On Tournaments with Domination Number Exactly k

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Abstract

In this paper we establish that for arbitrary positive integers k and m, where $k > 1$, there exists a tournament which has exactly m minimum dominating sets of order k. A construction of such tournaments will be given.

Dedicated to Professor Stephen T. Hedetniemi on the occasion of his 60th birthday.

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1 Introduction

We will consider only finite simple digraphs and graphs (no multiple arcs or edges and no loops), and throughout this paper k will represent an arbitrary positive integer.

Although we will define some notation for digraphs, we will generally use standard notation for digraphs and graphs from [4]. For a digraph D, V(D) represents the set of vertices of D and A(D) represents the set of arcs of D. For $x,y \in V(D)$, $xy \in A(D)$ and $x \to y$ will be used interchangeably. In such a case, x is said to *dominate* y. A digraph T is called a *tournament* if for all distinct pairs $x,y \in V(T)$, exactly one of xy or $yx \in A(T)$. A tournament produced by randomly directing the edges of a complete graph is called a *random tournament.*
The concept of domination in graphs and digraphs is a well-studied concept [4]. We include some useful notation for our presentation. A *dominating set* S of D is a subset of V(D) such that for each $x \in V(D) - S$, $\exists\, y \in S$ such that $yx \in A(D)$. For the sake of brevity, we will refer to the digraph (subdigraph, subgraph, etc.) D and the set of vertices V(D) of D interchangeably as D, when the context is clear. The order of a minimum dominating set of D is called the *domination number* of D and is written $\gamma(D)$. For digraphs (or graphs) D_1 and D_2, $D_1 \cup D_2$ represents the graph with the following properties: $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$, and $xy \in A(D_1 \cup D_2)$ if and only if $xy \in A(D_1)$ or $xy \in A(D_2)$.

2 Pertinent Previous Results

In 1963, Erdös used probabilistic methods [1] on random tournaments to demonstrate the existence of tournaments with domination number greater than k for an arbitrary positive integer k. Nearly a decade later, in 1971, R.L.Graham and J.H.Spencer published a paper [2] outlining a method by which, for an arbitrary positive integer k, tournaments T could be constructed for which $\gamma(T) > k$. Furthermore, in [2] it was shown that if p is a prime, $p \equiv 3 \pmod 4$ and $p > k^2 2^{2k-2}$, then $\gamma(T_p) > k$ where T_p is the quadratic residue tournament on p vertices.

Therefore, for an arbitrary positive integer k, there exists a tournament T_p such that $\gamma(T_p)>k$ (in fact there are infinitely many such tournaments).

3 There exists a tournament with domination number exactly k

Theorem 1 *For every positive integer k there exists a tournament T such that $\gamma(T) = k$.*

Proof. Let T be a tournament such that $\gamma(T) > k$. We can be sure that such tournaments exist due to [1]. Note that deleting any vertex $x \in V(T)$ will result in a new tournament $T - \{x\}$ whose domination number may be greater than $\gamma(T)$ but cannot be smaller than $\gamma(T) - 1$. That is, $\forall x \in V(T), \gamma(T - \{x\}) \geq \gamma(T) - 1$. We prove this assertion by contradiction. Assume that $\exists T^*$ and $v \in V(T^*)$ such that $\gamma(T^* - \{v\}) < \gamma(T^*) - 1$. Let D be any digraph and $w \in V(D)$. Let S be a dominating set of $D - \{w\}$, then $S \cup \{w\}$ dominates D. It follows that $\gamma(D) \leq \gamma(D - \{w\}) + 1$. Therefore,

$$\gamma(T^*) \leq \gamma(T^* - \{v\}) + 1 < \gamma(T^*) - 1 + 1 = \gamma(T^*)$$

(contradiction). Clearly, we may continue deleting vertices of T until we arrive at a new tournament T′ with less than k vertices. Therefore, $\gamma(T') < k$. Since each deletion of a vertex can reduce the domination number of the resultant tournament by at most one, it follows that some intermediate tournament T_j, between T and T′, must have domination number exactly k. □

Unfortunately, the actual structure of the tournament is unknown. In the remainder of this paper we give a constructive proof of Theorem 1. Furthermore, our construction gives exactly m, $m \geq 1$, different minimum dominating sets.

4 The Inheritance Digraph

Let D be a digraph such that $|D| = n$, and for convenience denote $V(D) = \{(1,0), (2,0), \ldots, (n,0)\}$. We define the *Inheritance Digraph on g generations of the digraph D*, written $J_g(D)$, to be that digraph constructed as follows:

1. Generate g copies (generations) of D, and label them: D_1, D_2, ...,D_g. (D_i is called the i^{th} generation of D.) Label the vertex of D_i corresponding to (a,0) in D as (a,i), $\forall a$: $1 \leq a \leq n$ and $\forall i$: $1 \leq i \leq g$.

2. Let $\{(a,i) \rightarrow (b,j)$ in $J_g(D)\}$ if and only if $\{[(a,0) \rightarrow (b,0)$ in D , for $a \neq b]$ or $[\, i < j$, for $a = b]\}$. For D_j, we will call D_i a "previous generation" if and only if $i < j$.

Let $S \subset J_g(D)$. Define $\breve{S}$ as follows: $\{(a,1) \in \breve{S}\}$ if and only if $\{(a,i) \in S$ for some $i : 1 \leq i \leq g\}$. That is, $\breve{S}$ is the projection of S onto D_1. Note that $|\breve{S}| \leq |S|$. In this paper, we will use for sets A and B the notation $A \Rightarrow B$ to mean that set A dominates set B, that is to say for every $b \in B\text{-}A$ there is an element $a \in A$ such that a dominates b.

Lemma 1 *Let $S,H \subset J_g(D)$ for some digraph D. If $S \Rightarrow H$ then $\breve{S} \Rightarrow H$.*

Proof. Let D be a digraph such that $|D| = n$, and $V(D) = \{(1,0), (2,0), \ldots, (n,0)\}$. Also, let $S \Rightarrow H$, where $S,H \subset J_g(D)$. If $S - \breve{S} = \emptyset$, then clearly $\breve{S} \Rightarrow S$. If $S - \breve{S} \neq \emptyset$, then for all $(a,i) \in S - \breve{S}$, $i \neq 1$, it follows that $(a,1) \in \breve{S}$ and $(a,1) \rightarrow (a,i)$, so $\breve{S} \Rightarrow S$. If $H \subset S$, then we are done. Assume that $H \not\subset S$. Consider $(h,i) \in H - S$ for some i: $1 \leq i \leq g$. Since $(h,i) \in H - S$ and $S \Rightarrow H$, there exists $(b,j) \in S$ so that $(b,j) \rightarrow (h,i)$. If $b = h$ then $j<i$, and since (h,1) dominates all (h,i) for all $1 \leq i \leq g$, (h,1) dominates (h,i). If $b \neq h$ then $(b,0) \rightarrow (h,0)$ in D. This implies that $(b,1) \rightarrow (h,i)$ in $J_g(D)$, therefore, it follows that $\breve{S} \Rightarrow H$. ❑

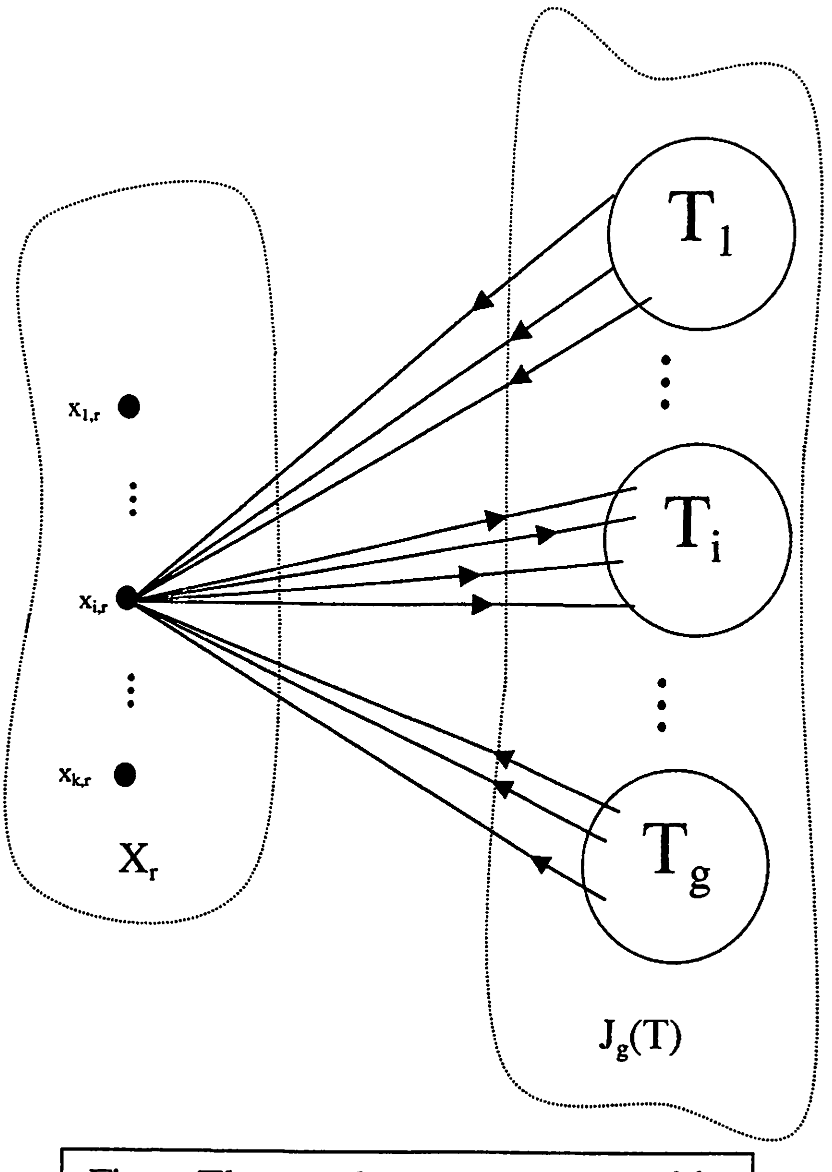

Fig. The arcs between $x_{i,r} \in X_r$, $\forall i$: $1 \leq i \leq k$, and $J_g(T)$.

Lemma 2 *Let D be a digraph and let $S \subset J_g(D)$. If $S \Rightarrow D_i$ for any i: $1 \le i \le g$ where $D_i \subset J_g(D)$, then $|S| \ge \gamma(D)$.*

Proof. Let D be a digraph such that $|D| = n$ and $\gamma(D) = k$. Let $V(D) = \{(1,0), (2,0), ..., (n,0)\}$. Let $S \subset J_g(D)$, such that $S \Rightarrow D_i$ for some i: $1 \le i \le g$. Then, by Lemma 1, $\breve{S} \Rightarrow D_i$. Assume $|S| < k$. This implies that $|\breve{S}| \le |S| < k$. But $|\breve{S}| < k$ implies that $\breve{S}$ cannot dominate D_1. Therefore, $\exists$ $(a,1) \in D_1$ such that $(a,1) \notin \breve{S}$ and $\breve{S}$ does not dominate $(a,1)$ in D_1. Therefore, $\breve{S}$ does not dominate (a,i), which implies, by Lemma 1, that S does not dominate (a,i), which is a contradiction. Therefore, $|S| \ge k = \gamma(D)$. ❑

Lemma 3 *Let D be a digraph. Then $\gamma[J_g(D)] = \gamma(D)$.*

Proof. Let D be a digraph, and let $S \subset J_g(D)$ be a minimum dominating set of $J_g(D)$. $S \Rightarrow J_g(D)$ implies that $S \Rightarrow D_1$, also. By Lemma 2, $|S| \ge \gamma(D)$. Therefore $\gamma[J_g(D)] \ge \gamma(D)$.

Let S_1 be a minimum dominating set of D_1. By definition, $|S_1| = \gamma(D_1) = \gamma(D)$. Clearly, $S_1 \Rightarrow \cup S_i$ (where S_i is the projection of S_1 onto D_i). Let $(a,i) \in D_i - S_i$, for some i: $1 \le i \le g$. Then $(a,1) \in D_1 - S_1$, which implies that there exists $(b,1) \in S_1$ such that $a \ne b$ and $(b,1) \rightarrow (a,1)$. Therefore $(b,1) \rightarrow (a,i)$ for all i: $1 \le i \le g$. This implies that $S_1 \Rightarrow \cup(D_i - S_i)$. Therefore $S_1 \Rightarrow J_g(D)$. This implies that $\gamma[J_g(D)] \le |S_1| = \gamma(D)$.

Therefore $\gamma[J_g(D)] = \gamma(D)$, for every positive integer g. ❑

Thus, the Inheritance Digraph on g generations of the digraph D, $J_g(D)$, has the following properties:

(1) $\gamma[J_g(D)] = \gamma(D)$ for all positive integers g,

and

(2) If $S \Rightarrow D_i$, for any i: $1 \le i \le g$, then $|S| \ge \gamma(D)$.

5 Construction of a tournament which has m minimum dominating sets of order k.

Theorem 2 *For all positive integers k and m, where k > 1, there exists a tournament with exactly m minimum dominating sets of order k.*

Proof. Let T be a tournament constructed by the Graham-Spencer method such that $\gamma(T) > k$.

Consider $J_k(T)$. Note that $J_k(T)$ is, of course, a tournament. We will construct a new tournament X to join with $J_k(T)$ to produce a new tournament T* with exactly m minimum dominating sets of order k. Let V(X) $= \{x_{1,1}, x_{1,2}, \dots, x_{1,m}, x_2, \dots, x_k\}$. Observe that $|V(X)| = k + m - 1$.

In order to simplify notation, we will now develop a number of conventions. For j: $2 \le j \le k$, let x_j also be written as $x_{j,r}$, $\forall r$: $1 \le r \le m$. Let $M = \{x_{1,1}, x_{1,2}, \dots, x_{1,m}\}$ and $X_r = \{x_{1,r}, x_2, \dots, x_k\} = \{x_{1,r}, x_{2,r}, \dots, x_{k,r}\}$.

Choose X to be any tournament for which $(X - M) \Rightarrow M$. This can clearly be done, for example let $x_2 \rightarrow x_{1,r}$, $\forall r$: $1 \le r \le m$ and choose all other arcs of X randomly.

Now define T*: $V(T^*) = V(X) \cup V(J_k(T))$. Let T* preserve all arcs of X and $J_k(T)$ and for every $x_{i,r} \in X$ and $y \in J_k(T)$, let $x_{i,r} \rightarrow y$ if and only if $y \in T_i$ (else $y \rightarrow x_{i,r}$) (see Fig.). Note that T* is a well-defined tournament.

Obviously, X_r is a dominating set of T* of order k, $\forall r$: $1 \le r \le m$. It only remains to show that T* contains no smaller dominating sets.

Let X* be any dominating set of T* which does not contain X_r, $\forall r$: $1 \le r \le m$.

Case I. $|X^* \cap M| = 0$. Then no element of T_1 is dominated by any element of $(X \cap X^*)$.

Case II. $|X^* \cap M| \neq 0$. Then $x_j \notin X^*$, for some j: $2 \leq j \leq k$, otherwise $x_{1,r} \in X^* \cap M$ would imply that $X_r \subset X^*$. Therefore, no element of T_j is dominated by any element of $(X \cap X^*)$.

Therefore, there exists t: $1 \leq t \leq k$ such that no element of T_t is dominated by any element of $(X \cap X^*)$. Note that $X^* \Rightarrow T^*$ implies that T_t is dominated by elements of $(J_k(T) \cap X^*) \subset J_k(T)$. But, by Lemma 2, $|J_k(T) \cap X^*| \geq \gamma(T) > k$. But this would imply that $|X^*| > k$.

This implies that S is a minimum dominating set of T^* if and only if it is X_r for some r: $1 \leq r \leq m$, and that T^* contains exactly m minimum dominating sets of order k. ❑

Note that this construction gives a constructive proof of the existence of a tournament with domination number k. Also, the case when $m = 1$ gives a tournament with a unique minimum dominating set of order k. Furthermore, in the case k=1, there are many tournaments, for example the transitive tournament, which have a single vertex dominate, but it is only possible to have one such dominating set in a tournament, since these vertices would have to dominate each other.

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