A reduction principle concerning minimum dominating sets in graphs

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Abstract

In this note we consider finite, undirected, and simple graphs. A subset D of the vertex set of a graph G is a dominating set if each vertex of G is either in D or adjacent to some vertex of D. A dominating set of minimum cardinality is called a minimum dominating set. A vertex v of a graph G is called a cut-vertex of G if $G-v$ has more components than G. A block of a graph is a maximal connected subgraph having no cut-vertex. A block-cactus graph is a graph whose blocks are either complete graphs or cycles, and we speak of a cactus if the complete graphs consists of only one edge. In our main theorem we shall show that the minimum dominating set problem of an arbitrary graph can be reduced to its blocks. This theorem provides a linear time algorithm for determining a minimum dominating set in a block-cactus graph, and thus, it can be seen as a supplement to a linear time algorithm for finding a minimum dominating set in a cactus, presented by S.T. Hedetniemi, R.C. Laskar, and J. Pfaff in 1986.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday

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1. Terminology and introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$. For a vertex x of a graph G, the neighborhood is denoted by $N(x, G)$ and the closed neighborhood by $N[x, G]$. For a subset X of the vertex set $V(G)$, we define $N(X, G) = \bigcup_{x \in X} N(x, G)$ and $N[X, G] = N(X, G) \cup X$. A subset $D \subseteq V(G)$ is an X-dominating set of G if $X \subseteq N[D, G]$. An X-dominating set D of minimum cardinality is a minimum X-dominating set, and $|D|$ is the X-domination number, denoted by $\gamma(G, X)$. Note that the case $X = V(G)$ leads to the ordinary dominating sets, minimum dominating sets, and domination number $\gamma(G) = \gamma(G, V(G))$. A vertex v of a graph G is called a cut-vertex of G if $G - v$ has more components than G. A connected graph without a cut-vertex is called a block. A block of a graph G is a subgraph of G, which is itself a block and which is maximal with respect to that property. A block B of a graph G is called an end block of G if B contains at most one cut-vertex of G. A block-cactus graph is a graph in which each block is either a cycle or a complete graph, and we speak of a cactus graph if the complete graphs consist of only one edge. A graph is a block graph if every block is complete.

It is well-known that the problem of determining a minimum dominating set in an arbitrary graph is NP-hard, and the analogous decision problem is NP-complete, see e.g. the books of Garey and Johnson [5] or Haynes, Hedetniemi, and Slater [6]. Therefore, most of the work has been devoted to solving this problem for special families of graphs. The first domination algorithm was a linear time algorithm for computing $\gamma(T)$ for an arbitrary tree T, by Cockayne, Goodman, and Hedetniemi [2] in 1975. A graph is chordal if it contains no cycle of length greater than three as an induced subgraph. Booth and Johnson [1] were the first to show that the minimum dominating set problem remains NP-complete when restricted to chordal graphs. However, Farber [4] has shown that for strongly chordal graphs, a subclass of chordal graphs, the domination problem is solvable in linear time. Since block graphs are strongly chordal, as one can see by one of the characterizations of this class given by Farber [3], we deduce that the domination problem for block graphs is linear. A further efficient algorithm for block graphs can be found in [9]. In 1986, Hedetniemi, Laskar, and Pfaff [8] have given a linear time algorithm for locating a minimum dominating set in a cactus. With a so called "reduction theorem" we show that the minimum dominating set problem of an arbitrary graph can be reduced to its blocks. This theorem provides a linear time algorithm for determining a minimum dominating set in a block-cactus graph. Since block-cactus graphs are a common generalization of block graphs and cactus graphs, this can be seen as a supplement to some of the above mentioned results.

The books by Haynes, Hedetniemi, and Slater [6] and [7] on domination in graphs are an excellent source of additional information.

2. A reduction principle

Reduction Theorem. Let G be a graph and $X \subseteq V(G)$. If B is an end block of G with the unique cut-vertex v of G and $X_B = X \cap V(B)$, then $\gamma(G, X_B - v) \leq \gamma(G, X_B)$. Now we distinguish two cases.
Case 1. Assume that

$$\gamma(G, X_B - v) < \gamma(G, X_B). \tag{1}$$

If D' is a minimum X'-dominating set of G with $X' = X - (X_B - v)$, and D_{B-v} a minimum $(X_B - v)$-dominating set of G, then $D = D' \cup D_{B-v}$ is a minimum X-dominating set of G.
Case 2. Assume that

$$\gamma(G, X_B - v) = \gamma(G, X_B). \tag{2}$$

Subcase 2.1. There exists a minimum X_B-dominating set D_v of G with the property that $v \in D_v$. If D'' is a minimum X''-dominating set of G with $X'' = X - (V(B) \cup N(v, G))$, then $D = D'' \cup D_v$ is a minimum X-dominating set of G.
Subcase 2.2. There exists no minimum X_B-dominating set D_v of G with $v \in D_v$. If D^* is a minimum X^*-dominating set of G with $X^* = X - V(B)$ and D_B is a minimum X_B-dominating set of G, then $D = D^* \cup D_B$ is a minimum X-dominating set of G.

Proof. Every X_B-dominating set is an $(X_B - v)$-dominating set and thus, it follows that $\gamma(G, X_B - v) \leq \gamma(G, X_B)$.
Case 1. First, we note that the inequality (1) implies immediately that $v \in X_B$. Now let D_0 be an arbitrary minimum X-dominating set of G. By the definition of $D = D' \cup D_{B-v}$, it is evident that D is an X-dominating set of G and therefore, it is enough to show that $|D| \leq |D_0|$. Next, we consider two further subcases.
Subcase 1.1. Let $v \in D_0$. According to (1), we observe that $v \notin D_{B-v}$ and hence, we conclude that

$$|D_0 \cap (V(B) - v)| \geq |D_{B-v}|, \tag{3}$$

because otherwise, $D_0 \cap V(B)$ would be an X_B-dominating set with the property that $|D_0 \cap V(B)| \leq |D_{B-v}|$, a contradiction to (1). In addition, with $G' = G - (V(B) - v)$, it follows at once that

$$|D_0 \cap V(G')| \geq |D'|. \tag{4}$$

Combining the two inequalities (3) and (4), we obtain the desired estimation

$$|D| = |D'|+|D_{B-v}| \leq |D_0 \cap V(G')|+|D_0 \cap (V(B)-v)| = |D_0 \cap V(G)| = |D_0|.$$

Subcase 1.2. Let $v \notin D_0$. If we define the vertex set D_1 by $D_1 = D_0 \cap V(B)$, then we investigate three cases.
If $|D_1| < |D_{B-v}|$, then $D_1 \cup \{v\}$ is an X_B-dominating set with $|D_1 \cup \{v\}| \leq |D_{B-v}|$, a contradiction to (1).
In the case $|D_1| = |D_{B-v}|$, the inequality (1) yields $v \notin N[D_1, G]$. But since $v \in X$, we deduce that $v \in N[D_0 - D_1, G]$ and hence, $D_0 - D_1$ is an X'-dominating set of G. From the fact that D_1 is an $(X_B - v)$-dominating set of G, we conclude

$$|D| = |D'| + |D_{B-v}| \leq |D_0 - D_1| + |D_1| = |D_0|.$$

In the remaining case $|D_1| \geq |D_{B-v}| + 1$, it is evident that $D_0 - D_1$ is an $(X' - v)$-dominating set of G and therefore, $(D_0 - D_1) \cup \{v\}$ is an X'-dominating set of G. This leads us to

$$|D| = |D'| + |D_{B-v}| \leq |D_0 - D_1| + 1 + |D_1| - 1 = |D_0|,$$

which completes the proof of Case 1.
Case 2. Let again D_0 be an arbitrary minimum X-dominating set of G.
Subcase 2.1. Since $D = D'' \cup D_v$ is an X-dominating set of G, it is sufficient to show that $|D| \leq |D_0|$.
If $v \in D_0$, then it follows analogously to Subcase 1.1 that $|D_0 \cap V(B)| \geq |D_v|$ and $|D_0 - V(B)| \geq |D''|$, and therefore, $|D| = |D_v| + |D''| \leq |D_0|$.
If $v \notin D_0$, then, $D_1 = D_0 \cap V(B)$ is an $(X_B - v)$-dominating set of G and hence, the hypothesis (2) yields $|D_1| \geq |D_v|$. Clearly, $D_0 - V(B)$ is an X''-dominating set of G and so we obtain $|D_0 - D_1| \geq |D''|$. The last two inequalities imply

$$|D| = |D''| + |D_v| \leq |D_0 - D_1| + |D_1| = |D_0|.$$

Subcase 2.2. Since $D = D^* \cup D_B$ is an X-dominating set of G, it is again enough to show that $|D| \leq |D_0|$.

In the case $v \in D_0$, our condition that there doesn't exist a minimum X_B-dominating set containing the cut-vertex v, leads to $|D_0 \cap V(B)| \geq |D_B|+1$. In addition, since $(D_0 - V(B)) \cup \{v\}$ is an X^*-dominating set, we deduce that

$$|D^*| \leq |(D_0 - V(B)) \cup \{v\}| = |D_0 - V(B)| + 1.$$

Altogether, we find the desired estimation

$$|D| = |D^*| + |D_B| \leq |D_0 - V(B)| + 1 + |D_0 \cap V(B)| - 1 = |D_0|.$$

In the remaining case $v \notin D_0$, the set $D_1 = D_0 \cap V(B)$ is an $(X_B - v)$-dominating set of G and hence, in view of (2), we have $|D_1| \geq |D_B|$. Furthermore, $D_0 \cap (V(G) - V(B))$ is an X^*-dominating set and we see, as before that $|D| \leq |D_0|$. This completes the proof of the reduction theorem.

We will now give some comments on the application of our reduction theorem to the minimum dominating set problem in block-cactus graphs.
In the introduction we have surveyed several complexity results for the minimum dominating set problem restricted to special classes of graphs. These results do not immediately imply that the more general minimum X-dominating set problem is of similar complexity. However, it is straightforward to verify that for every graph $G = (V, E)$ and every set $X \subseteq V$ there is a graph $G' = (V', E')$ such that G is an induced subgraph of G', $|V'| = O(|V|)$, $|E'| = |E| + O(|V|)$, all cycles in G' are cycles in G and if $D \subseteq V'$ is a minimum dominating set of G' then $D \cap V$ is a minimum X-dominating set of G. (One way of constructing G' could be to join a complete graph K_2 to all vertices of $V \setminus X$.) Hence, we see that the minimum X-dominating set problem can be solved in linear time for trees and cactus graphs.
Applying our reduction theorem, we see that we can solve the minimum X-dominating set problem for the graph in linear time if we can solve it in linear time for its blocks. Hence our approach and the use of the previously known complexity results immediately implies the existence of a linear time algorithm to solve the minimum X-dominating set problem for block-cactus graphs.

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