Queens on Hexagonal Boards

A. P. Burger and C. M. Mynhardt
Department of Mathematics
University of South Africa,
P. O. Box 392, Pretoria
0003 South Africa

Abstract

A queen on a hexagonal board with hexagonal cells is defined as a piece that moves along three lines, namely along the cells in the same row, up diagonal or down diagonal. A queen dominates a cell if the cell is in the same line as the queen. We show that hexagonal boards with $n \geq 1$ rows and diagonals, where $n \equiv 3 (\mathrm{mod}\, 4)$, have only two types of minimum dominating sets. We also determine the irredundance numbers of the boards with 5 and 7 rows.

Dedicated to Steve Hedetniemi on the occasion of his 60th birthday

1 Introduction

The study of combinatorial problems on chessboards can be traced back to the middle of the 19th century, when a German chess player, Max Bezzel [1], first posed the problem of placing n queens on an $n \times n$ chessboard so that no two queens attack each other. The study of chessboard domination problems dates back to 1862, when C. F. de Jaenisch [4] first considered the queens domination problem, that is, the problem of determining the minimum number of queens required to dominate every square on an $n \times n$ chessboard. Since then many papers concerning combinatorial problems on chessboards have appeared in the literature and surveys of the topic are given in [5, 7]. The queens domination problem is also studied in depth in the Ph. D. thesis [2].

In this paper we consider hexagonal boards (hives) consisting of hexagonal cells (see Figure 1). We define a queen on the hexagonal board as a piece that moves along three *lines*, namely along the cells in the same row, up diagonal or down diagonal. The *edge* consists of all the cells on the edge of the hive. A cell or a line is *empty* if there is no queen on the cell or line, otherwise it is *occupied*. A queen *dominates* or *covers* a cell if the cell is in the same line as the queen. (Thus a queen also dominates the cell on which she is placed.) A cell is *open* if it is not dominated. The problem is to determine the minimum number of queens necessary to dominate all the cells on the board. Note that domination by queens on a square beehive was studied by Theron and Geldenhuys in [9].

This problem can also be considered as a graph domination problem in the following way: The *hexagonal queens graph* H_n has the cells of a board with n rows and diagonals as its vertices. Two vertices are *adjacent* if the two corresponding cells are in the same row or diagonal. A set D of vertices (cells) is a *dominating set* of H_n if every cell of H_n is either in D or adjacent to a vertex in D. If no two cells of a set I are adjacent, then I is an *independent* set. Let $\gamma(H_n)$ denote the minimum size of a dominating set of H_n, and let $i(H_n)$ denote the minimum size of an independent dominating (*i.e.* a maximal independent – see [6, p. 70]) set of H_n.

We also study irredundance numbers of some small boards, and for this purpose we need some definitions. The *closed neighbourhood* $N[v]$ of the vertex v in a graph $G = (V, E)$ consists of v and the set of vertices adjacent to v. The *closed neighbourhood* of a set $S \subseteq V$ is defined by $N[S] = \cup_{v \in S} N[v]$. We define the *private neighbourhood* of $v \in S$ as $pn(v, S) = N[v] - N[S - \{v\}]$. If $pn(v, S) \neq \emptyset$ for some vertex v, then every vertex in $pn(v, S)$ is called a *private neighbour* of v. Note that a vertex can be its own private neighbour. We say that a set S of vertices is *irredundant* if for every vertex $v \in S$, v has at least one private neighbour. Note that a minimal dominating set is also irredundant. An irredundant set S is *maximal irredundant* if for every vertex $u \in V - S$, the set $S \cup \{u\}$ is not irredundant, which means that there exists at least one vertex $w \in S \cup \{u\}$ which does not have a private neighbour. The minimum cardinality of a maximal irredundant set in a graph G is called the *irredundance number* and is denoted by $ir(G)$. Note that for any graph G, $ir\,(G) \leq \gamma(G) \leq i(G)$ (see [6, p. 58]).

If a vertex u is added to a set S and it destroys all the private neighbours of some vertex w in S (*i.e.*, $pn(w, S) \neq \emptyset$ and $pn(w, S \cup \{u\}) = \emptyset$), we call u an *annihilator*, and say that u *annihilates* w. If $u \in V$ has no private neighbours with respect to S, we say u is *pn-less* with respect to S. We say a vertex v (or a cell or a square in the case of hexagonal boards or chessboards) is *open* (with respect to S) if it is not dominated by S.

We only consider hives with a centre cell, *i.e.*, hives with an odd number of rows and diagonals. The values for $\gamma(H_n)$ and $i(H_n)$ (n odd) were first determined by Theron and Burger [8] (these turn out to be equal). In this paper we show that there are only two types of minimum dominating sets for H_{4k+3}. We also determine $ir\,(H_5)$ and $ir\,(H_7)$.

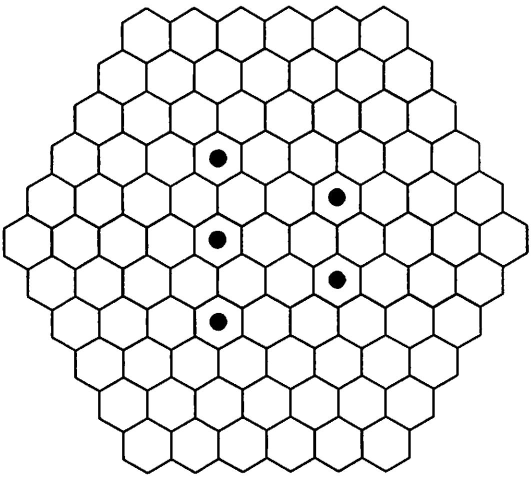

Figure 1: A dominating set for H_{11}

2 Domination and independent domination numbers

Theron and Burger [8] first showed that $\gamma(H_{4k+1}) = i(H_{4k+1}) = \gamma(H_{4k+3}) = i(H_{4k+3})$. In order to show that there are only two types of minimum dominating sets of H_{4k+3} for all $k \geq 1$, we need to repeat some of their results here. The proofs given here are slightly different.

The lines of the hive are labelled as shown in Figure 2. Each cell has three coordinates, namely row (r), up diagonal (u) and down diagonal (d), which we denote as (r, u, d). We begin by noting the following.

Remark 1 *For all cells we have* $r + u + d = 0$

Remark 2 *A line with a negative (positive) label intersects an edge line with a positive (negative) label.*

We now describe a dominating set of queens on H_{4k+3} which was first discovered in [8]. The placement consists of two columns with $k + 1$ and k queens respectively - Figure 1 shows the case $k = 2$. In general, the coordinates for $k \geq 0$ are given below (see Figure 3), where the second set of coordinates is undefined (and to be ignored) when $k = 0$:

$$(2a - k, -a, k - a) \text{ for } a = 0, 1, \ldots, k$$

and

$$(2a + 1 - k, k - a, -1 - a) \text{ for } a = 0, 1, \ldots, k - 1.$$

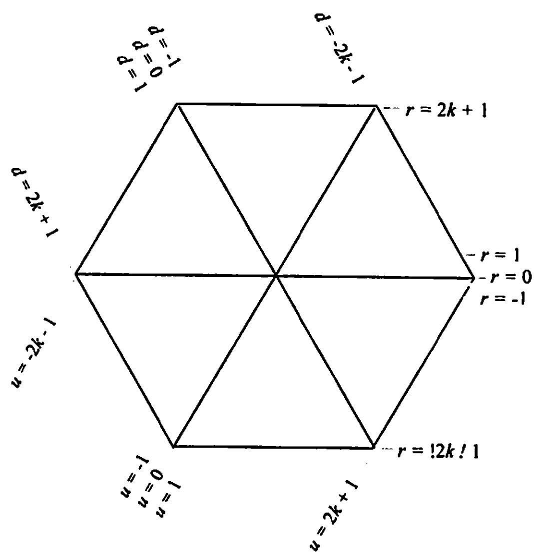

Figure 2: Numbering of rows and diagonals

We refer to this placement as the *Double Column Placement* (*DCP*). We see that each of the rows, up diagonals and down diagonals covered by a DCP has the labels

$$-k, -k+1, \ldots, -1, 0, 1, \ldots, k-1, k.$$

Thus the $2k+1$ lines closest to the centre are all covered. This is sufficient to dominate the whole hive. Note that the queens form an independent set. Since the case $k = 0$ is trivial, we assume henceforth that $k \geq 1$. We state the following lemma without proof.

Lemma 3 *For all* $k \geq 1$, $\gamma(H_{4k+3}) \leq i(H_{4k+3}) \leq 2k+1$.

We define a *ring* as a six-sided convex polygon formed by the union of six lines, where each line consists of at least two cells. The edge is an example of a ring. The ring can be made smaller by replacing one line of the ring with a line closer to the centre, as long as the ring has six sides. For any set of queens on a hive we define the *Biggest Empty Ring* (*BER*), if it exists, as the ring formed by the edge lines, if they are unoccupied, or by replacing each of the occupied edge lines with the empty parallel line closest to the edge line concerned (see Figure 4). Let the *distance* of a line of the BER from the edge be the number of lines outside that side of the

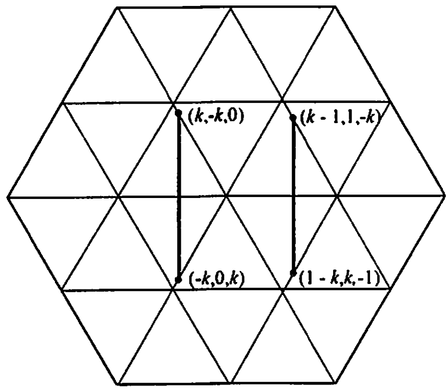

Figure 3: Double Column Placement for H_{4k+3}

BER. Let δ be the sum of all the distances of the six lines of the BER from the edge. Note that δ equals the number of cells in the edge minus the number of cells in the BER, because if a line of the ring is replaced by a line just closer to the centre, the number of cells in the ring decreases by one. We now have the following lemma.

Lemma 4 *For all $k \geq 1$, if $2k$ or fewer queens are placed on H_{4k+3}, then the BER exists.*

Proof. We only have to verify that the BER always has six sides. If a line of any ring is replaced by a line immediately closer to the centre, the number of cells in two lines of the ring decreases by one. When constructing the BER, each queen outside the BER caused either one or two such replacements (depending on whether the queen is at a "corner" or not). It is easy to see that each queen outside the BER caused the number of cells in any side to decrease by at most one. There are $2k + 2$ cells in each edge line. Thus, if there are $2k$ queens, each side of the BER must have at least two cells. □

Let c be the total number of times the BER is dominated by all the queens. Thus if one BER cell is covered m times, it must be counted m times. Let q be the number of queens on the board. We have the following lemma:

Lemma 5 *If the BER exists for a set of q queens on H_{2n+1}, then $c \leq 6q - 2\delta$, where equality holds if and only if the queens are independent.*

Proof. There are two types of queens outside the BER (see Figure 4):

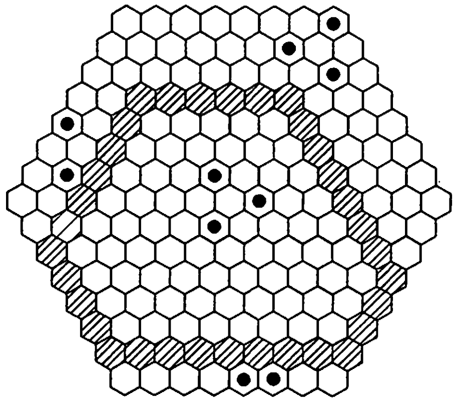

Figure 4: Biggest empty ring

(1) Queens that lie on the outside of only one line of the BER. Each of them covers four cells of the BER.

(2) Queens that lie on the outside of two lines of the BER. Each of them covers two cells of the BER.

Let there be b_1 and b_2 queens of each type respectively. It is easy to see that

$$\delta \leq b_1 + 2b_2,$$

with equality if and only if the queens are independent. Now,

$$\begin{aligned} c &= 6(q - b_1 - b_2) + 4b_1 + 2b_2 \\ &= 6q - 2b_1 - 4b_2 \\ &= 6q - 2(b_1 + 2b_2) \\ &\leq 6q - 2\delta. \end{aligned}$$

Again, equality holds if and only if the queens are independent. □

Lemma 6 *If a set of n queens dominates H_{2n+1}, then*
(a) *the BER is the edge.*
(b) *each edge cell is covered exactly once.*

Proof. From Lemma 5 we have $c \leq 6n - 2\delta$. Also $|BER| = 6n - \delta$. For the BER to be dominated we must have $c \geq |BER|$. Thus

$$6n - \delta \leq c \leq 6n - 2\delta$$

and so $\delta = 0$. Thus the BER must be the edge. To prove (b) we note that there are $6n$ edge cells and each of the n queens can cover six edge cells. Therefore each edge cell must be dominated exactly once. □

The proofs of the following two results, first proved in [8], now follow easily:

Theorem 7 [8] *For all $k \geq 0$, $i(H_{4k+3}) = \gamma(H_{4k+3}) = 2k+1$.*

Proof. As noted before we need only consider the case $k \geq 1$. We first show that $\gamma(H_{4k+3}) \geq 2k+1$. Consider any set of $2k$ queens. There are $6(2k+1) - \delta$ cells in the BER. But by Lemma 5, $c \leq 6(2k) - 2\delta$. Thus we have $c \leq 6(2k) - 2\delta < 6(2k+1) - \delta = |BER|$. Therefore the BER cannot be dominated. The result now follows from Lemma 3. □

Theorem 8 [8] *For all $k \geq 0$, $i(H_{4k+1}) = \gamma(H_{4k+1}) = 2k+1$.*

Proof. H_{4k+1} is H_{4k+3} with the edge removed. Therefore the Double Column Placement also dominates H_{4k+1}, which establishes $\gamma(H_{4k+1}) \leq i(H_{4k+1}) \leq 2k+1$. To show that $\gamma(H_{4k+1}) \geq 2k+1$, $k \geq 1$, we show that $2k$ queens cannot dominate H_{4k+1}. Suppose we have a set of $2k$ queens dominating H_{4k+1}. From Lemma 6(b) we see that each cell on the edge is dominated exactly once. The corner cells can only be dominated by a queen on a main diagonal. Consider any main diagonal. It must contain a queen, and the remaining $2k-1$ queens are on either side of (but not on) the diagonal. A queen in a specific half of the board dominates four cells of the edge in that half and two cells of the edge in the other half. To dominate the same number of edge cells on the two sides, there must be the same number of queens in the two halves. This is a contradiction, because an odd number of queens remains. □

3 Minimum dominating sets of H_{4k+3}

In this section we show that there are only two types of minimum dominating sets of H_{4k+3} for all $k \geq 1$. From Lemma 6 we see that any dominating set of H_{4k+3} consisting of $2k+1$ queens leaves the edge empty. We use the fact that each edge cell must be dominated exactly once to prove the following lemmas.

Lemma 9 *If H_{4k+3} is dominated by $2k+1$ queens, then lines with the same label are either all occupied or all empty.*

Proof. Each of the edge cells is dominated exactly once. Thus if the row $r = a (a > 0)$ is occupied (respectively empty), then $d = 2k+1-a$ and

$u = 2k + 1 - a$ are empty (respectively occupied). But then $u = 2k + 1 - (2k + 1 - a) = a$ and $d = a$ are occupied (respectively empty). The arguments for the diagonals are the same. Also, if $a < 0$, the arguments are similar. The lines with label 0 must be occupied, because the corner cells can only be dominated by these lines. □

From Lemma 9 we see that we do not have to distinguish between labels of rows and labels of diagonals. Consequently, we will only refer to the set of labels

$$L = \{-2k - 1, -2k, \ldots - 2, -1, 0, 1, \ldots, 2k, 2k + 1\}.$$

This set can be partitioned into two disjoint sets: the set representing all the occupied lines (O) and the set representing all the empty lines (E).

Lemma 10 *If $a \in O$, then*

$$-a + 2k + 1 \in E \text{ if } a > 0$$

$$-a - 2k - 1 \in E \text{ if } a < 0.$$

Proof. The edge lines are labelled $2k+1$ or $-2k-1$. Suppose $a \in O$. Then since each edge cell is dominated exactly once, it follows from Remarks 1 and 2 that $-a + 2k + 1 \in E$ if $a > 0$ and $-a - 2k - 1 \in E$ if $a < 0$. □

Lemma 11 *If $a, b \in E$ and $|a + b| < 2k + 1$, then*
(a) $-a - b \in O$
(b) $a + b + 2k + 1 \in E$ *if* $a + b < 0$
(c) $a + b - 2k - 1 \in E$ *if* $a + b > 0$.

Proof. If lines a and b are empty and they intersect inside the edge, the third line going through the intersection must be occupied. From Remark 1 this line must be $-a - b$. Statements (b) and (c) follow from (a) and Lemma 10. □

Lemma 12 *If $2a \in E$, then $-a \in O$.*

Proof. Suppose $2a \in E$ and $-a \in E$. Then from Lemma 11(a), $-2a + a = -a \in O$. This is a contradiction. Therefore $-a \in O$. □

Lemma 13 *If $1 \in E$, then all odd elements of L are in E.*

Proof. If $1 \in E$, then from Lemma 11(c), $1 + 1 - 2k - 1 = 1 - 2k \in E$. If $1, 1 - 2k \in E$, then from Lemma 11(b), $1 + 1 - 2k + 2k + 1 = 3 \in E$. If $1, 3 \in E$, then from Lemma 11(c), $1 + 3 - 2k - 1 = 3 - 2k \in E$. Continuing in this way, we find the following elements in E:

$$1 - 2k, 3, 3 - 2k, 5, 5 - 2k, \ldots, 2k - 3, -3, 2k - 1, -1.$$

These are all the odd elements of L. □

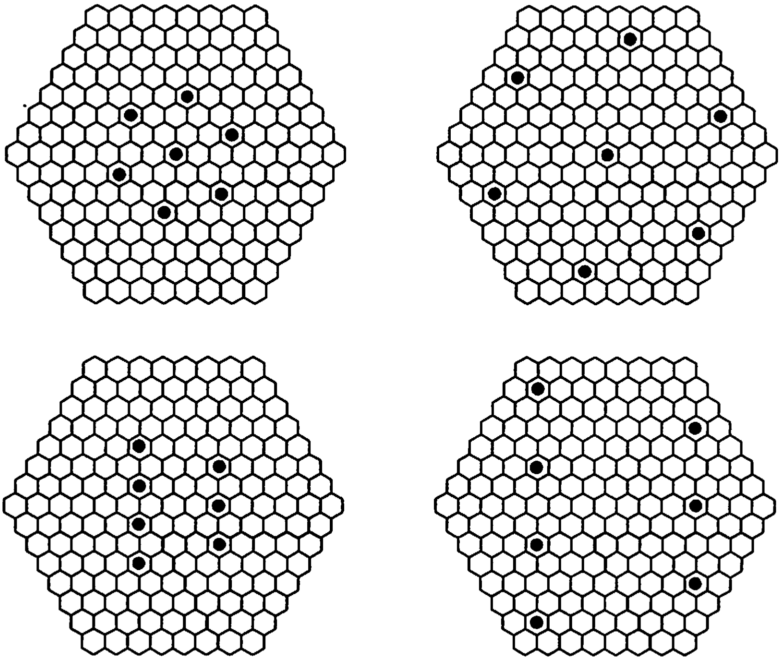

Figure 5: Dominating sets of different types for H_{15}

Theorem 14 *There are only two types of dominating sets of cardinality* $2k+1$ *for* H_{4k+3}, $k \geq 1$:

(a) $O = \{-2k, -2k+2, ..., -2, 0, 2, ..., 2k-2, 2k\}$

(b) $O = \{-k, -k+1, ..., -1, 0, 1, ..., k-1, k\}$

Proof. Either $1 \in E$ or $1 \in O$. If $1 \in E$, then from Lemma 13 we have (a). We must show that every set of vertices satisfying (a) is dominating. A cell can only be open if three empty lines intersect in that cell. All empty lines have odd labels. Thus the sum of the coordinates of such a cell would be odd. This is impossible because the sum must be 0.

If $1 \in O$, then by Lemma 10 we have $2k \in E$. It follows from Lemma 12 that $-k \in O$, and then from Lemma 10 that $-k-1 \in E$. If $2k \in E$ and $-k-1 \in E$, then it follows from Lemma 11(c) that $-k-2 \in E$. If $2k \in E$ and $-k-2 \in E$, then again from Lemma 11(c), $-k-3 \in E$. Continuing in this way, we find that

$$-k-1, -k-2, -k-3, ..., -2k+1, -2k \in E.$$

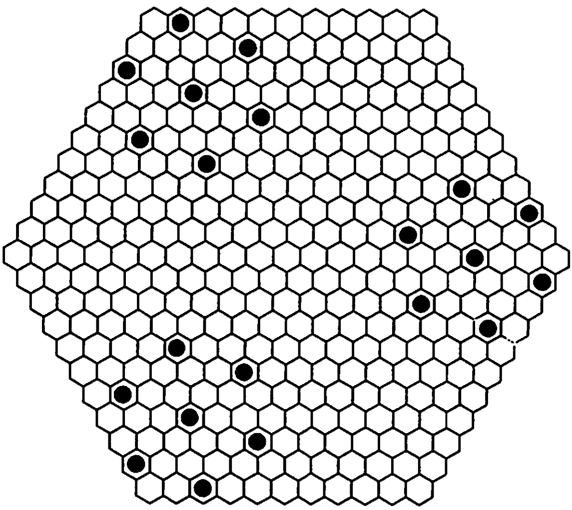

Figure 6: The central section of H_{43}

The whole argument can be repeated with $-2k \in E$ to show that

$$k+1, k+2, ..., 2k-1, 2k \in E.$$

Thus (b) follows, and every set of vertices satisfying (b) is also dominating as explained in the case of a DCP. □

We note that in Theorem 14 the labels in (a) are double the labels in (b). Thus if we take the coordinates of a dominating set of type (b) and multiply it by two, we have the coordinates of a dominating set of type (a). The reverse can also be done. We therefore have a one-to-one correspondence between all the dominating sets of type (a) and (b). Figure 5 shows a few examples.

Table 1 lists the number of dominating sets found by computer. We see that the number of dominating sets is large for large boards. Dominating sets for H_{4k+1} are even more numerous, and they are not restricted to two types of minimum dominating sets.

We can construct minimum dominating sets of larger boards using the dominating sets of smaller boards. In Figure 6 a dominating set of H_{43} is obtained by repeating the pattern of a dominating set of H_{15}. Note that only the central section of the board is shown. This method has so far not been successful for minimum dominating sets of queens on $n \times n$ chessboards. As illustrated in Figure 7, the hexagonal queen domination problem is the same as the queen domination problem for chessboards with

k	$4k+3$	γ	number
1	7	3	1
2	11	5	1
3	15	7	5
4	19	9	56
5	23	11	540
6	27	13	6996

Table 1: Number of dominating sets found

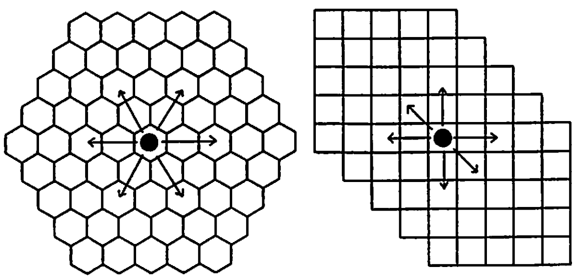

Figure 7: Relation between hexagonal boards and chessboards

the queens' domination restricted to three lines (row, column and one diagonal) and with two corners of the board cut off. The diagonal not used by queens on the hive will be useful to dominate squares in those corners of the $n \times n$ board that have been cut off, so it may be possible that dominating sets of queens on hives can lead to dominating sets of queens on chessboards.

4 Irredundance

Before we consider irredundance on hexagonal boards, we state the following general results on irredundance. For a vertex subset S of a graph G we denote the set of all open vertices (vertices not dominated by S) by R_S.

Theorem 15 [3] *An irredundant set S of a graph G is maximal irredundant if and only if each $v \in N[R_S]$ is an annihilator.*

Since every vertex in an independent set is its own private neighbour, it follows directly from the definition of irredundance that every (maximal) independent set is (maximal) irredundant. We obtain the following result as an immediate corollary.

Proposition 16 *If S is a maximal irredundant set in a graph G and $|S| < i(G)$, then S is not independent.*

We now return to hexagonal boards. Recall that a cell or line is empty if there is no queen on the cell or line, and that a cell is open if it is not dominated by any queen.

Lemma 17 *If $S = \{q_1, q_2\}$ is a maximal irredundant set of two queens on H_5, then:*

(a) *There are at least two open cells.*
(b) *The number of pn-less cells is at most four.*
(c) *Each queen has at least two private neighbours.*
(d) *Each queen can be annihilated from at most six cells.*

Proof. The queens q_1 and q_2 are adjacent (Proposition 16); assume without loss of generality their coordinates are (r, u_1, d_1) and (r, u_2, d_2) respectively.

(a) Suppose firstly that q_1 (say) is on the central cell, in which case its coordinates are $(0,0,0)$. Then q_1 covers 13 cells. There are at most six cells on u_2 and d_2 which are not on $r = 0$, and u_1 (d_1, respectively) intersects d_2 (u_2, respectively) in exactly one cell. Hence there are at most four cells that are covered by q_2 but not q_1 and since H_5 has 19 cells, at least two cells are open. Now suppose there is no queen with coordinates $(0,0,0)$. A queen on the edge dominates at most nine cells; hence we may assume without loss of generality that q_1 is not on the edge. It is now easy to see that q_1 covers 11 cells and that exactly one of r, u_1 and d_1 is equal to 0. If $r = 0$, then as above q_2 covers at most four cells not covered by q_1, while if (say) $u_1 = 0$, then q_2 covers at most five cells not covered by q_1. In each case at least three cells are open.

(b) Since S is irredundant, the pn-less cells are the empty cells not in line with the open cells and thus they are empty cells on occupied lines. There are at least two open cells. Even if there are exactly two open cells, it is easy to check that there are at most six cells not in line with them. (The extremal case occurs when new queens on the open cells cover as few cells as possible and share the longest possible line – see Figure 8, where the open cells are indicated by open circles.) This leaves at most four pn-less cells, because q_1 and q_2 are also not in line with the open cells.

(c) On each of the two lines occupied by q_1 but not q_2, there are at least two cells that can possibly be private neighbours of q_1. But q_2 dominates at most one cell on such a line. Thus q_1 (and similarly q_2) has at least two private neighbours.

(d) Consider the $t \geq 2$ private neighbours of q_1 and suppose firstly that they are on the same line l. If q_1 is also on this line, then q_1 can be annihilated from at most four cells (the empty cells) on l, and from at most two cells not on l (since any two cells on the same line can be dominated

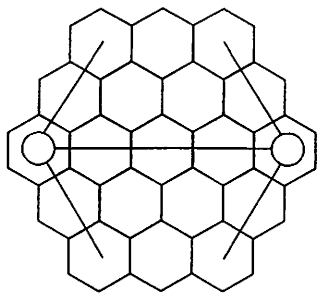

Figure 8: There are at most four pn-less cells not in line with open cells.

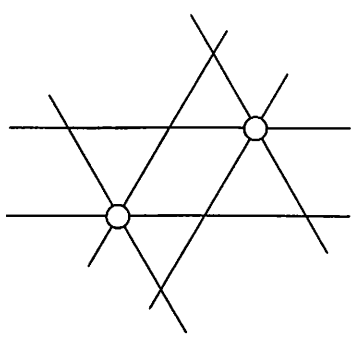

Figure 9: If $ir(H_5) = 2$, then there are at most six pn-destroyers

simultaneously from at most two cells not on the line). If q_1 is not on l, that is, if $l \notin \{u_1, d_1\}$, then q_1 can be annihilated from at most five cells on l and at most one cell not on l. Suppose the private neighbours are not all on the same line. Then the cells which are annihilators are those cells that lie on the intersection of t lines, each of which contains a private neighbour. (Some of the lines may be parallel.) There are at most six such positions – the maximum occurs when $t = 2$. (See Figure 9.) □

Theorem 18 $ir(H_5) = 3$.

Proof. We know that $ir(H_5) \leq \gamma(H_5) = 3$. Suppose $ir(H_5) < 3$. It is easy to see that $ir(H_5) > 1$, so consider a maximal irredundant set of H_5 consisting of two queens. There are 17 unoccupied cells. All of them must be either annihilators or pn-less. This is impossible, because each queen's private neighbours can be annihilated from at most 6 cells, and the number of pn-less cells is at most 4, *i.e.* the total number of annihilators or pn-less cells is $6 + 6 + 4 < 17$. □

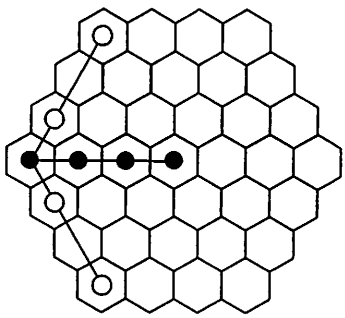

Figure 10: If $ir(H_7) = 2$, there are no pn-destroyers

Lemma 19 *If $S = \{q_1, q_2\}$ is a maximal irredundant set of two queens on H_7, then:*

(a) *There are at least eight open cells.*

(b) *Each queen has at least four private neighbours.*

(c) *There are no annihilators.*

Proof. The two queens are adjacent (Proposition 16) and therefore occupy a total of five distinct lines. Suppose without loss of generality that q_1 and q_2 have coordinates (r, u_1, d_1) and (r, u_2, d_2) respectively.

(a) By Lemma 4, the BER exists. There are 18 edge cells and each line through a cell not on the edge intersects the edge in exactly two cells. Therefore if the BER is the edge, there are at least $18 - 2(5) = 8$ open cells in the BER. Suppose the BER is not the edge. Then $\delta \geq 1$ and the BER contains $18 - \delta$ cells. By Lemma 5, the BER is dominated at most $12 - 2\delta - 1$ times and hence the BER alone contains at least $7 + \delta \geq 8$ cells.

(b) Since the queens are adjacent, each queen q_i has the two lines u_i and d_i that can contain its private neighbours. Proceeding as in the proof of Lemma 17(c), it is easy to see that each line has at least two private neighbours.

(c) For any line l and any cell a not on l, a queen on a dominates at most two cells on l. The private neighbours of q_i lie on the lines u_i and d_i, with at least two private neighbours on each line. Consider any empty cell a on u_1. Then a (new) queen q on a dominates the cell on d_1 containing q_1 and at most one other cell on d_1. Hence q does not annihilate q_1. Hence no new queen on u_i (d_i, respectively) annihilates q_i because it does not dominate more than one private neighbour of q_i on d_i (u_i, respectively). A queen not on u_i or d_i dominates at most two private neighbours of q_i on each line. Thus, in order for a new queen to annihilate q_1 (say), q_1 has exactly four private neighbours and the only possibility is that q_1 lies on a corner cell with q_2 not on one of the edge lines, *i.e.*, $r = 0$. (See Figure 10, where alternative positions of q_2 are indicated by shaded circles.) It is now easy to see that in this case there are also no annihilators. □

Theorem 20 $ir(H_7) = 3$.

Proof. We know $ir \leq \gamma = 3$. Suppose $ir(H_7) = 2$ and consider a maximal irredundant set of H_7 with two queens. By Lemma 19 there are no annihilators and there are at least eight open cells. But all open cells must be annihilators (Theorem 15), a contradiction. Thus the theorem follows. □

References

[1] M. Bezzel. Schachfreund, *Berliner Schachzeitung* **3**(1848), 363.

[2] A. P. Burger. *The Queens Domination Problem*, Ph. D. thesis, University of South Africa, 1998.

[3] E. J. Cockayne, P. J. P. Grobler, S. T. Hedetniemi and A. A. McRae. What makes an irredundant set maximal? *J. Combin. Math. Combin. Comput.* **25**(1997), 213-223.

[4] C. F. de Jaenisch. *Applications de l'Analyse Mathematique au Jeu des Echecs.* Petrograd, 1862.

[5] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, C. K. Wallis, M. S. Jacobson, H. W. Martin and W. D. Weakley. Combinatorial Problems on Chessboards: A Brief Survey. In Y. Alavi and A. J. Schwenk, editors, *Graph Theory, Combinatorics, and Algorithms, Proceedings of the Seventh Quadrennial International Conference on the Theory and Application of Graphs*, (Kalamazoo, MI 1992), Volume 1, Wiley-Interscience, New York, 1995, 507-528.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[7] S. M. Hedetniemi, S. T. Hedetniemi and R. Reynolds. Combinatorial problems on chessboards: II. In T. W. Haynes, S. T. Hedetniemi and P. J. Slater, editors, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998, 133-162.

[8] W. F. D. Theron and A. P. Burger. Queen domination of hexagonal hives. *J. Combin. Math. Combin. Comput.* **30**(1999), to appear.

[9] W. F. D. Theron and G. Geldenhuys. Domination by queens on a square beehive. *Discrete Math.* **178**(1998), 213-220.