Extremal domination insensitive graphs

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Abstract

A connected graph G is (γ, k)-insensitive if the domination number $\gamma(G)$ is unchanged when an arbitrary set of k edges is removed. The problem of finding the least number of edges in any such graph has been solved for $k = 1$ and for $k = \gamma(G) = 2$. Asymptotic results as n approaches infinity are known for $k \geq 2$ and $k+1 \leq \gamma(G) \leq 2k$. Note that for $k = 2$, this bound holds only for graphs G with $\gamma(G) \in \{3, 4\}$. In this paper, we present an asymptotic bound for the minimum number of edges in an extremal (γ, k)-insensitive graph G, where $k = 2$ and $n \geq 3\gamma(G)^2 - 2\gamma(G) + 3$ that holds for $\gamma(G) \geq 3$. For small n, we present tighter bounds (in some cases exact values) for this minimum number of edges.

Dedicated to Prof. Stephen T. Hedetniemi
on the occasion of his 60th birthday.

1 Introduction

In a graph $G = (V, E)$ the open neighborhood of a vertex $v \in V$ is $N(v) = \{x \in V | vx \in E\}$, the set of vertices adjacent to v. The closed neighborhood is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S, that is, $V = \bigcup_{s \in S} N[s]$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set; and a minimum dominating set of a graph G is called a $\gamma(G)$-set, or simply a γ-set if the graph G is clear from the context. For a thorough study of domination and terminology not defined here, see [5].

Many studies have considered the effects on $\gamma(G)$ when G is modified by deleting a vertex or deleting or adding an edge. For surveys of such results, see Chapter 5 of [5] and Chapters 16 and 17 of [6].

Here we are interested in the effect that edge removal has on the domination number of a graph G. Obviously, the removal of an edge from G

cannot decrease the domination number and can increase it at most by 1. The graphs G for which the domination number changes upon the removal of an arbitrary edge, that is, $\gamma(G-e) = \gamma(G) + 1$ for every edge $e \in E(G)$, were first investigated by Walikar and Acharya in [11]. Bauer, Harary, Nieminen, and Suffel [1] observed that such a graph is a union of stars. On the other hand, obtaining a descriptive characterization of the graphs for which the domination number does not change is not as easy. Such graphs, that is, graphs G for which $\gamma(G-e) = \gamma(G)$ for every edge $e \in E(G)$, were called *γ-insensitive* by Dutton and Brigham [2]. Walikar and Acharya also studied these graphs and Hartnell and Rall [4] gave a constructive characterization of γ-insensitive trees.

Haynes, Brigham, and Dutton [7, 9] extended the notion of γ-insensitive graphs to (γ, k)-insensitive graphs by considering the removal of $k \geq 1$ arbitrary edges. A graph G is *(γ, k)-insensitive* if for every arbitrary set $F \subseteq E(G)$ of $k \geq 1$ edges, $\gamma(G-F) = \gamma(G) = \gamma$. Research on (γ, k)-insensitivity has been concerned mainly with extremal graphs. In this context, a connected graph of order n is extremal if it is (γ, k)-insensitive and has the minimum number of edges among all such graphs of order n.

It is proposed in [8] that extremal (γ, k)-insensitive graphs have applications in network design, where the vertices represent computers and an edge represents a direct link between two computers. For example, if graph G represents a communication network, a γ-set of G is a minimized core group that could function in a variety of ways, including as "masters", fileservers, or repositories for a global database essential to the other computers in the network. If it is desirable that the number of processors in the core group stay the same even after k links (edges) fail, then a network corresponding to an extremal (γ, k)-insensitive graph has minimum link cost (minimum number of edges) and the desired fault tolerant property.

Since our main result depends on and improves known results, we present a brief review of the studies of (γ, k)-insensitive extremal graphs. In particular, in Section 2 we present some background and discuss an existing asymptotic result. Our first new results appear in Section 3, where structural properties of (γ, k)-insensitive graphs are presented. These results will be used in subsequent sections. Recalling our network application, it seems logical that the problem is most practical for small k, that is, the case where a small number of links fail. Thus, for the remaining sections, we restrict our attention to $k = 2$ (that is, removing two edges).

Noting that for $k = 2$, the existing asymptotic result holds only for graphs G with $\gamma(G) = 3$ or $\gamma(G) = 4$, in Section 4 we present an asymptotic result that holds for all values of $\gamma(G) \geq 3$. Then in Section 5, better bounds and in some cases exact values are found for $k = 2$ and small n. Finally, we summarize the results and conclude with open problems in Section 6.

2 Background

2.1 Known extremal results

Dutton and Brigham [2] studied connected extremal $(\gamma, 1)$-insensitive graphs. The minimum number of edges in any (γ, k)-insensitive graph of order n is denoted $E_k(n, \gamma)$. Extremal graphs were determined in [2, 7, 9] for the case when $\gamma(G) = 1$.

Theorem 1 [7] *If G is a $(1, k)$-insensitive graph with order $n > 2k \geq 2$, then*

$$E_k(n, 1) = (2k + 1)(n - k - 1).$$

Dutton and Brigham [2] calculated $E_k(n, \gamma)$ for the case when $k = 1$. Note that Theorem 1 takes care of the case where $\gamma(G) = 1$, so we summarize their results for $\gamma(G) \geq 2$.

Theorem 2 [2] *For a $(\gamma, 1)$-insensitive graph G with $\gamma(G) \geq 2$,*

$$E_1(n, \gamma) = \begin{cases} n - 1 & \text{if } n \leq 3\gamma(G) - 2, \\ n & \text{if } n = 3\gamma(G) - 1, \\ 2n - 3\gamma(G) & \text{if } n \geq 3\gamma(G). \end{cases}$$

The value of $E_k(n, \gamma)$ and extremal graphs were determined in [10] for the case when $\gamma(G) = k = 2$.

Theorem 3 [10] *For a $(2, 2)$-insensitive graph G with $n \geq 11$,*

$$E_2(n, 2) = \lfloor (5n - 10)/2 \rfloor.$$

For an example of a family of extremal $(2, 2)$-insensitive graphs, see Figure 1.

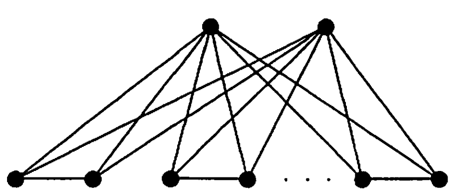

Figure 1: A family of $(2, 2)$-insensitive graphs.

2.2 Known asymptotic bound

Although extremal (γ, k)-insensitive graphs and the exact value of $E_k(n,\gamma)$ have not been found for the cases $\gamma(G) \geq 3, k = 2$ and $\gamma(G) \geq 2, k \geq 3$, upper and lower bounds on $E_k(n,\gamma)$ were determined in [9] and an asymptotic bound on $E_k(n,\gamma)$ for $k \geq 2$ was derived in [7].

Theorem 4 [7] *For $k+1 \leq \gamma(G) \leq 2k$, $E_k(n,\gamma)$ is asymptotically equal to $(k+3)n/2$ as n approaches infinity.*

Since our new result in Section 4 is dependent on the understanding of Theorem 4, we discuss the proof here. First an upper bound was established by constructing a family of (γ, k)-insensitive graphs. Let k and γ be fixed positive integers such that $k+1 \leq \gamma \leq 2k$, $n \geq \gamma(k+1)$, $t = \lfloor (n-\gamma)/k \rfloor$, and $r = (n-\gamma) \bmod k$. Note that $t \geq \gamma$. Construct graph G as follows:

(1) $V = A \cup B_1 \cup B_2 \cup ... \cup B_t$ where $A = \{a_1, a_2, ..., a_\gamma\}$, $B_i = \{b_{i1}, b_{i2}, ..., b_{ik}\}$ for $1 \leq i \leq t-1$ and $B_t = \{b_{t1}, b_{t2}, ..., b_{t,k+r}\}$.

(2) Each B_i, $1 \leq i \leq t$, induces a complete subgraph.

(3) Each vertex b_{ij} is adjacent to exactly two vertices of A, one of which is a_1 and the other is from $A - \{a_1\}$, such that for each B_i, $1 \leq i \leq t$, $|N(B_i) \cap A - \{a_1\}| \geq k$.

(4) Each vertex in $A - \{a_1\}$ is adjacent to a vertex b_{ij} for at least $\gamma - k$ distinct values of i.

Figure 2 shows a graph G having $n = 17$, $\gamma(G) = 4$, and $k = 3$ which has been constructed according to the specifications described above.

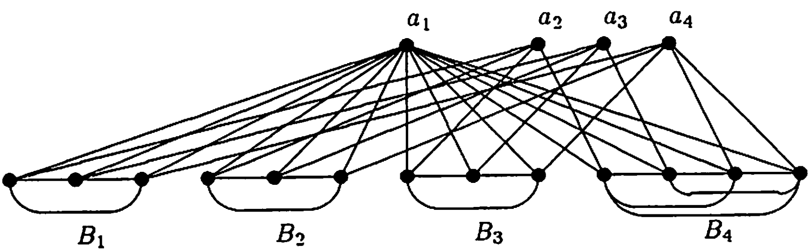

Figure 2: A (4,3)-insensitive graph.

Haynes, Brigham, and Dutton [7] proved that the graphs G obtained from this construction are connected, (γ, k)-insensitive, have $\gamma(G) = \gamma$, and have

$$(k+3)n/2 - [(k+3)\gamma - 2kr - r^2 + r]/2$$

edges. They also noted that the edge count is maximized when r has its largest value of $k-1$, and hence the following bound is established.

Lemma 5 [10] *For a (γ,k)-insensitive graph G with $2 \le k+1 \le \gamma(G) \le 2k$ and $n \ge \gamma(G)(k+1)$,*

$$E_k(n,\gamma) \le \frac{(k+3)n - ((k+3)\gamma(G) - 3k^2 + 5k - 2)}{2}.$$

Therefore, for fixed k and $\gamma(G)$, the upper bound is asymptotically equal to $(k+3)n/2$ as n approaches ∞.

Let S be a γ-set of an extremal (γ,k)-insensitive graph G and denote the maximum number of vertices in $V-S$ having degree at most k by $f(k)$. The lower bound is based on the following lemma.

Lemma 6 [10] *If G is a (γ,k)-insensitive graph with $k \ge 2$, $\gamma(G) \ge 3$, and $n \ge \gamma(G)^2 + 2\gamma(G) + f(k)$, then*

$$E_k(n,\gamma) \ge \frac{(k+3)n - [2(k+2)\gamma(G) + (k-1)(\gamma(G)^2 + f(k))]}{2}.$$

Finally, they showed that $f(k)$ is bounded by an expression that is independent of n, and hence the lower bound is asymptotically equal to $(k+3)n/2$ (the same asymptotic value as the upper bound), completing the proof of Theorem 4.

3 Properties of (γ,k)-insensitive graphs

Our first observation is straightforward, but useful.

Observation 7 *Let x be a vertex in a (γ,k)-insensitive graph G, where $1 \le deg(x) \le k$. If F contains the set of edges incident to x, then for every γ-set S of $G-F$, it follows that $x \in S$ and $N(x) \cap S = \emptyset$.*

Next we develop another useful property of (γ,k)-insensitive graphs.

Proposition 8 *If G is a (γ,k)-insensitive graph, $k \ge 1$, with vertices u and v where the distance $d(u,v) \le 2$, then $deg(u) + deg(v) \ge k+2$.*

Proof. Let G be a (γ,k)-insensitive graph with vertices u and v such that $d(u,v) \le 2$. Suppose that $deg(u) + deg(v) \le k+1$. Since $d(u,v) \le 2$ and $deg(u) + deg(v) \le k+1$, either $uv \in E(G)$ or u and v share a common neighbor implying that there are at most k edges incident to u or v. Let F be the set of edges incident to u or v. Then both u and v are isolates in $G-F$, and hence u and v are in every γ-set S of $G-F$. Since $\gamma(G) = \gamma(G-F)$, it follows that $\gamma(G)-2$ vertices dominate $G-\{u,v\}$. But since $d(u,v)\leq 2$, one vertex will dominate $\{u,v\}$ in G. Hence, $\gamma(G)\leq |S|-2+1=\gamma(G)-1$, a contradiction. □

Corollary 9 *An extremal (γ,k)-insensitive graph G has at most $\gamma(G)$ endvertices.*

The neighbor of an endvertex is called a *support vertex*.

Corollary 10 *Let G be a (γ,k)-insensitive graph. If $x\in V$ is an endvertex with support vertex v, then $deg(u)\geq k+1$ for every $u\in N(v)-\{x\}$.*

4 New asymptotic results

The asymptotic result of Theorem 4, which is independent of n, was derived from upper and lower bounds for $E_k(n,\gamma)$ that differ by $O(\gamma^k)$. We note that for $k=2$, Theorem 4 applies only for graphs G with $\gamma(G)=3$ or $\gamma(G)=4$. This restriction on $\gamma(G)$ comes from the graphs G that yield the upper bound. We construct a family of graphs to yield a suitable upper bound that holds for $\gamma(G)\geq 3$. Also, we improve the lower bound of Lemma 6 slightly to obtain the following asymptotic result.

Theorem 11 *For $\gamma(G)\geq 3$, $E_2(n,\gamma)$ is asymptotically equal to $5n/2$ as n approaches infinity.*

The proof to Theorem 11 follows directly from the following two lemmas. Let N_i be maximum number of vertices of degree at most k having at least i common neighbors in a (γ,k)-insensitive graph, $1\leq i\leq k$. It is shown in [7] that $N_k\leq 2$. Our first lemma improves the lower bound given in Lemma 6 by improving the known bound on $f(2)$.

Lemma 12 *If G is a $(\gamma,2)$-insensitive graph with $n\geq 3\gamma(G)^2-2\gamma(G)+3$ and $\gamma(G)\geq 3$, then*

$$E_2(n,\gamma)\geq \frac{5n-3\gamma(G)^2-4\gamma(G)-3}{2}.$$

Proof. Let G be a $(\gamma,2)$-insensitive graph with $\gamma(G)\geq 3$ and $n\geq 3\gamma(G)^2-2\gamma(G)+3$. Substituting $k=2$ into the inequality of Lemma 6 yields

$$E_2(n,\gamma)\geq \frac{5n-\gamma(G)^2-8\gamma(G)-f(2)}{2}.$$

The next two claims help verify a bound for $f(k)$.

Claim 1 $N_1 \leq 2\gamma(G) - 2$.

Proof. Assume that $X = \{x_1, x_2, ..., x_t\}$ is a set of vertices such that $deg(x_i) \leq 2$ for $1 \leq i \leq t$ and vertex v is a common neighbor of all the vertices in X. Let $Y = \{y_1, y_2, ..., y_j\}$ be the other neighbors of the vertices in X. If $x_i \in X$ is an endvertex, then Proposition 8 implies that x_i is the only neighbor of v in X, that is, $|X| = 1$. Thus, assume that $deg(x_i) = 2$ for $1 \leq i \leq t$.

Since $N_2 \leq 2$, we have that $1 \leq |N(y_i) \cap X| \leq 2$ for $1 \leq i \leq j$. Assume that $|N(y_i) \cap X| = 1$ for all $y_i \in Y$. Then each x_i is adjacent to y_i and $|Y| = t$. Let $G' = G - x_i y_i - x_i v$ and S be a $\gamma(G')$-set. Then $x_i \in S$ and for each $x_l \in X - \{x_i\}$, either x_l or y_l is in S. There can be at most $\gamma(G) - 1$ of these vertices implying that $t \leq \gamma(G)$.

Without loss of generality, assume that $N(y_i) \cap X = \{x_1, x_2\}$ for some $y_i \in Y$. Let S be a γ-set of $G - x_1 v - x_1 y_i$. By Observation 7, $x_1 \in S$ and neither v nor y_i is in S. Hence, $x_2 \in S$. Furthermore, each $x_i \in X$, for $i \notin \{1, 2\}$, or its neighbor in Y must be in S; and S can include at most $\gamma(G) - 2$ vertices from $X \cup Y - \{x_1, x_2\}$. Since $N_2 \leq 2$, it follows that t is maximized if each vertex in Y has exactly two neighbors in X and $S = \{x_1, x_2\} \cup Y - \{y_i\}$. Thus, $t \leq 2\gamma(G) - 2$. □

Claim 2 $f(2) \leq 2\gamma(G)^2 - 4\gamma(G) + 3$.

Proof. Assume that x is a vertex such that $deg(x) \leq 2$. Let G' be the graph obtained by removing all edges incident to x. Then by Observation 7, for any $\gamma(G')$-set S, $x \in S$. Now $|S - \{x\}| = \gamma(G) - 1$ and by Claim 1 each vertex in $S - \{x\}$ can dominate at most $2\gamma(G) - 2$ vertices having degree at most 2. Thus, $f(2) \leq (\gamma(G) - 1)(2\gamma(G) - 2) + 1 = 2\gamma(G)^2 - 4\gamma(G) + 3$. □

From Claim 2 we have $f(2) \leq 2\gamma(G)^2 - 4\gamma(G) + 3$. Since $E^2(n, \gamma)$ is minimized when $f(2)$ is maximized, substituting the maximum value of $f(2)$ into the bound of Lemma 6 gives a lower bound that holds for $n \geq 3\gamma(G)^2 - 2\gamma(G) + 3$. □

Our next lemma gives the desired upper bound.

Lemma 13 *If G is a $(\gamma, 2)$-insensitive graph with $\gamma(G) \geq 3$ and $n \geq \gamma(G)^2 + 1$, then*

$$E_2(n, \gamma) \leq \frac{5n - \gamma(G)^2 - 4\gamma(G) + 2}{2}.$$

Proof. For a fixed positve integer γ, we construct a graph G with order $n \geq \gamma^2 + 1$ and $\gamma(G) = \gamma$ as follows. Let A be a set of γ vertices labeled $a_1, a_2, ..., a_\gamma$. Let PA be the set of all possible pairs of vertices in A. For each pair $\{a_i, a_j\} \in PA - \{a_1, a_2\}$, add two vertices u_{ij} and v_{ij} such that $deg(u_{ij}) = deg(v_{ij}) = 2$ and $N(u_{ij}) = N(v_{ij}) = \{a_i, a_j\}$ for a total of $\gamma(\gamma - 1) - 2$ new vertices. Add a set B of $n - \gamma^2 + 2$ vertices labeled b_i, for $1 \leq i \leq n - \gamma^2 + 2$, and add the edges $a_1 b_i$, $a_2 b_i$ for $1 \leq i \leq n - \gamma^2 + 2$. First assume that $n - \gamma^2 > 1$. If $n - \gamma^2$ is even, we add edges $b_i b_{i+1}$ for $i = 1, 3, 5, \ldots, n - \gamma^2 + 1$ and if $n - \gamma^2$ is odd, add edges $b_i b_{i+1}$ for $i = 1, 3, \ldots, n - \gamma^2$ and $b_r b_{r+1}$ for $r = n - \gamma^2 + 1$. Then G has a total of $2\gamma^2 - 2\gamma - 4 + 2(n - \gamma^2 + 2) + \lceil (n - \gamma^2 + 2)/2 \rceil$ edges, which reduces to the bound of the theorem. Figures 3 and 4 illustrate the construction for $\gamma = 4$. For the special case of $n - \gamma^2 = 1$, $|B| = 3$ and the only additional edge is $b_1 b_2$. Thus, $deg(b_3) = 2$ and G has $\lfloor (5n - \gamma^2 - 4\gamma + 2)/2 \rfloor$ edges. Figure 5 gives an example of this construction for $n = 17$ and $\gamma = 4$. In the remainder of this proof, we shall refer to this case as the exceptional case.

Certainly, A dominates G, so $\gamma(G) \leq |A| = \gamma$. To see that any dominating set has at least γ vertices, observe that each set $\{a_i, a_j\} \cup \{u_{ij}, v_{ij}\}$ induces a C_4 and each C_4 requires two vertices to dominate it. Since each $\{a_i, a_j\}$ dominates at least as many vertices as $\{u_{ij}, v_{ij}\}$ does, it follows that A is a γ-set for G. It remains to be shown that $\gamma(G - e_1 - e_2) = \gamma(G)$ for arbitrary edges e_1 and e_2.

If each vertex in $V - A$ is adjacent to a vertex in A in $G' = G - e_1 - e_2$, then A dominates G' and the result follows. Thus we need only consider cases where a vertex, say x, has both its edges to A removed. If $x = u_{ij}$, then $S = A - \{a_i, a_j\} \cup \{x, v_{ij}\}$ dominates G'. Notice that S dominates B, since at least one of a_1 and a_2 is not in $N(u_{ij})$. If $x = b_3$ and we are discussing the exceptional case, then $A - \{a_1, a_2\} \cup \{b_1, b_3\}$ dominates G'. Finally, if $x = b_i$ and $deg(b_i) \geq 3$, then $A - \{a_2\} \cup \{b_j\}$ dominates G', where $b_j \in N(b_i)$. □

Since the upper and lower bounds differ by $O(\gamma^2)$, the asymptotic result of Theorem 11 follows.

We note that graphs constructed as described in the proof to Theorem 11 are generalizations of the extremal $(2,2)$-insensitive graphs of Theorem 3 that have $\lfloor (5n - 10)/2 \rfloor$ edges. Although we have not been able to prove that the graphs from Theorem 11 are extremal, both the fact that they are generalizations of extremal $(2,2)$-insensitive graphs and the asymptotic result imply they are promising candidates.

5 Bounds on $E_2(n, \gamma)$ for small values of n

We now consider small values of n. It is well-known that for any graph G without isolates, $\gamma(G) \leq n/2$. Hence, $E_k(n, \gamma)$ is not defined for $n < 2\gamma(G)$. For $n \geq 2\gamma(G)$, we derive lower bounds. Our first two lower bounds are straightforward and come from known results.

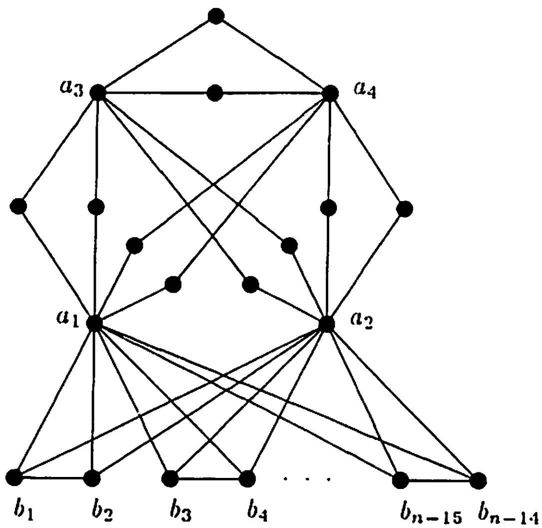

Figure 3: $n - \gamma(G)^2$ is even.

Studies of a related concept (called the bondage number) [1, 3] showed that no tree is domination insensitive relative to the removal of two or more edges, that is, they showed that if a tree T is (γ, k)-insensitive tree, then k must be 1. Hence, the (γ, k)-insensitive trees are precisely the trees that were characterized by Hartnell and Rall in [4]. Thus, for $k \geq 2$, we observe the following lower bound on $E_k(n, \gamma)$.

Observation 14 *If G is a (γ, k)-insensitive graph with $k \geq 2$, then $E_k(n, \gamma) \geq n$.*

Since for any (γ, k)-insensitive graph G with $k \geq 2$, $G - F$ is $(\gamma, 1)$-insensitive for any arbitrary set F of $k-1$ edges of G, our next lower bound follows from Theorem 2.

Proposition 15 *If G is a (γ, k)-insensitive graph with $\gamma(G) \geq 2$ and $k \geq 2$, then*

$$E_k(n, \gamma) \geq \begin{cases} n + k - 2 & \text{if } n \leq 3\gamma(G) - 2, \\ n + k - 1 & \text{if } n = 3\gamma(G) - 1, \\ 2n - 3\gamma(G) + k - 1 & \text{if } n \geq 3\gamma(G). \end{cases}$$

To present a slight improvement of the lower bound of n, we construct the only $(\gamma, 2)$-insensitive graphs having exactly n edges. A corona $G \circ K_1$

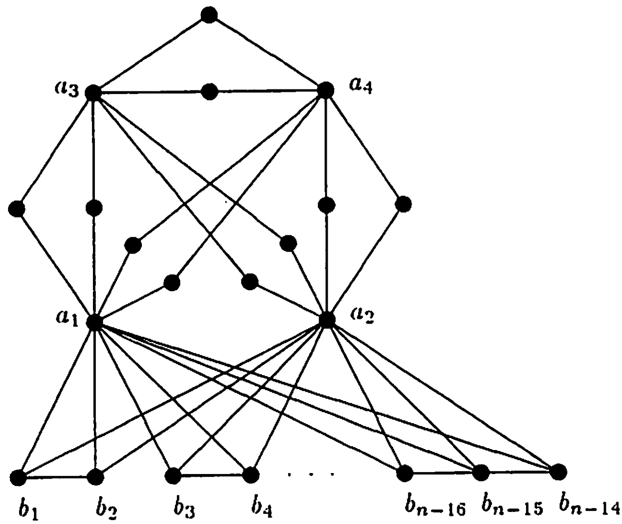

Figure 4: $n - \gamma(G)^2$ is odd.

is the graph obtained from a graph G, where for each vertex $v \in V(G)$, a new vertex v' and the edge vv' are added.

Theorem 16 *If G is a $(\gamma, 2)$-insensitive graph with $\gamma(G) \geq 3$, $n \neq 2\gamma(G)$ and $n \neq 3\gamma(G) - 2$, then $E_2(n, \gamma) \geq n + 1$.*

Proof. Let G be an extremal $(\gamma, 2)$-insensitive graph with $\gamma(G) \geq 3$. For $n \in \{2\gamma(G), 2\gamma(G) - 2\}$, we show that $E_2(n, \gamma) = n$.

Claim 3 *If $n = 2\gamma(G)$, then $E_2(n, \gamma) = n$.*

Proof. Since no tree it (γ, k)-insensitive for $k \geq 2$, any connected $(\gamma, 2)$-insensitive graph must have at least n edges. It suffices to find such a graph having the insensitive property with n edges. Consider the corona $G = C_t \circ K_1$. Clearly, $\gamma(G) = t = n/2$ and $|E(G)| = n$. All that remains to be shown is that G is $(\gamma, 2)$-insensitive. Let F be two arbitrary edges of $E(G)$ and let I be the set of isolated vertices in $G - F$. If $|I| = \emptyset$, then the vertices of the cycle (or the set of endvertices) dominate G.

Otherwise, a dominating set with cardinality t can be found that includes the vertices of I and vertices on the cycle which, in G, are not in $N(I)$. □

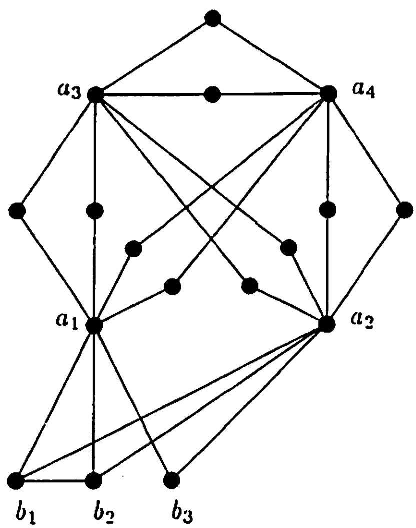

Figure 5: Exceptional case.

Claim 4 *If* $n = 3\gamma(G) - 2$, *then* $E_2(n,\gamma) = n$.

Proof. Again it suffices to give a $(\gamma, 2)$-insensitive graph having order $n = 3\gamma(G) - 2$ and n edges. Consider the cycle C_{3t-2} vertices. Clearly, C_{3t-2} has n edges and domination number $\gamma(C_{3t-2}) = t$. Remove two arbitrary edges e_1 and e_2 to create disjoint paths P_i and P_{n-i}, where $1 \leq i \leq n/2$. Then $\lceil i/3 \rceil$ vertices dominate P_i and $\lceil (n-i)/3 \rceil$ vertices dominate P_{n-i}. Since $\lceil i/3 \rceil + \lceil (3t-2i)/3 \rceil = t = \gamma(C_{3t-2})$, the cycle is $(\gamma, 2)$-insensitive. □

Next we show that the graphs in Claims 3 and 4 are the only $(\gamma, 2)$-insensitive graphs having n edges, and thereby establish that $n+1$ is a lower bound for $E_2(n,\gamma)$ when $n \neq 2\gamma(G)$ and $n \neq 3\gamma(G)-2$, $\gamma(G) \geq 3$. Since any graph having exactly n edges must be unicyclic, let G be a $(\gamma, 2)$-insensitive graph having n edges and a cycle subgraph C_t, and let T be a subtree rooted at a vertex v on the cycle. Let u be the endvertex on a longest path from v in T. If $d(u,v) \geq 2$ and w is the support vertex of u, then Proposition 8 implies that $deg(y) \geq 3$ for all $y \in N(w) - \{u\}$. But this is impossible since u is an endvertex on a longest path from v. Therefore, $d(u,v) = 1$ and Proposition 8 implies that u is the only endvertex in $N(v)$. Furthermore, Proposition 8 implies that either every vertex on the cycle is adjacent to an endvertex or no vertex on the cycle is adjacent to an endvertex. Since either the endvertex or its support vertex must be in any γ-set of G, the only situation where each vertex on the cycle is incident to a endvertex is the graph G of order $n = 2\gamma(G)$ as described in Claim 3. Therefore, in what is remaining, we consider only cycles $G = C_n$. If $n < 3t - 2$ or $n > 3t$, then $\gamma(G) \neq t$, a contradiction. Suppose that $n = 3t - 1$ or $3t$, and remove two adjacent edges to form $G' = P_1 \cup P_{n-1}$. Then $\gamma(G') = \gamma(G)+1$, contradicting the fact that G is $(\gamma, 2)$-insensitive. Hence the only unicyclic graphs that are $(\gamma, 2)$-insensitive are the ones described in Claims 3 and 4. □

Thus, $E_2(n, \gamma)$ is determined for graphs with order $n \leq 2\gamma(G)$ and $n = 3\gamma(G) - 2$. The next two theorems give upper bounds on $E_2(n, \gamma)$ for $2\gamma(G) + 1 \leq n \leq 3\gamma(G) - 3$ and $3\gamma(G) - 1 \leq n \leq \gamma(G)^2 - 1$.

Proposition 17 *If G is a $(\gamma, 2)$-insensitive graph with $2\gamma(G) + 1 \leq n \leq 3\gamma(G) - 3$, then $E_2(n, \gamma) \leq 2n - 2\gamma(G)$.*

Proof. Let γ be a fixed positive integer such that $2\gamma + 1 \leq n \leq 3\gamma - 3$. Note that $\gamma \geq 4$. We construct a $(\gamma, 2)$-insensitive graph G having $2n - 2\gamma$ edges as follows. Begin with $V = B \cup C$ where $B = \{b_1, b_2, ..., b_\gamma\}$ and $C = \{c_1, c_2, ..., c_{n-\gamma}\}$, such that, the vertices of C form a cycle and each $b_i \in B$ is adjacent to either one vertex or two adjacent vertices on the cycle and each vertex on the cycle has exactly one neighbor in B. Observe that this construction can always be accomplished since $\gamma + 1 \leq n - \gamma \leq 2\gamma - 3$. It remains to be shown that G is $(\gamma, 2)$-insensitive. Obviously, B dominates G and since $N[b_i] \cap N[b_j] = \emptyset$ for $i \neq j$, it follows that at least one vertex for each $b_i \in B$ must be in every γ-set. Thus, $\gamma(G) = \gamma$. Since $\gamma \geq 4$ implies that $n - \gamma \neq 4$, it is a simple exercise to see that any pair of edges can be removed from G with no effect on the domination number. □

Corollary 18 *If G is a $(\gamma, 2)$-insensitive graph with $n = 2\gamma(G) + 1$ and $\gamma(G) \geq 4$, then $E_2(n, \gamma) = n + 1$.*

Proposition 19 *If G is a $(\gamma, 2)$-insensitive graph with $3\gamma(G) - 1 \leq n \leq \gamma(G)^2 - 1$, then $E_2(n, \gamma) \leq 3n - 6\gamma(G) + 4$.*

Proof. We construct a $(\gamma, 2)$-insensitive graph G from a cycle C_{3t-2} as follows. Let c_1, c_2, and c_3 be three consecutive vertices on the cycle. Add $n - 3t + 2$ vertices such that each of them is adjacent to c_1, c_2, and c_3. Then G has $3n - 6t + 4$ edges. Obviously, t vertices dominate G and t vertices are necessary since any vertex dominates at most three vertices on the cycle. Hence, $\gamma(G) = t$ and it is straightforward to show that G is $(\gamma, 2)$-insensitive by considering the possible ways of removing any two edges. □

Next we consider graphs with order $n = \gamma(G)^2$. See Figure 6 for an example where $n = 16$ and $\gamma(G) = 4$.

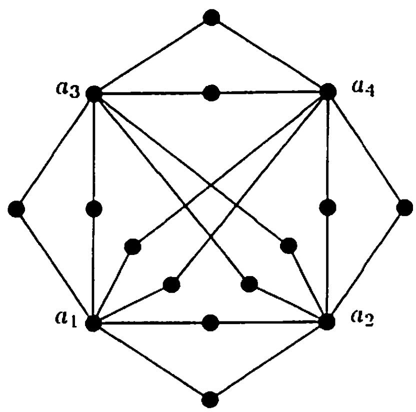

Figure 6: A (4, 2)-insensitive graph with $n = \gamma(G)^2 = 16$.

Proposition 20 *If G is a $(\gamma, 2)$-insensitive graph with $\gamma(G) \geq 3$ and $n = \gamma(G)^2$, then*

$$E_2(n, \gamma) \leq 2n - 2\gamma(G).$$

Proof. For a fixed positve integer γ, we construct a graph G with order $n = \gamma^2$ and $\gamma(G) = \gamma$ as follows. Let A be a set of γ vertices labeled $a_1, a_2, ..., a_\gamma$, and let PA be the set of all possible pairs of vertices in A. For each pair $\{a_i, a_j\} \in PA$, add two vertices u_{ij} and v_{ij} such that $deg(u_{ij}) = deg(v_{ij}) = 2$ and $N(u_{ij}) = N(v_{ij}) = \{a_i, a_j\}$ for a total of $\gamma(\gamma - 1)$ new vertices. An argument similar to the one in the proof of Lemma 13 shows that G has $\gamma(G) = \gamma$, $2n - 2\gamma(G)$ edges, and is $(\gamma, 2)$-insensitive. □

6 Concluding Remarks

Table 1 summarizes the results of this paper concerning $E_2(n, \gamma)$ for $\gamma(G) = \gamma \geq 3$ and all values of n. Let $g(\gamma) = 3\gamma^2 - 2\gamma + 3$ and $f(n, \gamma) = \lceil (5n - \gamma^2 - 4\gamma + 2)/2 \rceil$.

n	$E_2(n,\gamma)$	Difference in Bounds
	Lower, Upper	
$< 2\gamma$	undefined	—
2γ	$2\gamma,\ \ 2\gamma$	0
$2\gamma + 1$	$2\gamma + 2,\ \ 2\gamma + 2$	0
$2\gamma + 2 \leq n \leq 3\gamma - 3$	$n + 1,\ \ 2n - 2\gamma$	$O(\gamma)$
$3\gamma - 2$	$3\gamma - 2,\ \ 3\gamma - 2$	0
$3\gamma - 1$	$n + 1,\ \ n + 2$	1
$3\gamma \leq n \leq \gamma^2 - 1$	$2n - 3\gamma + 1,\ \ 3n - 6\gamma + 4$	$O(\gamma^2)$
γ_2	$2n - 3\gamma + 1,\ \ 2n - 2\gamma$	$O(\gamma)$
$\gamma^2 + 1 \leq n < g(\gamma)$	$2n - 3\gamma + 1,\ \ f(n,\gamma)$	$O(\gamma^2)$
$n \geq g(\gamma)$	$(5n - 3\gamma^2 - 4\gamma - 3)/2,\ \ f(n,\gamma)$	$O(\gamma^2)$

We conclude with open problems.

1. Determine $E_k(n,\gamma)$ for $k = 2$ and $\gamma \geq 3$.
2. Determine $E_k(n,\gamma)$ for $k \geq 3$ and $\gamma \geq 2$.
3. Determine properties of extremal (γ, k)-insensitive graphs G, such as the diameter and independent domination number. (We think that $\gamma(G)$ may equal $i(G)$ for such graphs.)
4. Study extremal (γ, k)-insensitive graphs G with the added restriction that G remains connected when k arbitrary edges are removed.
5. For a given value of k, characterize the extremal (γ, k)-insensitive graphs.

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