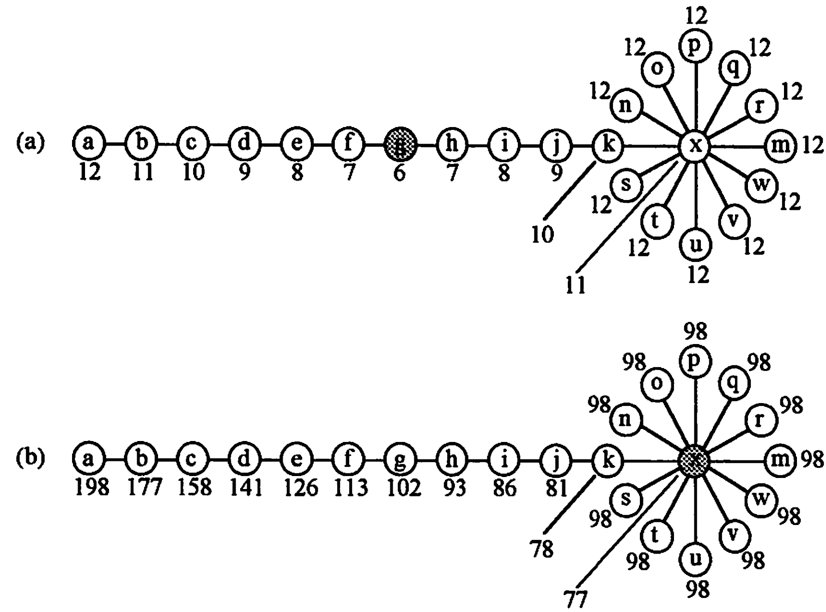


Figure 5. Illustration of 1-Center and 1-Median Problems.

To this point, we have examined facility problems in which only one facility is to be placed. There are numerous problems, however, that require locating multiple facilities; many of these are described by Tansel, Francis and Lowe in [6]. One such problem, introduced by Hakimi [1], is the p-Center problem for similar facilities. The goal in this problem is to minimize the largest distance from a customer location to the nearest of p indistinguishable facility locations. For example, if we are to place p hospitals in a city, assuming the hospitals all serve the same function, we need only minimize the distance from a customer to the closest hospital instead of from a customer to a particular hospital. Figure 6 displays several configurations for a 3-Center problem, with facilities displayed as shaded vertices. The maximum (MAX) distance from each non-facility vertex to the nearest facility is labeled. Since the maximum distance from a non-facility vertex to a facility vertex is lowest in Figure 6(a), it is apparent that the configuration in Figure 6(a) is the best of the three configurations shown in terms of our mimimax criterion. For tree T of Figure 5, each of {d,j}, {d,k}, and {c,j} is a 2-Center solution. For the p-Median problem presented by Hakimi in [1] for graph G with |V(G)|=n vertices, one selects p facility locations and n-p customer locations, the objective being to minimize the SUM of the n-p values obtained by taking the distance of each customer to the nearest facility. Again, see Figure 5. For this minisum criterion, a 2-Median solution for tree T of Figure 5 is {d,x} or {e,x}. In addition, in Figure 6, it is apparent that Figure 6(a) is the best of the three configurations shown in terms of the minisum criterion.

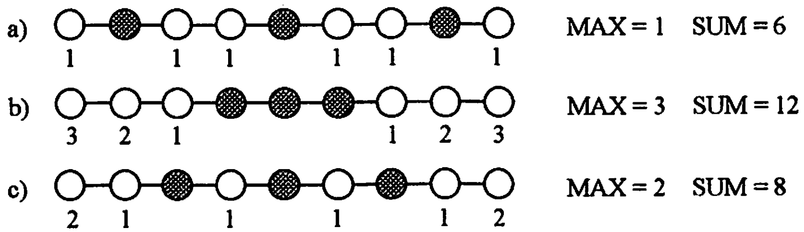


Figure 6. Illustration of 3-Center and 3-Median Problems.

In addition to the problems mentioned above, a problem that requires locating multiple different facilities in a network was introduced in Hulme and Slater [3] and Slater [5]. For this problem, it is assumed that each of the n-p customers must make separate trips to each of the p facilities. For each customer we compute a total distance, which equals the sum of the distances to the p different facilities. Such problems, in which p different facilities are to be placed, are referred to as p-Mean Median problems. Figure 7 displays three configurations for a 3-Mean Median problem, with the shaded vertices denoting facilities. Each non-facility vertex is labeled with its total distance to all of the p different facilities. By adding up the total distance for all of the non-facility vertices, we get an overall total distance which will enable us to compare configurations. Since the total distance for each of Figure 7(a) and 7(b) is 54, while the total distance for Figure 7(c) is 52, it is apparent that the configuration in Figure 7(c) is the best of the three configurations shown in terms of the minimization criterion of total distance. Equivalently, we are minimizing the average distance from a facility to a non-facility. This particular type of facility problem with average/total distance as the minimization criteria will be the principal focus of this paper for the special case where the sites are vertices of a grid.

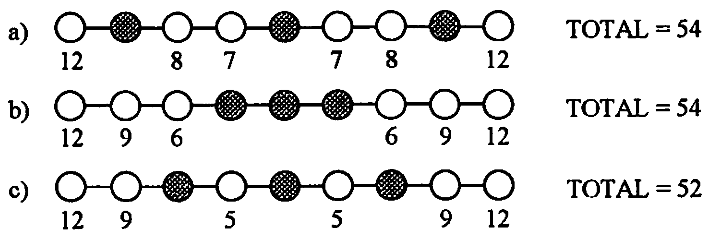
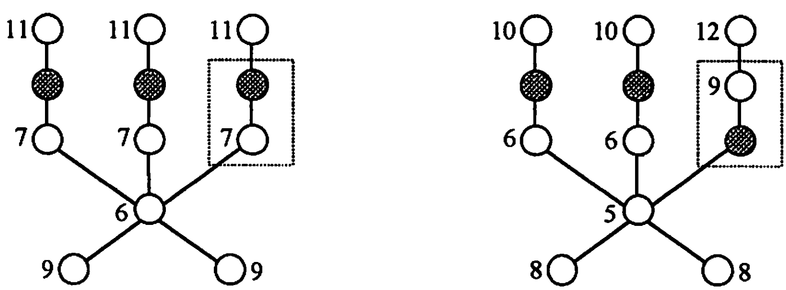


Figure 7. Illustration of 3-Mean Median Problem.

In obtaining minimized solutions to facility location problems, it is often convenient to begin with a starting configuration and to then move facilities about the graph in a manner which better suits the minimization criterion. One way of accomplishing this moving is by switching a pair of adjacent facility and non-facility vertices. If facilities are moved about until no adjacent switch will improve the minimization, we say that a *locally minimal solution* has been reached. There may be numerous locally optimal solutions which produce different values. Nevertheless, there is only one globally optimum solution value. To reach a globally optimum solution from a locally optimal solution, we might have to make adjacent switches that would at first go against the optimization goal and would appear to be counterintuitive. Hence, it is often difficult to discern the globally optimum solution for a graph.

A switching lemma for graphs can be used as a tool to determine when an adjacent switch between a facility and a non-facility vertex will produce a configuration closer to a locally optimal solution. For multi-facility problems where the facilities are different and the minimization criterion is average distance, we would want to make an adjacent switch if the combined benefit from the facility moving closer to some customer vertices and the customer moving closer to some facility vertices exceeds any concomitant increases. This switching lemma for graphs is illustrated in Figure 8. Figure 8(b) was obtained from Figure 8(a) by switching the highlighted facility and non-facility. Each non-facility vertex is labeled with its distance to the three facility vertices. Note that the switched facility and customer remain at distance one, that the facility is one unit closer to seven customers and one unit farther from one customer, and that the switched customer is one unit farther away from the other two facilities. The net change is (-7+1) + (2) = -4, from 78 to 74. Thus, the switch was a favorable one which brought the configuration closer to a locally optimal solution.



a) Total = 78 b) Total = 74

Figure 8. Illustration of Switching Lemma for Graphs.

3. MAD($G_{m,n}$)

Several facility location problems have been studied for restricted classes of graphs. This section will focus on a special class of facility location problems in which the structure upon which the facilities are to be positioned can be modeled by a grid. Lemmas specific to this problem type will be developed and then used to find globally optimum solutions for the special class of equipartitioned grid problems. First, we formally define the p-MAD problem for arbitrary graphs.

For two vertices u and v in V(G), d(u,v) denotes the distance between them. For $S \subseteq V(G)$, let $TD(S)=\Sigma\{d(u,v): u \in S, v \in V(G)-S\}$. The p-mean median problem is to minimize TD(S) for vertex sets of order p, and we let $MTD(G;p) = MIN\{TD(S): S \subseteq V(G), |S|=p\}$. Note that if TD(S)=MTD(G;p) with |S|=p, then S is a p-set for which we have the Minimum Average Distance between a (facility) vertex in S and a (customer) vertex in V(G)-S. This minimum average distance is actually MAD(G;p)=MTD(G;p)/p(n-p) where n=|V(G)| is the order of G.

A grid $G_{m,n}$ refers to a group of m*n blocks that are organized into m rows and n columns. In this model, vertices correspond to the 1x1 square blocks. For our study, we will be interested in the placement of black and white vertices, representing facility and non-facility vertices respectively, on the grid. B will refer to the set of black vertices on the grid, and W will refer to the set of white vertices. When referring to a specific black vertex, b will be used, while w will denote a specific white vertex. In this paper, grids will be presented in a binary form in which a black block will be denoted by 1 while a white block will be denoted by 0. Figure 9 depicts two 5-by-8 grids in binary form. For each grid, the additional row, denoted the summary row, will contain the numbers of black vertices in the columns of the grid. Likewise, the additional right column, denoted the summary column, will contain the numbers of black vertices in the rows of the grid. In general, for an m-by-n grid, b_i will represent the number of black vertices in row i, $1 \leq i \leq m$, and a_j will represent the number of black vertices in column j, $1 \leq j \leq n$. As such, the summary row will hold all the a_j values for $1 \leq j \leq n$, and the summary column will hold all the b_i values for $1 \leq i \leq m$.

Grids with corresponding numbers of rows and columns can differ due to the placement of black and white vertices. A specific arrangement of black and white vertices within a grid will be defined as a *configuration*. The *distribution* for a grid configuration will be defined by the number of black and white vertices in each column and row of the grid. As an example, Figure 9 depicts two 5x8 grids having the same distribution but different configurations. Since the summary rows and summary columns for the two grids are the same, the grids are said to have the same distribution; they have the same number of black and white vertices in their corresponding columns and rows. Although the grids have the same distribution, their configurations are different since the black and white vertices are arranged differently within the rows and columns.

0	1	1	1	0	0	1	0	$4 = b_1$		1	0	1	1	1	0	0	0	4
0	0	0	1	0	0	1	0	$2 = b_2$		0	0	0	0	0	0	1	1	2
1	1	0	1	0	0	1	0	$4 = b_3$		1	1	0	1	0	0	1	0	4
1	0	1	0	0	0	0	1	$3 = b_4$		0	1	0	1	0	0	1	0	3
1	0	0	1	1	0	1	1	$5 = b_5$		1	0	1	1	0	0	1	1	5
$a_1 =$ 3	2	2	4	1	0	4	2 $= a_8$			3	2	2	4	1	0	4	2	

Figure 9. Illustration of Grids With the Same Distribution But Different Configurations.

Since we are attempting to minimize the overall distance from each non-facility vertex to each of the p different facility vertices, we will be concerned with the total distance from each non-facility, or white vertex, to all of the different facilities, or black vertices. Assume that B-W distance denotes the total distance between all the black vertices and white vertices in the grid. The B-W distance for a particular grid configuration will also be denoted by TD(B,W), where B and W again represent the set of black and white vertices in the grid. The distance from a black vertex to a white vertex can be calculated by adding the vertical distance between the rows of the two vertices to the horizontal distance between the columns of the two vertices. As such, TD(B,W) corresponds to adding the horizontal and vertical differences between all the black and white vertices in the grid. A globally optimum solution for a grid will have the lowest possible TD(B,W). We will denote TD(B,W) for a globally optimum solution on a grid G to be the Minimum Total Distance MTD(G, j, k), where j represents the number of black vertices and k the number of white vertices in grid G. (The actual average distance can be obtained by dividing the Minimum Total Distance by the number of possible black:white pairs, MAD(G,j,k)=MTD(G,j,k)/(jk).)

In pursuing an optimal solution, we will often want to reposition black and white vertices on a grid as a means of decreasing the B-W distance. As an example, Figure 10 shows two 5x6 grids, in which Grid 2 is obtained from Grid 1 by switching the adjacent black and white vertices (3,4) and (3,5). The set of black vertices in Grid 2, B_2, consists of the same set of black vertices as in Grid 1, B_1, only with the addition of the new black vertex (3,5) and with the removal of the old black vertex (3,4); thus, $B_2 = B_1 - (3,4) + (3,5)$. In the same way, $W_2 = W_1 - (3,5) + (3,4)$, where W_1 and W_2 represent the set of white vertices in Grid 1 and Grid 2, respectively. As indicated in the figure, $TD(B_2,W_2) < TD(B_1,W_1)$, revealing that the switch of adjacent vertices lowered the B-W distance. The following Switching Lemma for grids tells us precisely when a switch involving adjacent black and white vertices will yield a decrease in the B-W distance.

Grid 1

$TD(B_1,W_1) = 764$

1	0	0	1	1	1	**4**
0	0	1	1	0	1	**3**
1	0	0	1	0	1	**3**
1	1	1	0	0	0	**3**
1	1	1	1	0	1	**5**
4	**2**	**3**	**4**	**1**	**4**	

Grid 2

$TD(B_2,W_2) = 760$

1	0	0	1	1	1	**4**
0	0	1	1	0	1	**3**
1	0	0	0	1	1	**3**
1	1	1	0	0	0	**3**
1	1	1	1	0	1	**5**
4	**2**	**3**	**3**	**2**	**4**	

Figure 10. Illustration of the Switching Lemma for Grids.

Switching Lemma. *Suppose there is a grid $G_{m,n}$ with a B-W configuration producing row distribution $(b_1, b_2, ..., b_m)$ and column distribution $(a_1, a_2, ..., a_n)$ and that a black block at (j,k) is switched with a white block at $(j,k+1)$. The change in $TD(B,W)$ will be:*

$$2km-mn-2(a_1+a_2+...+a_k)+2(a_{k+1}+a_{k+2}+...+a_n)+2.$$

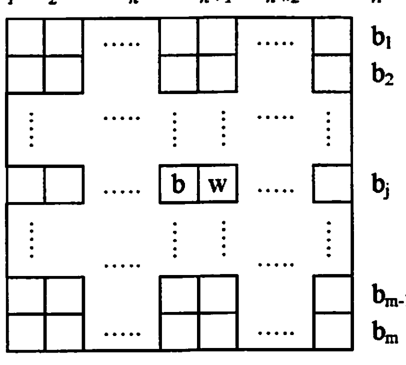


Figure 11. Grid $G_{m,n}$ To Be Used in the Switching Lemma Proof.

Proof: Consider the black block that is switched. (See Figure 11.) Since a_i equals the number of black blocks in column i and there are m blocks in a column, $(m-a_i)$ is the number of white blocks in column i. Since the black block is moving from column k to column (k+1), the black block is getting closer to all the white blocks in the columns from column (k+1) to column n, except for the white block with which it is switching (where the distance remains the same). Hence, the black block is getting closer to $(m-a_{k+1})+(m-a_{k+2})+...+(m-a_n)-1$ white blocks, which can be written alternatively as m(n-k)-

$(a_{k+1}+a_{k+2}+...+a_n)$-1 white blocks. It gets farther from all the white blocks in the first k columns. As such, it gets farther from $(m-a_1)+(m-a_2)+...+(m-a_k)$ white blocks, or, alternatively, $km-(a_1+a_2+...+a_k)$ white blocks. Thus, in summary, the distance from the black block increases by one to $km-(a_1+a_2+...+a_k)$ white blocks but also decreases by one to $m(n-k) - (a_{k+1}+a_{k+2}+...+a_n)-1$ white blocks.

Consider the white block that is switched. It gets farther from the black blocks in the columns from column (k+1) to column n. Thus, the white block gets farther from $(a_{k+1}+a_{k+2}+...+a_n)$ black blocks. Since the white block is getting closer to all the black blocks in the first k columns except for the black block with which it is switching (where the distance remains the same), the white block gets closer to $(a_1+a_2+...+a_k)-1$ black blocks. Thus, the distance from the white block increases by one to $a_{k+1}+a_{k+2} + ... + a_n$ black blocks but also decreases by one to $a_1+a_2 + ... +a_k-1$ black blocks.

Now, adding the distance increases and decreases for the black and white block, it is apparent that the change in TD(B,W) is: $km-(a_1+a_2 + ... +a_k)+(a_{k+1}+a_{k+2}+ ... +a_n) - [m(n-k)-(a_{k+1}+a_{k+2}+ ... +a_n)-1 +(a_1+a_2 + ... +a_k) - 1]$

$$= km-mn+km-2(a_1+a_2+...+a_k)+2(a_{k+1}+a_{k+2}+...+a_n)+2$$
$$= 2km-mn-2(a_1+a_2+...+a_k)+2(a_{k+1}+a_{k+2}+...+a_n)+2. \blacksquare$$

The following results follow in a similar manner.

(1) If a white block at (j,k) is switched with a black block at (j,k+1), the change in TD(B,W) will be $mn-2km+2m-2(a_k+a_{k+1}+...+a_n)+2(a_1+a_2+...+a_{k-1})+2$.

(2) If a black block at (j,k) is switched vertically with a white block at (j+1,k), the change in TD(B,W) will be $2jn-nm-2(b_1+b_2+...+b_j)+2(b_{j+1}+b_{j+2}+...+b_m)+2$.

(3) If a white block at (j,k) is switched vertically with a black block at (j+1,k), the change in TD(B,W) will be $mn-2jn-2(b_{j+1}+b_{j+2}+...+b_m)+2(b_1+b_2+...+b_j)+2$.

By applying the Switching Lemma to a pair of adjacent black and white vertices, we can tell whether switching the adjacent vertices will lower TD(B,W). Since the Switching Lemma tells us what the change in TD(B,W) will be if the switch is performed, we know that if the calculated change is negative, we should make the switch since TD(B,W) will be decreasing. Similarly, if the change is positive, we should not make the switch since TD(B,W) will increase. If the change is zero, the switch will not affect TD(B,W).

The following lemma establishes the relationship between the distribution of a grid configuration and TD(B,W).

Distribution Lemma. *Suppose there is a grid $G_{m,n}$ with a B-W configuration producing row distribution $(b_1, b_2, ..., b_m)$ and column distribution $(a_1, a_2, ..., a_n)$. TD(B,W) is given by:*

$$\sum_{1\le i<j\le m} [(n-b_i)b_j|j-i|+(n-b_j)b_i|j-i|]+\sum_{1\le i<j\le n}[(m-a_i)a_j|j-i|+(m-a_j)a_i|j-i|].$$

In particular, TD(B,W) will be the same for any two bicolored grid configurations having the same number of black and white blocks in their corresponding rows and columns.

Proof: TD(B,W) can be found by calculating the distance from all of the white blocks in the grid to all of the black blocks in the grid.

Suppose that a white block "travels" to a black block by first moving in its column from its row to the row of the black block and then moving from its column to the column of the black block. As such, the distance from a white block to a black block is the sum of the vertical distance between the two blocks' rows and the horizontal distance between the two blocks' columns. Thus, the distance from all the white blocks in the grid to all the black blocks in the grid is simply the sum of the vertical distances from the white blocks to the black blocks and the horizontal distances from the white blocks to the black blocks.

Consider a white block in row i. To reach a black block in row j, the white block must travel a vertical distance of $|j-i|$. However, since there are b_j black blocks in row j, the white block must travel a vertical distance of $b_j|j-i|$ to reach all the black blocks in row j. Now, since one white block in row i travels a distance of $b_j|j-i|$ to reach all the b_j black blocks in row j and there are $(n-b_i)$ white blocks in row i, the vertical distance from all the white blocks in row i to all the black blocks in row j is $(n-b_i)b_j|j-i|$. In a similar manner, since there are $(n-b_j)$ white blocks in row j and b_i black blocks in row i, the vertical distance from all the white blocks in row j to all the black blocks in row i is $(n-b_j)b_i|i-j|=(n-b_j)b_i|j-i|$. Thus, the total vertical distance between the white blocks and black blocks in rows i and j is $(n-b_i)b_j|j-i| + (n-b_j)b_i|j-i|$.

As such, the total vertical distance from all the white blocks to all the black blocks in the grid is

$$\sum_{1\le i<j\le m}[(n-b_i)b_j|j-i|+(n-b_j)b_i|j-i|].$$

Similarly, the horizontal distance from all the white blocks to all the black blocks is

$$\sum_{1\le i<j\le n}[(m-a_i)a_j|j-i|+(m-a_j)a_i|j-i|].$$

Thus, the total distance from the white blocks to the black blocks is

$$\sum_{1\le i<j\le m} [(n-b_i)b_j|j-i|+(n-b_j)b_i|j-i|]+\sum_{1\le i<j\le n}[(m-a_i)a_j|j-i|+(m-a_j)a_i|j-i|]. \blacksquare$$

The Switching Lemma and Distribution Lemma can be used in conjunction with each other to find locally minimal solutions to grid problems. A heuristic computer program, based on the two lemmas, was implemented to identify locally minimal solutions. Heuristic programs do not necessarily yield optimum answers; typically, they follow certain guidelines to produce an optimal, but possibly suboptimum, answer. Given the size specifications for a grid, our program implementation generated a random grid configuration and then switched adjacent facility and non-facility vertices when the Switching Lemma revealed that the switch would reduce the total/average distance. The program produced locally optimal solutions in which no switch between an adjacent facility and non-facility vertex for the final grid configuration would decrease the total distance. By applying the Distribution Lemma to the final solution grid, the total distance for the resulting grid configuration was calculated, and we kept track of both the best and worst configurations found. Figures 12(a) and 12(b) show the best and worst locally optimal configurations found for a 10x10 grid with 16 facilities in which 500 solutions were generated. Although Figure 12(a) shows the best locally optimal configuration found from among the 500 solutions, this solution is not a globally optimum solution. In fact, Figure 13 shows a 10x10 grid with 16 facilities which has a lower total average distance than the configuration shown in Figure 12(a). In the next section, we will examine a special class of grids for which all of the locally optimal solutions will actually be globally optimum solutions.

a) BEST CASE: TD(B,W) = 7696

0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	1	1	1	0	0	0	**3**
0	0	1	1	1	1	1	0	0	0	**5**
0	0	0	1	1	1	1	0	0	0	**4**
0	0	0	0	1	1	1	0	0	0	**3**
0	0	0	0	1	0	0	0	0	0	**1**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	**0**	**1**	**2**	**5**	**4**	**4**	**0**	**0**	**0**	

b) WORST CASE: TD(B,W) = 7902

0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	1	0	0	0	0	**1**
0	0	0	0	0	1	1	0	0	0	**2**
0	0	0	0	1	1	1	1	1	0	**5**
1	1	1	1	1	0	0	0	0	0	**5**
0	0	0	1	1	1	0	0	0	0	**3**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
1	**1**	**1**	**2**	**3**	**4**	**2**	**1**	**1**	**0**	

Figure 12. Best and Worst Locally Minimal Solutions Found Among 500 Solutions.

TD(B,W) = 7680

0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	1	1	1	1	0	0	0	**4**
0	0	0	1	1	1	1	0	0	0	**4**
0	0	0	1	1	1	1	0	0	0	**4**
0	0	0	1	1	1	1	0	0	0	**4**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	0	0	0	0	0	0	0	0	0	**0**
0	**0**	**0**	**4**	**4**	**4**	**4**	**0**	**0**	**0**	

Figure 13. A Better Locally Minimal Solution.

4. EQUIPARTITIONED GRIDS

With the Switching Lemma and Distribution Lemma as tools, we can determine minimized solutions for the special class of equipartitioned grids. The following four theorems present global minimized solutions for equipartitioned grids, that is, where the number of black and white vertices is equal or as close as possible to equal.

Theorem 1. *Suppose there is a grid $G_{m,2k}$ and that there are km black blocks and km white blocks in the grid, where m can be either even or odd. TD(B,W) will be a minimum only if there are an equal number of black and white blocks in each row of the grid. (Likewise, for a grid $G_{2j,m}$ with jm black blocks and jm white blocks, TD(B,W) will be a minimum only if there are an equal number of black and white blocks in each column of the grid.) (Figure 14(a) illustrates the globally optimal distribution for a grid $G_{m,2k}$)*

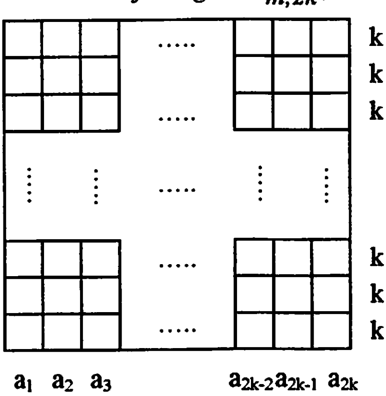


Figure 14(a). Illustration of the Globally Optimal Distribution for a Grid $G_{m,2k}$.

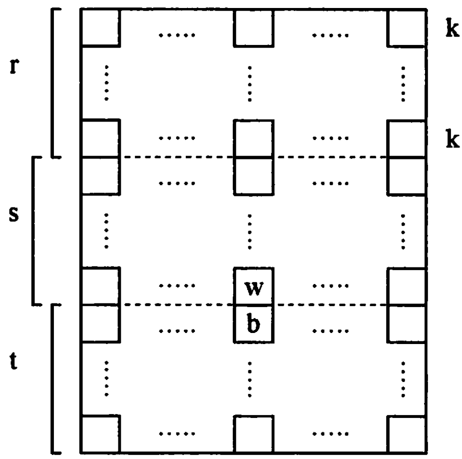


Figure 14(b). Grid $G_{m,2k}$ To Be Used in the Proof.

Proof: Assume a minimum solution and suppose that there are not an equal number of white and black blocks in each row of the grid. It will be shown that switching some pair of black and white blocks that are adjacent will lessen TD(B,W), contradicting the minimality.

Suppose that there are an equal number of black and white blocks in each row of the grid for the first r rows, but not in row r+1. (See Figure 14(b).) As such, in the $(r+1)^{st}$ row, there are either more white blocks than black blocks or more black blocks than white blocks. Suppose that the $(r+1)^{st}$ row has more white blocks than black blocks. Now, since in total there are an equal number of white and black blocks in the grid, there must be a row below the $(r+1)^{st}$ row that has more black blocks than white blocks. As such, there must be a row below the $(r+1)^{st}$ row where there is a black block in the same column as one of the white blocks in the $(r+1)^{st}$ row.

Now, this row may or may not be directly below the $(r+1)^{st}$ row. Suppose the first row below the $(r+1)^{st}$ row which has a black block in the same column as one of the white blocks in the $(r+1)^{st}$ row is in the $(r+s+1)^{st}$ row. This implies that, in rows (r+1) through (r+s), there was a white block in the same column as each of the white blocks in the $(r+1)^{st}$ row. Thus, in each of the rows (r+1) through (r+s), there are more white blocks than black blocks. Suppose that in the s rows from row (r+1) to row (r+s) there are ks + c white blocks, where $c \geq s \geq 1$. Now, since there are an equal number of black and white blocks in the grid and in the first r rows, there must be an equal number of black and white blocks in the final m-r rows of the grid. As such, since there are ks + c white blocks in the s rows from row (r+1) to row (r+s), there must be kt - c white blocks in the final t=m-(r+s) rows of the grid. Thus, in these s rows from row (r+1) to row (r+s), there are ks - c black blocks and ks + c white blocks. In the final t rows, there are kt + c black blocks and kt - c white blocks.

Suppose that the adjacent black and white block where the white block is in the (r+s) row and the black block is in the (r+s+1) st row are switched.

Consider the first r rows, where there are an equal number of black and white blocks in each row. Since there are kr black blocks in this region and the white block is moving one square away from these black blocks, the distance from the white block to each of the kr black blocks will increase by 1. But, since there are kr white blocks and the black block is moving one square closer to them, the distance to the black block will decrease by 1 for each of the kr white blocks. Since the distance from the black block to the kr white blocks decreases by 1 and the distance from the white block to the kr black blocks increases by 1, the net change in the distance from the white blocks to the black blocks in this region is zero and indicates that, for the blocks in the first r rows, the distance between black and white blocks is not affected by the switch.

Now, consider how the distance changes for the blocks in the remaining m-r rows.

Consider the white block. It gets farther from the (ks - c) black blocks in the s rows. There are a total of (kt + c) black blocks in the bottom t rows. Since the white block is getting closer to all those blocks except for the black block with which it is switching (where the distance remains the same), the white block gets closer to (kt + c -1) black blocks. Thus, in summary, the distance from the white block increases by 1 to (ks - c) black blocks but decreases by 1 to (kt + c - 1) black blocks.

Consider the black block. It gets closer to the (ks + c) white blocks in the s rows except for the white block with which it is switching (where the distance remains the same). As such, the black block gets closer to (ks + c - 1) white blocks. Since there are (kt - c) white blocks in the bottom t rows, the distance from the black block to each of those (kt - c) white blocks increases by one. Thus, the distance from the black block increases by 1 to (kt - c) white blocks but decreases by 1 to (ks + c - 1) white blocks.

Now, adding the distance increases and decreases for the black and white block, we see that the change is ks - c + kt - c - [kt + c - 1 + ks + c - 1] = 2 - 4c < 0 (since $c \geq s \geq 1$).

Thus, the overall distance between the black and white blocks in the grid decreases due to the switch. But, since we assumed a minimum solution, we have reached a contradiction.

(The proof for the 2jxm grid follows in a similar manner.) ■

Theorem 2: *Suppose there is a grid $G_{2j,2k}$ and that there are 2jk black blocks and 2jk white blocks in the grid. TD(B,W) will be a minimum if and only if there are k black and k white blocks in each row and j black and j white* *blocks in each column of the grid. (Figure 15 illustrates the unique globally optimal distribution for an even-by-even grid $G_{2j,2k}$.)*

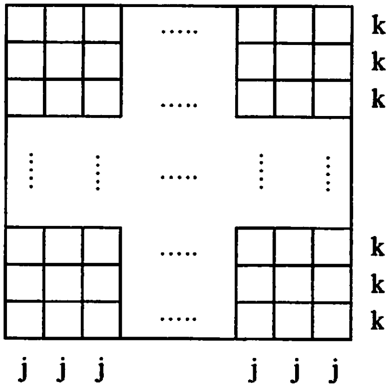


Figure 15. Illustration of the Globally Optimal Distribution for a Grid $G_{2j,2k}$.

Proof: Theorem 1 guarantees that TD(B,W) in an m x 2k grid will be a minimum only if there are an equal number of black and white blocks in each row of the grid. Similarly, it also ensures that TD(B,W) in a 2j x m grid will be a minimum only if there are an equal number of black and white blocks in each column of the grid. Thus, applying Theorem 1 to the grid $G_{2j,2k}$, we see that TD(B,W) in this grid will be a minimum only if there are an equal number of black and white blocks in each row and in each column of the grid.

The Distribution Lemma guarantees that TD(B,W) will be the same for any 2j x 2k grid with this distribution, ensuring that TD(B,W) will be minimum *if and only if* there are an equal number of black and white blocks in each row and in each column of the grid. ■

Applying the Distribution Lemma to grid $G_{2j,2j}$, we see that in particular

$$MTD(G_{2jx2j};2j^2) = 2[\sum_{1\le a<b\le 2j} 2j^2(b-a)] = 4j^2 \sum_{1\le a<b\le 2j}(b-a) = 4j^2\binom{2j+1}{3}.$$

Corollary. $MAD(G_{2k,2k};2k^2) = \frac{4k}{3} - \frac{1}{3k}$.

The arguments for Theorem 3 and Theorem 4 are more complicated than those for Theorem 1 and Theorem 2 but are similar in nature. Theorem 3 concerns globally optimum solutions for equipartitioned odd-by-odd grids, while Theorem 4 concerns globally optimum solutions for equipartitioned odd-by-even grids. The proof for each theorem consists of two parts. First, it is shown that any optimum configuration satisfies the distribution guidelines specified in the theorem. Second, the Distribution Lemma is applied to assure that any two configurations meeting the specified distribution have the same TD(B,W).

Theorem 3. *Suppose there is a 2j+1 x 2k+1 grid and that there are 2jk + j + k black blocks and 2jk + j + k + 1 white blocks in the grid. The distance between the black blocks and white blocks will be a minimum if and only if there are k black blocks and k+1 white blocks in each odd row, k+1 black blocks and k white blocks in each even row, j black blocks and j+1 white blocks in each odd column, and j+1 black blocks and j white blocks in each even column of the grid. Let this distribution be denoted the "alternating distribution." (Figure 16 illustrates this unique globally optimal distribution for an odd-by-odd grid $G_{2j+1,2k+1}$.)*

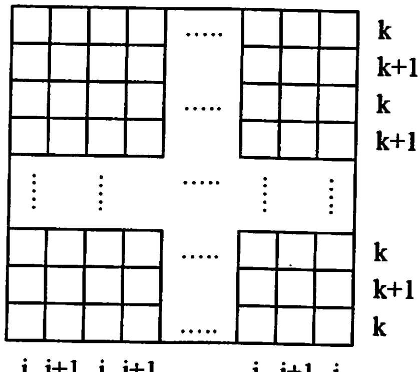


Figure 16. Illustration of the Globally Optimal Distribution For a Grid $G_{2j+1,2k+1}$.

Consider Figure 3. Figure 3 displays three equipartitioned 5x7 grids with different configurations but the same alternating distribution. By Theorem 3, since all three grids possess the alternating distribution, Figures 3(a), 3(b) and 3(c) are equivalent globally optimum solutions for equipartitioned grid $G_{5,7}$.

Note that in Figure 1 we have three distinct column distributions (a_1, a_2, ..., a_6), namely, (2,3,2,3,2,3), (2,3,2,3,3,2) and (3,2,2,3,2,3), producing the same value TD(B,W) = 799.

Theorem 4. *Suppose there is a 2j+1 x 2k grid $G_{2j+1,2k}$ and that there are k(2j+1) black blocks and k(2j+1) white blocks in the grid. TD(B,W) will be a minimum if and only if: (1) there are an equal number of black and white blocks in each row of the grid and (2) there are 2j+1 black blocks and 2j+1 white blocks in each pair of columns 2i-1 and 2i, for i = 1 to k and (3) for* *consecutive columns 2i-1 and 2i, for i = 1 to k, there are either j+1 black blocks and j white blocks in column 2i-1 and, hence, j black blocks and j+1 white blocks in column 2i or there are j black blocks and j+1 white blocks in column 2i-1 and j+1 black blocks and j white blocks in column 2i. (Figure 17 illustrates the 2^k distinct globally optimal distribution for an odd-by-even grid $G_{2j+1,2k}$)*

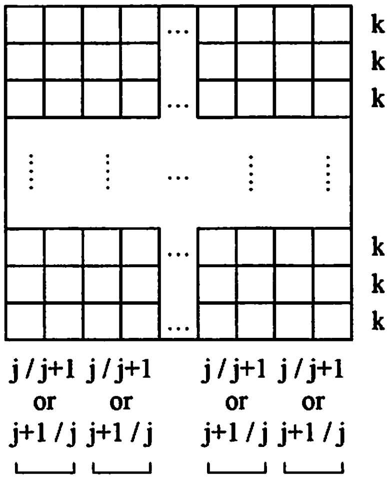


Figure 17. Illustration of the Globally Optimal Distribution For a Grid $G_{2j+1,2k}$.

Again consider Figure 1 which shows three distinct equipartitioned 5-by-6 grids, each having a distribution which satisfies the three conditions presented in Theorem 4. As such, Theorem 4 assures us that Figures 1(a), 1(b) and 1(c) are equivalent globally optimum solutions for equipartitioned grid $G_{5,6}$.

5. CONCLUSION

In this paper, useful tools and guidelines have been developed to help find locally optimal as well as globally optimum solutions to certain gridlike facility location problems. The Switching Lemma and Distribution Lemma can be used for all grid facility problems in identifying locally optimal solutions. The Switching Lemma identifies when a switch of adjacent black and white vertices will yield an improvement in the overall B-W distance and, hence, bring the configuration closer to a locally optimal solution, in which no adjacent switch between any two black and white vertices will lower the B-W distance. The Distribution Lemma assures us that grids with the same distribution but different configurations have the same B-W distance.

With the use of these tools, it has been shown that all locally optimal solutions for the equipartitioned grid problems are actually globally optimum solutions. Exact solutions for optimally placing facilities on a grid structure in which the number of facilities and customers are equal (or near equal for the odd by odd case) have been presented.

There are still a number of problems left to be considered even within the special class of grid location problems alone. Foremost in this area is to find globally optimum solutions for non-equipartitioned grids. Also under study is the more general problem of partitioning a vertex set V(G) into t teams of specified orders so as to minimize the average interteam distance.

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