Well-spread Sequences

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Abstract

A well-spread sequence is an increasing sequence of distinct positive integers whose pairwise sums are distinct. Some properties of these sequences are discussed.

Dedicated to Prof. Stephen T. Hedetniemi on the occasion of his 60th birthday.

Definitions

A well-spread sequence $A = (a_1, a_2, \cdots, a_n)$ of length n is a sequence with the following properties:

1. $0 < a_1 < a_2 < \cdots < a_n$;
2. $a_i + a_j \neq a_k + a_\ell$ whenever $i \neq j$ and $k \neq \ell$ (except, of course, when $\{a_i, a_j\} = \{a_k, a_\ell\}$).

We define

$$\sigma(A) = a_n - a_1 + 1$$

$$
\begin{aligned}
\rho(A) &= a_n + a_{n-1} - a_2 - a_1 + 1 \\
&= \sigma(A) + a_{n-1} - a_2 \\
\sigma^*(n) &= \min \sigma(A) \\
\rho^*(n) &= \min \rho(A)
\end{aligned}
$$

where the minima are taken over all well-spread sequences A of length n. σ is called the size of the sequence. Without loss of generality one can assume $a_1 = 1$ when constructing a sequence, and then the size equals the largest element.

Well-spread sequences have application in the study of edge-magic total labelings of graphs. Such a labeling on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \cdots, |V(G) \cup E(G)|$ with the property that, given any edge (x, y),

$$\lambda(x) + \lambda(x, y) + \lambda(y) = k$$

for some constant k. A graph with a such a labeling will be called a magic graph. It is shown in [4] that, if a magic graph G contains a complete subgraph H with n vertices, then the labels on the vertices of H form a well-spread sequence, A say, and the number of vertices and edges in G is at least $\rho(A)$. This has been used to show that no complete graph on more than six vertices can be magic (see [3, 4]) and there are also implications for the edge-magic total labeling of other dense graphs. For this reason, and because of their intrinsic mathematical interest, we would like to find out more about well-spread sequences and the functions $\sigma^*(n)$ and $\rho^*(n)$.

Evaluation, bounds

Theorem 1 [2] $\sigma^*(n) \geq 4 + \binom{n-1}{2}$ when $n \geq 7$.

Theorem 2 [2] $\rho^*(n) \geq 2\sigma^*(n-1)$ when $n \geq 4$.

Proof. (For completeness.) Consider the sequences

$$
\begin{aligned}
A &= (a_1, a_2, \cdots, a_n) \\
B &= (a_1, a_2, \cdots, a_{n-1}) \\
C &= (a_2, a_3, \cdots, a_n)
\end{aligned}
$$

where $n \geq 4$. Clearly

$$\rho * (n) \geq \rho(A)$$

$$\begin{aligned} &= a_n + a_{n-1} - a_2 - a_1 + 1 \\ &= (a_n - a_2 + 1) + (a_{n-1} - a_1 + 1) - 1 \\ &= \sigma(B) + \sigma(C) - 1 \\ &\geq 2\sigma^*(n-1) - 1. \end{aligned}$$

Moreover, equality can apply only if $\sigma(B) = \sigma(C) = \sigma^*(n-1)$. But

$$\begin{aligned} \sigma(B) = \sigma(C) &\Rightarrow a_n - a_2 = a_{n-1} - a_1 \\ &\Rightarrow a_{n-1} + a_2 = a_n + a_1, \end{aligned}$$

which is impossible for a well-spread sequence A. Since σ^* and ρ^* are integral,

$$\rho^*(n) \geq 2\sigma^*(n-1).$$

Calculations for small values of n

In practise, values of $\sigma^*(n)$ and $\rho^*(n)$ have been calculated using an exhaustive, backtracking approach. The following result is helpful in restricting the search for $\rho^*(n)$, once some σ^* values are known. Typically, the sequence B used will be one for which $\sigma(B) = \sigma^*(n)$.

Theorem 3 Suppose the sequence $(1, x, \cdots, y, z)$ attains the value $\rho^*(n)$, and suppose B is any well-spread sequence of length n. Then

$$\sigma^*(n) \leq z \leq \rho(B) - \sigma^*(n-2) + 1$$

and

$$x \leq (\rho(B) - \sigma^*(n-2) + 1) - \sigma^*(n-1) + 1.$$

Proof. Since $(x, \cdots, y)$ is a well-spread sequence,

$$y - x + 1 \geq \sigma^*(n-2).$$

But

$$\rho^*(n) = z + y - x$$

so

$$\begin{aligned} z &= \rho^*(n) - (y - x) \\ &\leq \rho^*(n) - \sigma^*(n-2) + 1 \\ &\leq \rho(B) - \sigma^*(n-2) + 1. \end{aligned}$$

Also $(x, \cdots, y, z)$ is well-spread, so

$$z - x \geq \sigma^*(n-1) - 1$$

and the second part of the Theorem follows from the upper bound for z.

We know the following small values of the two functions. The values for $n \leq 8$ are calculated in [2] and listed in [4].

$$\begin{array}{rclrcl}
\sigma^*(3) & = & 3 & \rho^*(3) & = & 3 \\
\sigma^*(4) & = & 5 & \rho^*(4) & = & 6 \\
\sigma^*(5) & = & 8 & \rho^*(5) & = & 11 \\
\sigma^*(6) & = & 13 & \rho^*(6) & = & 19 \\
\sigma^*(7) & = & 19 & \rho^*(7) & = & 30 \\
\sigma^*(8) & = & 25 & \rho^*(8) & = & 43 \\
\sigma^*(9) & = & 35 & \rho^*(9) & = & 62 \\
\sigma^*(10) & = & 46 & \rho^*(10) & = & 80 \\
\sigma^*(11) & = & 58 & \rho^*(11) & = & 110 \\
\sigma^*(12) & = & 72 & \rho^*(12) & = & 137
\end{array}$$

Sample sequences attaining the σ^* values are:

$\sigma^*(1)$ through $\sigma^*(6)$: 1 2 3 5 8 13 (or part thereof);

$\sigma^*(7)$: 1 2 3 5 9 14 19;

$\sigma^*(8)$: 1 2 3 5 9 15 20 25;

$\sigma^*(9)$: 1 2 3 5 9 16 25 30 35;

$\sigma^*(10)$: 1 2 8 11 14 22 27 42 44 46;

$\sigma^*(11)$: 1 2 6 10 18 32 34 45 52 55 58;

$\sigma^*(12)$: 1 2 3 8 13 23 38 41 55 64 68 72.

The same sequences attain $\rho^*(n)$ for $n, = 1, 2, 3, 4, 5, 6, 8$. For the other values, examples are

$\rho^*(7)$: 1 6 8 10 11 14 22;

$\rho^*(9)$: 1 5 7 9 12 17 26 27 40;

$\rho^*(10)$: 1 2 3 5 9 16 25 30 35 47;

$\rho^*(11)$: 1 2 3 5 9 16 25 30 35 47 65.

$\rho^*(12)$: 1 3 5 8 11 21 30 39 51 62 63 77.

Note: The only other sequence of length 7 with $\rho = 30$ is 1, 9, 12, 13, 15, 17, 22.

From Theorem 3, we have

For $n = 7$	$x \leq 12$	$19 \leq z \leq 24$
For $n = 8$	$x \leq 13$	$25 \leq z \leq 31$
For $n = 9$	$x \leq 21$	$35 \leq z \leq 45$
For $n = 10$	$x \leq 30$	$46 \leq z \leq 64$
For $n = 11$	$x \leq 32$	$58 \leq z \leq 77$
For $n = 12$	$x \leq 36$	$72 \leq z \leq 93$

and these bounds were used in calculating the example sequences for $\rho^*(n)$ when $n \geq 7$.

A greedy approach

Here is a simple observation. If $(a_1, a_2, \cdots, a_{n-1})$ is well- spread, then none of its sums can exceed $a_{n-2} + a_{n-1}$. Put $a_n = a_{n-1} + a_{n-2}$. Then all the sums $a_i + a_n$ are new, and (since the sequence is strictly monotonic) they are all different. So we have a new well-spread sequence. This will be useful in constructing the smallest well-spread sequences for small orders: for example, after observing that $(1, 2, 3, 5, 8, 13)$ is a minimal example for $n = 6$, one need not test any sequence in the case $n = 7$ which has size greater than 21. (Unfortunately, 21 is not small enough.)

Suppose $(a_1, a_2, \cdots, a_{n-1})$ had minimal size, and put $a_1 = 1$. Then $\sigma^*(n-1) = a_{n-1}$. Clearly $a_{n-2} < a_{n-1}$, so we have the (bad) bound

$$\sigma^*(n) \leq 2\sigma^*(n-1) - 1.$$

Another application of this idea comes from noticing that the recursive construction $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}$ gives a well- spread sequence. This is the Fibonacci sequence (f_n), except that the standard notation for the Fibonacci numbers has $f_1 = f_2 = 1, f_3 = 2$, etc. So we have a well-spread sequence with its size equal to the $(n+1)$- th term of the Fibonacci sequence. Therefore

$$\sigma^*(n) \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

The same reasoning shows that

$$\rho^*(n) \leq f_{n+1} + f_n - 2 = f_{n+2} - 2.$$

Note. For further information on the Fibonacci numbers, see for example Section 7.1 of [1].

References

[1] R. A. Brualdi, Introductory Combinatorics (3rd ed.), Prentice-Hall (1999).

[2] A. Kotzig, On well spread sets of integers, Publ. CRM–161 (1972).

[3] A. Kotzig and A. Rosa, Magic valuations of finite graphs, Canad. Math. Bull. 13 (1970), 451–461.

[4] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Publ. CRM–175 (1972).