Hamiltonian Cycle Decomposition of Kronecker Product of Some Cubic Graphs by Cycles

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ABSTRACT. A graph is said h-decomposable if its edge-set is decomposable into hamiltonian cycles. In this paper, we prove that if $G = L_1 \cup L_2 \cup L_3$ is a strongly hamiltonian bipartite cubic graph (where L_i is a perfect matching, for $1 \leq i \leq 3$ and (L_1, L_2, L_3) is a 1-factorization of G), then $G \times C_{2n+1}$ (n odd and $n \geq 1$) is decomposable. As a corollary, we show that for $r \geq 1$ odd and $n \geq 3$, $K_{r,r} \times K_n$ is h-decomposable. Moreover, in the case where G is a strongly hamiltonian non-bipartite cubic graph, we prove that the same result can be derived using a special perfect matching. Hence $K_{2r} \times K_{2n+1}$ will be h-decomposable, for $r, n \geq 1$.

To study the product of $G = L_1 \cup L_2 \cup L_3$ by even cycle, we define a dual graph G_C based on an alternating cycle subset of $L_2 \cup L_3$. We show that if a non-bipartite cubic graph $G = L_1 \cup L_2 \cup L_3$, with $|V(G)| = 2m$, admits $L_1 \cup L_2$ as a hamiltonian cycle and G_C is connected, then $G \times K_2$ is hamiltonian and $G \times C_{2n}$ has two edge-disjoint hamiltonian cycles. Finally, we prove that if $C = L_2 \cup L_3$ and $L_1 \cup L_3$ admits a particular alternating 4-cycle C', then $G \times C_{2n}$ is h-decomposable.

1 Introduction

The Kronecker product $G \times H$ of two graphs G and H is a graph K with vertex set $V(K) = V(G) \times V(H)$ and edge set $E(K) = E(G \times H) = \{(u_1, u_2)\ (v_1, v_2) : u_1 v_1 \in E(G) \text{ and } u_2 v_2 \in E(H)\}$. This implies that $d_{G \times H}(x, y) = d_G(x) \times d_H(y)$ (with $x \in V(G)$ and $y \in V(H)$). It is well-known that the Kronecker product is commutative and distributive with respect to the edge-disjoint union of graphs [2].

A graph G is said to be *decomposable* into hamiltonian cycles, i.e. h-decomposable, if its set of edges can be partitioned into edge disjoint hamiltonian cycles. A *linear factorization* of a graph $G = (V, E)$ is a partition of E, $E = L_1 \cup L_2 \cup \ldots \cup L_k$, such that for each $i = 1, 2, \ldots, k$, L_i is a linear factor of G (or a perfect matching) [3]. A graph $G = L_1 \cup L_2 \cup, \ldots \cup L_k$ is said *strongly hamiltonain* if and only if for each $1 \leq i \neq j \leq k$, $C_{i,j} = L_i \cup L_j$ is a hamiltonian cycle of G [3]. We denote an even and odd cycle respectively by C_{2n} and C_{2n+1}.

Let now $G = L_1 \cup L_2 \cup L_3$ be a cubic graph with $|V(G)| = 2m$. Let $C_{1,2}$ be a hamiltonian cycle with origin vertex z_0. Let $\sigma_{12} : V(G) \rightarrow \{0, 1\}$ be a mapping relative to the cycle $C_{1,2} = L_1 \cup L_2$, such that the vertices of $C_{1,2}$ are labeled alternatively by 0 and 1. By convention, we choose $\sigma_{12}(z_0) = 0$. In what follows, we say that$[x, y]$ is a *one-parity* (resp. *bi-parity*) edge (relatively to $C_{1,2}$) if $\sigma_{12}(x) = \sigma_{12}(y)$ (resp. $\sigma_{12}(x) \neq \sigma_{12}(y)$). We remark that if $[x, y] \in L_3$ is a one parity edge, then the end-vertices of each edge of $[x, y] \times C_{2n}$ are adjacent to the two components of $C_{1,2} \times C_{2n}$ in $G \times C_{2n}$, otherwise they belong to one only component of $C_{1,2} \times C_{2n}$.

A general problem on the Kronecker product is the following:

"If G_1 has a decomposition into hamiltonian cycles and a 1-factor, and G_2 has a decomposition into hamiltonian cycles, then does $G_1 \times G_2$ have a decomposition into hamiltonian cycles?"

The problem is more difficult to solve when $|V(G_2)|$ is even and G_1 is a $(2k+1)$-regular graph. Muthusamy and Paulraja proved that $K_{2r} \times C_{2n+1}$ is h-decomposable [4], by using Walecki's construction for the complete graph. Agnihotri and al. [1] introduced the notion of an alternating 4-cycle for decomposing into hamiltonian cycles the Kronecker product of the cycle C_{2n} and a 4-regular graph. In this paper, we solve this problem for the cubic graphs which allow to decompose the Kronecker product of a large family of graphs by cycles. In doing so we introduce the general notion of an alternating cycle, which is a cycle whose two perfect matchings or a cycle and a perfect matching.

The remainder of this paper is organized as follows. We start, in section 2, by recalling some important results on the Kronecker product. In Section 3, we prove that the product of a strongly hamiltonian bipartite cubic graph and an odd cycle is decomposable into three hamiltonian cycles. Consequently we show that, the Kronecker product of any complete bipartite graph with an odd cycle is h-decomposable. From this, we deduce that for every $r, n \geq 1$, $K_{r,r} \times K_{2n+1}$ is h-decomposable. Subsequently we establish another result for a strongly hamiltonian non-bipartite cubic graph $G = L_1 \cup L_2 \cup L_3$ which admits exactly two one-parity edges belonging to a perfect matching L_3, that $G \times C_n$ is h-decomposable. Finally we deduce that $K_{2r} \times K_{2n+1}$ is h-decomposable for $r, n \geq 1$.

The second part of section 3 is devoted to the Kronecker product of non-bipartite cubic graphs by even cycles. We introduce the notion of the *dual graph* G_C of an alternating cycle C. Let $G = L_1 \cup L_2 \cup L_3$ be a non-bipartite cubic graph, let $C_{1,2} = L_1 \cup L_2$ be a hamiltonian cycle and let $C = (x_0, x_1, x_2, \ldots, x_{k-1}, x_0)$ be an alternating cycle between $C_{1,2}$ and L_3. Let $g\colon V(C) \to \{0,1\}$ such that $g(x_0) = 0$ and

$$g(x_{i+1}) = \begin{cases} \overline{g(x_i)} & \text{if } [x_i, x_{i+1}] \in L_3 \text{ and } (\sigma_{12}(x_i) = \sigma_{12}(x_{i+1}) \\ g(x_i) & \text{otherwise} \end{cases}$$

for $i = 0, 1, 2 \ldots, k-2$.

We remark that if $[x, y] \in (L_1 \cup L_2) \cap E(C)$ then $g(x) = g(y)$. Let G_C be a graph associated to C and G such that $V(G_C) = V(G) \times \{0,1\}$ and

$$\begin{aligned} E(G_C) = &\{[(x, g(x))(y, g(y))] : [x, y] \in (E(C) \cap L_3)\} && E_1 \\ &\cup \{[(x, i)(y, i)] : i \in \{0,1\}, [x, y] \in (L_1 \cup L_2) \backslash E(C))\} && E_2 \\ &\cup \{[(x, i)(y, i)] : i \neq g(x), [x, y] \in (L_1 \cup L_2) \cap E(C))\} && E_3 \end{aligned}$$

G_C is said to be a *dual graph* of C (relatively to G). We may note that G_C is a 2-factor (see Lemma 3.3). We show that, if $G = L_1 \cup L_2 \cup L_3$ is a non-bipartite cubic graph and G_C is a connected dual graph of an alternating cycle C between $C_{1,2}$ and L_3, then $G \times K_2$ is hamiltonian. In particular, if $C \subseteq L_2 \cup L_3$ and G_C is connected, then $G \times C_{2n}$ has two disjoint hamiltonian cycles. Furthermore, if there exists an alternating cycle C' of size 4 between L_1 and L_3 such that all edges of L_3 are one-parity, then $G \times C_{2n}$ is h-decomposable.

Finally, we conclude with the description of three classes of graphs satisfying the hypotheses of this last result.

2 Preliminaries

In order to make our paper self contained, we recall in this section some previous results which characterize the connectivity of the Kronecker product of two graphs [2].

Theorem 2.1 [2]. *Let G and H be nontrivial connected graphs. If G and H are both bipartite then the graph $G \times H$ consists of exactly two connected components, otherwise it is connected.*

Theorem 2.2 [2]. *For $s, r \geq 3$, the graph $C_s \times C_r$ admits a hamiltonian decomposition if and only if either s or r is odd. If s and r are both even, then $C_s \times C_r$ consists of two isomorphic connected components, each of which admits a hamiltonian decomposition (Figure 5).*

3 Main Results

In what follows, we consider $C_{1,2} = L_1 \cup L_2 = (z_0, z_1, \ldots, z_{2m-1}, z_0)$ to be a hamiltonian cycle in $G = L_1 \cup L_2 \cup L_3$ such that $L_1 = \{[z_0, z_1], [z_2, z_3], \ldots, [z_{2m-2}, z_{2m-1}]\}$ and $L_2 = \{[z_1, z_2], [z_3, z_4], \ldots, [z_{2m-1}, z_0]\}$.

3.1 Kronecker product of a strongly hamiltonian cubic graph by an odd cycle

The purpose of this section is to present the h-decomposition of $G \times C_{2n+1}$. We discuss independently the bipartite and non-bipartite cases.

A. Bipartite Case

Theorem 3.1 *If G is a strongly hamiltonian bipartite cubic graph, then $G \times C_{2n+1}$ is h-decomposable.*

Proof: Let $G = (X, Y, E)$. Let $H = (X, Y, E')$ be a multigraph defined as follows: each edge $[x, y]$ of G is duplicated into two edges $[x, y]$ colored respectively by α_1 and α_2. We define $C_{ij}(\alpha_1, \alpha_2)$ to be the hamiltonian cycle alternating between the edges of L_i with color α_1 and those of L_j with color α_2, for $1 \leq i \neq j \leq 3$. We can associate a hamiltonian decomposition H_d of H to a linear factorization $D = (L_1, L_2, L_3)$ of G as follows: $H_d = (C_{1,2}(\alpha_1, \alpha_2), C_{1,3}(\alpha_2, \alpha_1), C_{2,3}(\alpha_1, \alpha_2))$ (Figure 1.a). We remark that $C_{1,2}(\alpha_1, \alpha_2), C_{1,3}(\alpha_2, \alpha_1)$ and $C_{2,3}(\alpha_1, \alpha_2)$ are pairwise edge disjoint.

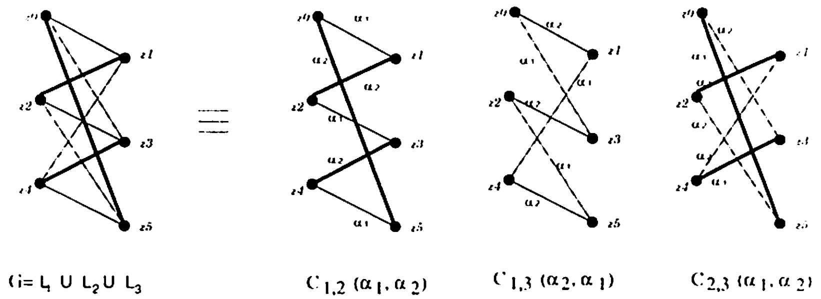

Figure 1. a
The decomposition H_d corresponding to the linear factorization of a graph G of order 6

Let $C_{2n+1} = (u_0, u_1, \ldots, u_{2n}, u_0)$. Let ϕ be a function which associates to each edge $[x, y]$ with color α_2 in H an edge in $G \times C_{2n+1}$ and to each edge $[x, y]$ with color α_1 a path in $G \times C_{2n+1}$, which is $([x, y] \times C_{2n+1}) \backslash \phi[[x, y]; \alpha_2]$.

Here $[[x, y]; \alpha_k]$ stands for an edge $[x, y]$ with color α_k. In other words,

$$
\begin{aligned}
\phi: \qquad E(H) &\rightarrow E(G \times C_{2n+1}) \\
[[x, y]; \alpha_1] &\rightarrow [(x, u_0) \dots (y, u_{2n})] := [(x, u_o)(y, u_1) \\
& \qquad (x, u_2)(y, u_3) \dots (y, u_{2n-1})(x, u_{2n}) \\
& \qquad (y, u_0)(x, u_1) \dots (x, u_{2n-1})(y, u_{2n})] \\
[[x, y]; \alpha_2] &\rightarrow [(x, u_0)(y, u_{2n})],
\end{aligned}
$$

for all $x \in X$ and $y \in Y$.

The path $[(x, u_0) \dots (y, u_{2n})]$ is a hamiltonian path of $[x, y] \times C_{2n+1}$. Since each $C_{ij}(\alpha_1, \alpha_2)$ alternates between edges $(e_p; \alpha_1)$ and $(e_{p+1}; \alpha_2)$ with e_p and e_{p+1} edges of H, then $\phi(C_{ij}(\alpha_1, \alpha_2))$ will alternate between the path $\phi(e_p, \alpha_1)$ and the edge $\phi(e_{p+1}, \alpha_2)$. It is easy to see that $\phi(C_{ij}(\alpha_1, \alpha_2))$ is a hamiltonian cycle of $G \times C_{2n+1}$, it covers all vertices of the product and it is connected. Note that ϕ is an injective correspondence, since $C_{1,2}(\alpha_1\alpha_2)$, $C_{1,3}(\alpha_2\alpha_1)$ and $C_{2,3}(\alpha_1\alpha_2)$ are pairwise edge-disjoint, thus $\phi(C_{1,2}(\alpha_1, \alpha_2)), \phi(C_{1,3}(\alpha_2, \alpha_1))$ and $\phi(C_{2,3}(\alpha_1, \alpha_2))$ are also pairwise edge-disjoint. The product $G \times C_{2n+1}$ will be then decomposed into three edge-disjoint hamiltonian cycles (H_1, H_2, H_3) (Figure 1.b). □

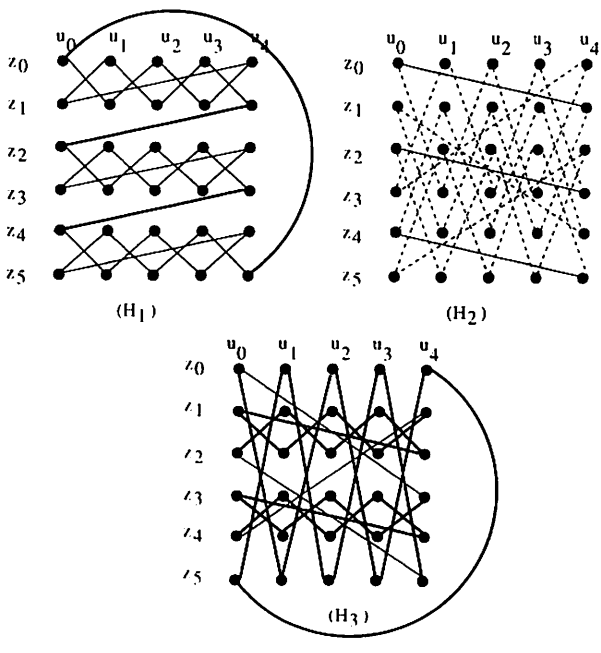

Figure 1. b
The three edge-disjoint hamiltonian cycles of $G \times C_5$ deduced from H_d.

Corollary 3.1.1 $K_{r,r} \times C_{2n+1}$ *is h-decomposable for* $r,\ n \geq 1$.

Proof: Case 1: r is odd.

Let $K_{r,r} = (A, B, E)$, such that $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_r\}$. Wallis [5] proved that $K_{r,r}$ admits a linear factorization into r one-factors, $D = (L_1, L_2, \ldots, L_r)$, such that $L_i = \{[a_j, b_{j-i+1}];\ j = 1, \ldots, r\}$; $(j-i+1)$ being taken as integers modulo r in the range $[1 \ldots r]$. It is easy to see that $L_i \cup L_{i+1} = (a_j, b_{j-i+1}, a_{j+1}, b_{j+1-i+1}, a_{j+2}, \ldots, a_{j+2r}, b_{j+2r-i+1}, a_j)$ is a hamiltonian cycle because we visit a_j the second time, crossing $L_i \cup L_{i+1}$, after r steps. Using the definition of L_i and since $|X| = |Y|$ is odd, we show that L_i is also a hamiltonian cycle. Consequently, $K_{r,r} = \{C_{1,2} \cup C_{3,4} \cup \ldots \cup C_{r-2,r-1} \cup L_r\}$. Furthermore, the product of each even cycle by C_{2n+1} is h-decomposable (Theorem 2.2), and $C_{r-2,r-1} \cup L_r$ is a strongly hamiltonian bipartite cubic graph, thus by Theorem 3.1, $K_{r,r} \times C_{2n+1}$ is h-decomposable.

Case 2: r is even.

$K_{r,r}$ is decomposable into $r/2$ even edge-disjoint hamiltonian cycles, each given by two consecutive 1-factors. Thus by theorem 2.2, $K_{r,r} \times C_{2n+1}$ is then h-decomposable. □

Corollary 3.1.2 $K_{r,r} \times K_{2n+1}$ *is h-decomposable for* $r, n \geq 1$.

Proof: It is well known that K_{2n+1} is decomposable into n odd hamiltonian cycles, then $K_{r,r} \times K_{2n+1} = K_{r,r} \times \{H_1 \cup H_2 \cup \ldots \cup H_n\}$, where $(H_i)_{1 \leq i \leq n}$ is a hamiltonian decomposition of K_{2n+1}. The result is an easy consequence of the above corollary. □

B. Non-Bipartite Case

We prove the following result, using another construction based on a special perfect matching L_3.

Theorem 3.2 *Let* $G = L_1 \cup L_2 \cup L_3$ *be a strongly hamiltonian non-bipartite cubic graph. Suppose there exist integers* h, t, q *and* t, *such that* $L_3 = \{[z_{2h}, z_{2p}], [z_{2t+1}, z_{2q+1}]\} \cup \{[z_i, z_j] : 0 \leq i \neq j \leq 2m-1$ *and* $i, j \neq 2p, 2q+1, 2h, 2t+1;\ \sigma_{12}(z_i) \neq \sigma_{12}(z_j)\ \}$, *then* $G \times C_{2n+1}$ *is h-decomposable.*

Proof: We construct three hamiltonian cycles in the product deduced from the three hamiltonian cycles in G given by the strongly hamiltonian property of G.

In order to construct the first hamiltonian cycle H_1 in $G \times C_{2n+1}$ from $L_1 \cup L_2$, we define an auxiliary cycle Γ_1.

Let Γ_1 be a cycle constructed between columns u_0 and u_{2n} from $C_{1,2} = (z_0, z_1, \ldots, z_{2m-1}, z_0)$, by duplicating each vertex z_i of $C_{1,2}$ into two adjacent vertices (z_i, u_0) and (z_i, u_{2n}). Thus, $\Gamma_1 = ((z_0, u_0), (z_0, u_{2n}), (z_1, u_0), (z_1, u_{2n}) \ldots, (z_{2m}, u_{2n}), (z_0, u_0))$ (Figure 3.a).

Let

$$\begin{aligned}\psi_1 \qquad & E(\Gamma_1) \rightarrow E(G\times C_{2n+1})\\ [(z_i,u_0)(z_i,u_{2n})] & \rightarrow [(z_i,u_0)\dots(z_i,u_{2n})] := [(z_i,u_0)(z_j,u_1)\\ & \qquad (z_i,u_2)\dots(z_j,u_{2n-1})(z_i,u_{2n})]\\ [(z_i,u_{2n})(z_{i+1},u_0)] & \rightarrow [(z_i,u_{2n})(z_{i+1},u_0)]\end{aligned}$$

For each $z_i \in V(C_{1,2})$ and $[z_i, z_j] \in L_3$.

$H_1 = \psi_1(\Gamma_1)$ covers all vertices of $G \times C_{2n+1}$ because for each $[z_i, z_j]$ of L_3, $\psi_1([(z_i,u_0)(z_i,u_{2n})]) \cup \psi_1([z_j,u_0)(z_j,u_{2n})])$ is the product $([z_i,z_j] \times C_n)\backslash\{[(z_i,u_0)(z_j,u_{2n})],[(z_i,u_{2n})(z_j,u_0)]\}$ (Figure 3 (b.1)). Furthermore, since each horizontal edge of Γ_1 is replaced by a path in H_1 then H_1 is an extension of Γ_1 which is a cycle, so H_1 will be connected.

The second hamiltonian cycle in $G \times C_{2n+1}$ is based on $L_1 \cup L_3$. We prove that for the vertices x_{2t+1} and x_{2h} (or x_{2q+1} and x_{2p}), we associate two vertex-disjoint paths $P_{z_{2t+1}}$ and $P'_{z_{2h}}$ which cover all internal vertices of $G \times C_{2n+1}$.

Lemma 3.2.1 *There exist two vertex-disjoint paths $P_{z_{2t+1}}$ and $P'_{z_{2h}}$ in $(L_1 \cup L_2) \times C_{2n+1}$ such that $P_{z_{2t+1}}$ covers all odd vertices of even columns and even vertices of odd columns (with ends (z_{2t+1},u_0) and (z_{2t+1},u_{2n})) except the odd vertices of u_{2n}. The second path $P'_{z_{2h}}$ covers all odd vertices of odd columns and even vertices of even columns (with ends (z_{2h},u_0) and (z_{2h},u_{2n})), except the even vertices of u_0.*

Proof: The two paths are constructed as follows.

$$\begin{aligned}P_{z_{2t+1}} &= [(z_{2t+1},u_0)\dots(z_{2t+1},u_{2n})]\\ &:= \bigcup_{r=0}^{n-1}\Big(\bigcup_{i=t}^{t+m-2}([(z_{2i+1},u_{2r})(z_{2i+2},u_{2r+1})]\\ &\quad \cup [(z_{2i+2},u_{2r+1})(z_{2i+3},u_{2r})]) \cup [(z_{2t-1},u_{2r})(z_{2t},u_{2r+1})]\\ &\quad \cup [(z_{2t},u_{2r+1})(z_{2t+1},u_{2r+2})]\Big)\end{aligned}$$

$$\begin{aligned}P'_{z_{2h}} &= [(z_{2h},u_0)\dots(z_{2h},u_{2n})] := \bigcup_{r=0}^{n-1}([(z_{2h},u_{2r})(z_{2h+1},u_{2r+1})]\\ &\quad \bigcup_{i=h}^{h+m-2}([(z_{2i+1},u_{2r+1})(z_{2i+2},u_{2r+2})]\\ &\quad \cup [(z_{2i+2},u_{2r+2})(z_{2i+3},u_{2r+1})]) \cup [(z_{2h-1},u_{2r+1})(z_{2h},u_{2r+2})])\end{aligned}$$

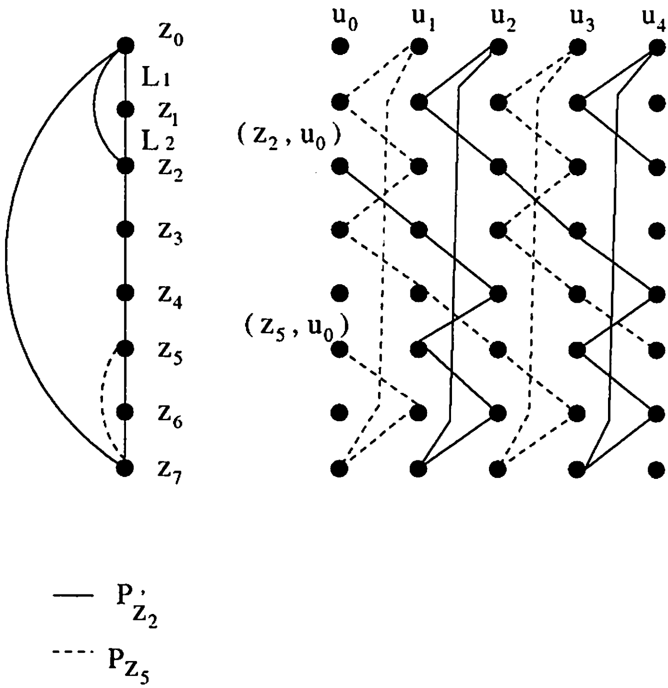

Figure 2. P'_{z_2} and P_{z_5} in $(C_{1,2} \cup \{[z_0, z_2], [z_5.z_7]\}) \times C_5$

The paths $P_{z_{2t+1}}$ and $P'_{z_{2h}}$ are vertex-disjoint (Figure 2), because $P'_{z_{2h}}$ is a right translation of $P_{z_{2t+1}}$ with a column i.e. the path $P'_{z_{2h}}$ is the union of edge $[(z_{2h}, u_0)(z_{2h+1}, u_1)]$ and $P_{z_{2h+1}}$ from u_1 to u_{2n}. These paths cover all internal vertices (each vertex has degree 2) of $G \times C_{2n+1}$ from columns u_1 to u_{2n-1} except

$$A = \{(z_{2i}, u_0) \ : \ 0 \leq i \leq m-1\} \cup \{(z_{2t+1}, u_0)\}$$
$$B = \{(z_{2i+1}, u_{2n}) \ : \ 0 \leq i \leq m-1\} \cup \{(z_{2h}, u_{2n})\}.$$

□

Consider now $C_{3,1} = (z_{2h}, z_{2p}, z_{2p+1}, \dots, z_{2q+1}, z_{2t+1}, \dots, z_{2h})$. Similarly we construct Γ_2 between columns u_0 and u_{2n} from $C_{3,1}$, by replacing z_{2h} by $e_0 = [(z_{2h}, u_0)(z_{2h}, u_{2n})]$ and z_{2t+1} by $e_1 = [(z_{2t+1}, u_0)(z_{2t+1}, u_{2n})]$. It follows that $\Gamma_2 = ((z_{2h}, u_{2n}), (z_{2p}, u_0), (z_{2p+1}, u_{2n}), \dots, (z_{2q+1}, u_{2n}), (z_{2t+1}, u_0), (z_{2t+1}, u_{2n}) \dots, (z_{2h}, u_0), (z_{2h}, u_{2n}))$[1]. It is easy to see that the

[1]If $C_{3,1} = (z_{2h}, z_{2p}, z_{2p+1}, \dots, z_{2t+1}, z_{2q+1}, \dots, z_{2h})$ then we choose $\Gamma_2 = ((z_{2h}, u_{2n}), (z_{2p}, u_0), (z_{2p+1}, u_{2n}), \dots, (z_{2t+1}, u_{2n}), (z_{2t+1}, u_0), (z_{2q+1}, u_{2n}), \dots, (z_{2h}, u_0), (z_{2h}, u_{2n}))$.

sets of vertices of columns u_0 and u_{2n} covered by Γ_2 are respectively A and B. This gives immediately the following Lemma.

Lemma 3.2.2 $H_2 = (P_{z_{2t+1}} \cup P'_{z_{2h}}) \cup (\Gamma_2 - \{e_0, e_1\})$ *is a hamiltonian cycle in* $G \times C_{2n+1}$.

Proof: By Lemma 3.2.1, $P_{z_{2t+1}}$ and $P'_{z_{2h}}$ are two vertex-disjoint paths which cover all internal vertices of $G \times C_{2n+1}$. The cycle Γ_2 covers all vertices of A and B (Figure 3.(b.2)), thus H_2 is a hamiltonian cycle of $G \times C_{2n+1}$. □

Finally, we construct the third hamiltonian cycle in $G \times C_{2n+1}$ from $L_2 \cup L_3$.

Let $C_{2n+1} = (y_0, y_1, \dots, y_{2n}, y_0)$ such that $y_k = u_{2n-k}$ with $0 \leq k \leq 2n$. Let $Q_{z_{2t+1}}$ and $Q'_{z_{2h}}$ be two vertex-disjoint paths from y_0 to y_{2n} constructed respectively as $P_{z_{2t+1}}$ and $P'_{z_{2h}}$. Let $D = \{(z_{2i+1}, u_0) : 0 \leq i \leq m-1\} \cup \{(z_{2h}, u_0)\}$ and $E = \{(z_{2i}, u_{2n}) : 0 \leq i \leq m-1\} \cup \{(z_{2t+1}, u_{2n})\}$ be the sets of vertices of columns u_0 and u_{2n} respectively not covered by $Q_{z_{2t+1}}$ and $Q'_{z_{2h}}$. Let $C_{3,2} = (z_{2h}, z_{2p}, z_{2p-1}, \dots, z_{2q+1}, z_{2t+1}, \dots, z_{2h})$. We construct Γ_3 from $C_{3,2}$ between D and E by replacing z_{2h} by $e_0 = [(z_{2h}, u_0), (z_{2h}, u_{2n})]$ and z_{2t+1} by $e_1 = [(z_{2t+1}, u_0), (z_{2t+1}, u_{2n})]$. This implies that $\Gamma_3 = ((z_{2h}, u_0), (z_{2p}, u_{2n}), (z_{2p-1}, u_0), \dots, (z_{2q+1}, u_0), (z_{2t+1}, u_{2n}), (z_{2t+1}, u_0), \dots, (z_{2h}, u_0))$[2]. Then H_3 obtained as $(Q_{z_{2t+1}} \cup Q'_{z_{2h}}) \cup (\Gamma_3 - \{e_0, e_1\})$ is a hamiltonian cycle in $G \times C_{2n+1}$ for the same reason as Lemma 3.2.2.

The hamiltonian cycle H_1 constructed from $C_{1,2}$ uses the edges of $L_1 \times C_{2n+1}$ and $L_2 \times C_{2n+1}$ of the form $[(z_i, u_{2n}), (z_{i+1}, u_0)]$ and the edges of $L_3 \times C_{2n+1} \setminus \{[(z_i, u_0), (z_j, u_{2n})] : [z_i, z_j[\in L_3\}$. However the edges of the form $[(z_i, u_0), (z_{i+1}, u_{2n})]$ of $C_{1,2} \times C_{2n+1}$ and those of the form $[(z_i, u_0), (z_j, u_{2n})]$ of $L_3 \times C_{2n+1}$ belong either to H_2 constructed from $C_{3,1}$ or to H_3 constructed from $C_{3,2}$ (Figure 3). Moreover the paths $(P_{z_{2t+1}} \cup P'_{z_{2h}})$ and $(Q_{z_{2t+1}} \cup Q'_{z_{2h}})$ are in the opposite order of columns, thus they are disjoint. Hence, the three hamiltonian cycles of $G \times C_{2n+1}$ are edge-disjoint. □

Example. Let $|V(G)| = 6$, $n = 2$, $h = 0$ and $t = 2$ (Figure 3).

From this theorem we deduce.

Corollary 3.2.3 $K_{2r} \times K_{2n+1}$ *is h-decomposable, for* $r, n \geq 1$. □

Remark. We note that the 1-factor used by Muthusamy and Paulraja [4] for decomposing $K_{2r} \times C_{2n+1}$ is a special case of our perfect matching L_3.

[2] As in [1].

The second part of this paper studies the h-decomposition of $G \times C_{2n}$. We consider G a non-bipartite graph. Otherwise, by Theorem 2.1 the product is not connected.

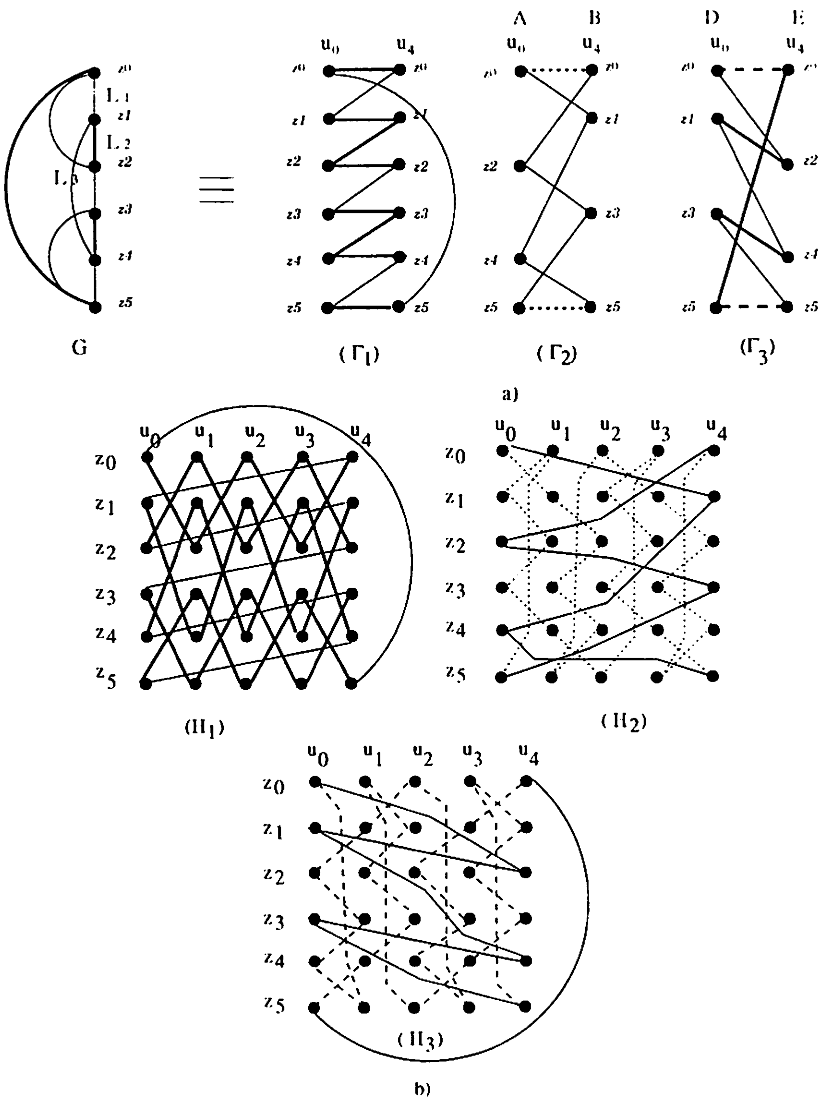

Figure 3. a) The cycles Γ_1, Γ_2 and Γ_3 correspond respectively to $C_{1,2}$, $C_{3,1}$ and $C_{3,2}$, b) The three edge-disjoint hamiltonian cycles of $G \times C_5$.

3.2 Kronecker product of non-bipartite cubic graphs by an even cycle

The connectedness of the dual graph introduced in this paper ensures the hamiltonicity of the product of non-bipartite cubic graph by an even cycle. Note that in the figures of section 3.2, the dotted lines represent the removing edges. We begin with this lemma.

Lemma 3.3 *The dual graph G_C is a 2-factor.*

Proof: Let (x, d) be any vertex of $V(G_C)$ with $d \in \{0,1\}$. Let y_1, y_2, y_3 the neighbors of x in G such that $[x, y_i] \in L_i$ in G for $1 \leq i \leq 3$. We consider two cases:
Case 1: If $x \notin V(C)$, then $[x, y_3] \notin E(C) \cap L_3$ and $\{[x, y_1], [x, y_2]\} \not\subset (L_1 \cup L_2) \cap E(C)$. So $[(x, d), (y_i, d_i')] \notin E_1 \cup E_3$, for $1 \leq i \leq 3$ and $d_i' \in \{0,1\}$. It remains that $\{[(x, d), (y_1, d)], [(x, d), (y_2, d)]\} \subset E_2$. Then $d_{G_C}((x, d)) = 2$.
Case 2: If $x \in V(C)$, then one and exactly one of the edges $[x, y_1]$ and $[x, y_2]$ belongs to $(L_1 \cup L_2) \cap E(C)$. Assume that $[x, y_2] \in (L_1 \cup L_2) \cap E(C)$. So $[(x, d), (y_2, d] \in E_2$. Furthermore, $[x, y_3] \in E(C) \cap L_3$, thus $[(x, d), (y_3, g(y_3)] \in E_1$, $d_{G_C}((x, g(x))) = 2$. □

Theorem 3.4 *Let $G = L_1 \cup L_2 \cup L_3$ be a non-bipartite cubic graph such that:*

- *$L_1 \cup L_2$ is a hamiltonian cycle.*
- *C is an alternating cycle between $C_{1,2}$ and L_3 such that G_C is connected.*

Then $G \times K_2$ is hamiltonian.

Proof: Since G_C is connected then by Lemma 3.3 it is a hamiltonian cycle of order $4m$ ($|V(G)| = 2m$). So let $G_C = ((x_0, d_0), (x_1, d_1), (x_2, d_2), \ldots, (x_{4m-1}, d_{4m-1}), (x_0, d_0))$ where $d_i \in \{0,1\}$ deduced from the definition of G_C. The hamiltonian cycle H in $G \times K_2$ is obtained from G_C as follows: $H = ((x_0, u_0), (x_1, u_1), (x_2, u_0), \ldots, (x_{4m-1}, u_1), (x_0, u_0))$, if $K_2 = [u_0, u_1]$ see (Figure 4).

Corollary 3.4.1 *Let $G = L_1 \cup L_2 \cup L_3$ be a non-bipartite cubic graph such that:*

- *$L_1 \cup L_2$ is a hamiltonian cycle.*
- *C is an alternating cycle of size 4 between $C_{1,2}$ and L_3 such that all edges of L_3 are one-parity edges.*

Then $G \times K_2$ is hamiltonian.

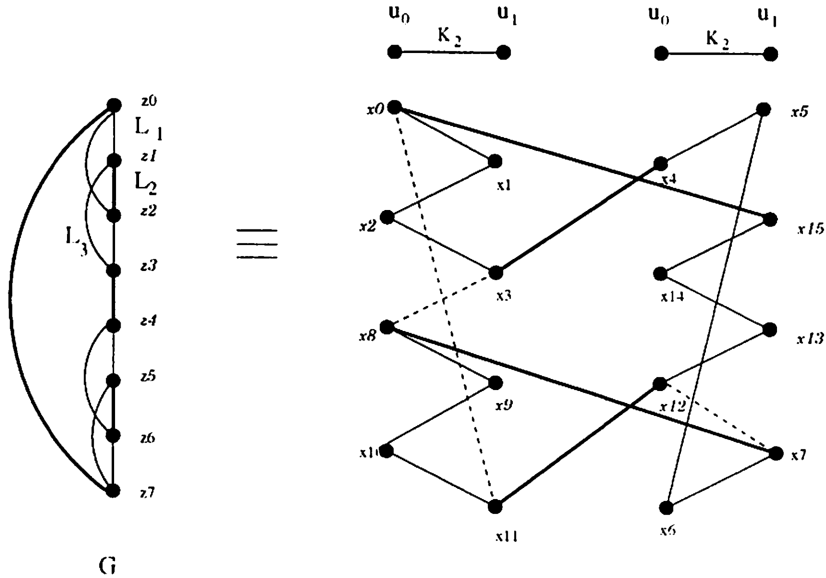

Figure 4.
The hamiltonian cycle of $G \times K_2$ with an alternating cycle $C = L_2 \cup L_3$

Proof: It is clear in this case that G_C is always connected. So the hamiltonian cycle is given from G_C as in Theorem 3.4. □

We should now put some restriction on the alternating cycle. We assume that it is a subset of only the two perfect matchings L_2 and L_3 ($C \subseteq L_2 \cup L_3$).

Theorem 3.5 *Let $G = L_1 \cup L_2 \cup L_3$ be a non-bipartite cubic graph, where (L_1, L_2, L_3) is a factorization of G such that:*

- *$L_1 \cup L_2$ is a hamiltonian cycle.*
- *C is an alternating cycle between L_2 and L_3 such that G_C is connected.*

Then $G \times C_{2n}$ contains two hamiltonian edge-disjoint cycles (these two cycles contain $L_1 \times C_{2n}$).

Proof: By Theorem 2.2, $C_{1,2} \times C_{2n}$ has two components CP_1 and CP_2 such that CP_1 (respectively CP_2) admits an h-decomposition (A, D) (respectively (B, E)) (Figure 5). Let $A = [(z_0, u_0)(z_1, u_1)(z_0, u_2)(z_1, u_3) \ldots$

$(z_1, u_{2n-1})][(z_1, u_{2n-1})(z_2, u_0)][(z_3, u_0)(z_4, u_1)(z_3, u_2)\dots(z_4, u_{2n-1})][(\cdot\cdot)]$ $[\dots[\dots]\dots[(z_{2m-1}u_{2n-1})(z_0, u_0)]$ and $D = CP_1 \backslash A$. Let $h : CP_1 \to CP_2$ such that $h((x_i, u_j)) = (x_i, u_{2n-j-1})$. This implies that $h(A) = B$ and $h(D) = E = CP_2 \setminus B$.

First of all let $G_C = (v_0, v_1, v_2, \dots, v_{4m-1})$. By the definition of the dual graph, we know that each v_i corresponds a pair (z_j, d_j) with $z_j \in V(C_{1,2}), d_j \in \{0, 1\}$ and $0 \le i \le 4m-1$.

We define two functions f_1 and f_2 which construct the two hamiltonian cycles recursively in $G \times C_{2n}$.

Let

$$\begin{array}{rcl} f_1: \quad E(G_C) & \to & E(G \times C_{2n}) \\ [v_i, v_{i+1}]_{([z_r, z_s] \in L_1)} & \to & [(z_r, u_0)(z_s, u_1)(z_r, u_2)(z_s, u_3)\dots(z_s, u_{2n-1})] \\ [v_i, v_{i+1}]_{([z_r, z_s] \in L_2 \cup L_3)} & \to & [(z_r, u_{2n-1})(z_r, u_0)] \end{array}$$

$$\begin{array}{rcl} f_2: \quad E(G_C) & \to & E(G \times C_{2n}) \\ [(v_i, v_{i+1}]_{([z_r, z_s] \in L_1)} & \to & [(z_r, u_1)(b_i, u_2)(z_r, u_3)(b_i, u_3)\dots(z_r, u_{2n-1})(z_s, u_0)] \\ [(v_i, v_{i+1}]_{([z_r, z_s] \in L_2 \cup L_3)} & \to & [(z_i, u_0)(z_{i+1}, u_1)] \end{array}$$

such that if $r < s$ then $b_i = z_{r-1}$, else $b_i = z_{r+1}$, with s and r in modulo $2m$.

We remark that each function allows to replace any edge of L_1 of G_C by a path in $G \times C_{2n}$. By definition, each function ($f_1(G_C)$ and $f_2(G_C)$) covers all vertices of $G \times C_{2n}$. Furthermore, G_C is connected. Consequently the graphs $\gamma_1 = f_1(E(G_C))$ and $\gamma_2 = f_2(E(G_C))$ are two hamiltonian cycles of $G \times C_{2n}$. We may note that G_C connects each time two cycles, each one in different components of $C_{1,2} \times C_{2n}$ (Figure (6,7)). Let $C_a = (\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_{k-1}, \beta_{k-1})$ (with $2k$ is a size of C and $k \le m$) be isomorphic to the cycle C in $G \times C_{2n}$ between columns u_0 and u_{2n-1} with edges $\{(\alpha_i)_{0 \le i \le k-1}\} \subset L_2 \times C_{2n}$, $\{(\beta_i)_{0 \le i \le k-1}\} \subset L_3 \times C_{2n}$. Let (x_0, u_0) be the origin vertex of C_a. Let $C_b = (\alpha'_0, \beta'_0, \alpha'_1, \beta'_1, \dots, \alpha'_{k-1}, \beta'_{k-1})$ be isomorphic to the cycle C in $G \times C_{2n}$ between columns u_0 and u_1 with edges $\{(\alpha'_i)_{0 \le i \le k-1}\} \subset L_2 \times C_{2n}$, $\{(\beta'_i)_{0 \le i \le k-1}\} \subset L_3 \times C_{2n}$. Let (x_0, u_1) be origin vertex of C_b. It is easy to see that $\gamma_1 = (A \cup B) \setminus \{(\alpha_i)_{0 \le i \le k-1} \cup \{(\beta_i)_{0 \le i \le k-1}\}$ and $\gamma_2 = (D \cup E) \setminus \{(\alpha'_i)_{0 \le i \le k-1}\} \cup \{(\beta'_i)_{0 \le i \le k-1}\}$. This implies that γ_1 and γ_2 are two edge-disjoint cycles in $G \times C_{2n}$. □

Finally, we hold to the main result of the product by even cycle. Using the two hamitonian cycles obtained in the preceding theorem (with slight modification of γ_1), we get, by an additional hypothesis, the h-decompostition of $G \times C_{2n}$.

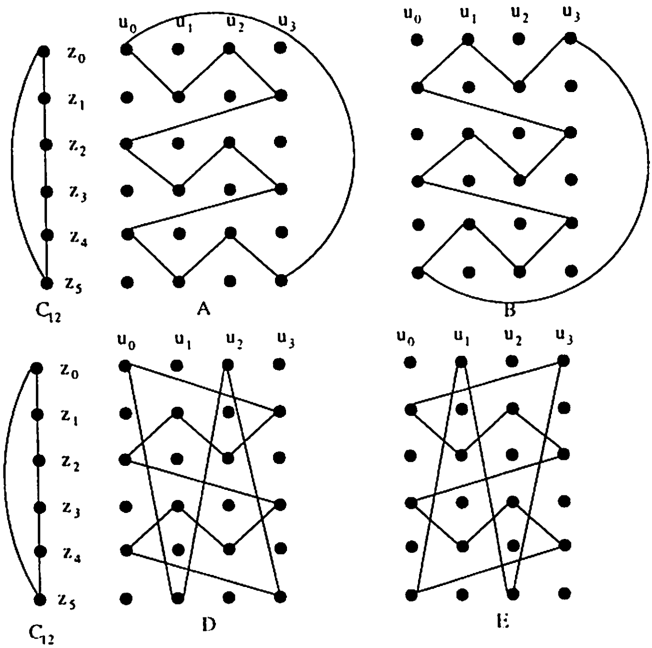

Figure 5.
$C_{1,2} \times C_{2n} = CP_1 \cup CP_2$, with $CP_1 = (A \cup D)$ and $CP_2 = (B \cup E)$

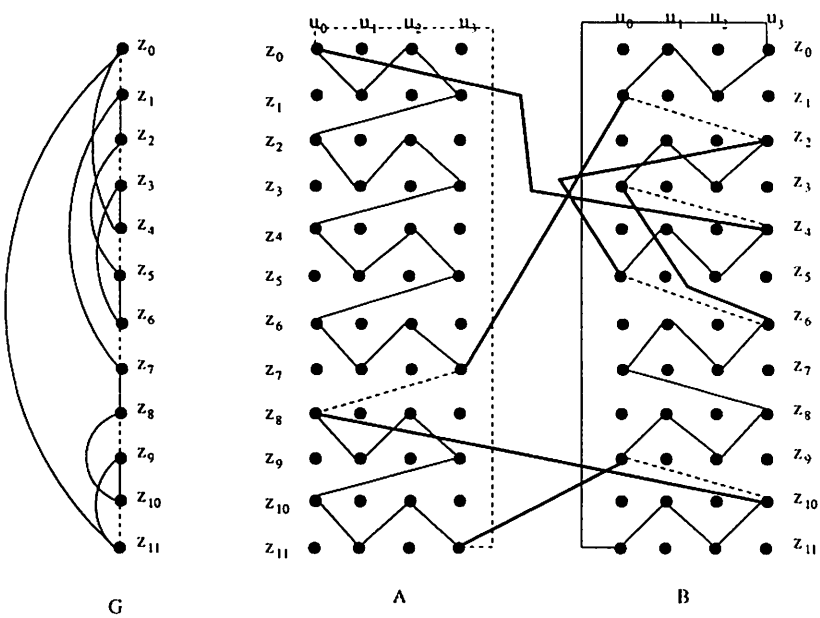

Figure 6. γ_1

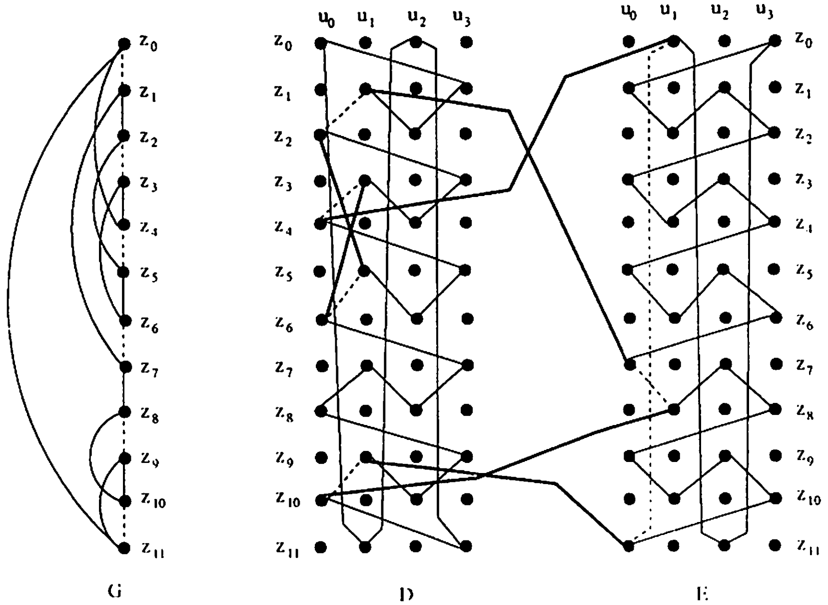

Figure 7. $\gamma_2 = H_2'$

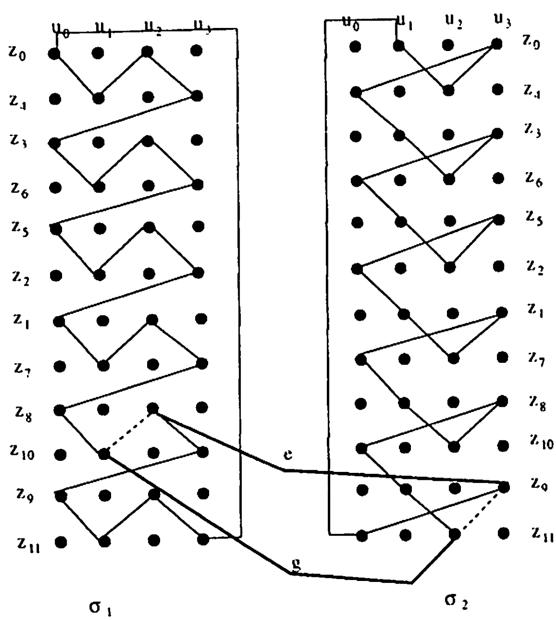

Figure 8. H_3'

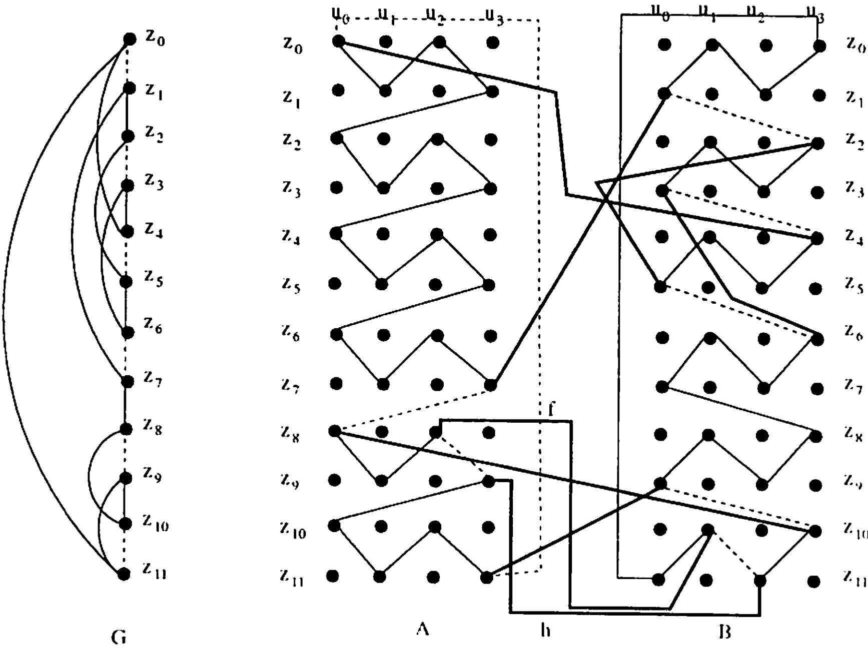

Figure 9. H_1'

Theorem 3.6 *Let $G = L_1 \cup L_2 \cup L_3$ be a non-bipartite cubic graph such that:*

- *$L_1 \cup L_2$ and $C = L_2 \cup L_3$ are two hamiltonian cycles.*
- *G_C is connected.*
- *$L_1 \cup L_3$ contains an alternating 4-cycle C', such that the edges of $L_3 \cap E(C')$ are one-parity edges.*

Then $G \times C_{2n}$ is h-decomposable.

Proof: Let $L_2 \cup L_3 = (y_0, y_1, \dots, y_{2m-1})$. One can see that each edge $e_i \in L_3$, $e_i \times C_{2n}$ gives two disjoint cycles $\beta_i \cup l_i$ and $\beta_i' \cup l_i'$, such that l_i and l_i' of size $2n-1$, $\beta_i \in C_a$ and $\beta_i' \in C_b$.

In fact, the spanning subgraph $(G \times C_{2n}) \setminus (\gamma_1 \cup \gamma_2)$ contains two cycles σ_1 and σ_2, such that σ_1 is obtained from C_a by replacing β_i by l_i, and σ_2 is obtained from C_b by replacing β_i' by l_i'. Then $(E(G \times C_{2n}) \setminus (\gamma_1 \cup \gamma_2)) = (\alpha_0, l_0, \alpha_1, l_1, \dots, \alpha_{k-1}, l_{k-1}) \cup (\alpha_0', l_0', \alpha_1', l_1', \dots, \alpha_{k-1}', l_{k-1}')$. These two cycles of $C \times C_{2n}$, are also given as follows:

$\sigma_1 \quad = \quad ([(y_0, u_0)(y_1, u_1)(y_0, u_2)(y_1, u_3) \dots (y_1, u_{2n-1})][(y_1, u_{2n-1})(y_2, u_0)]$

$[(y_2,u_0)(y_3,u_1)(y_2,u_2)(y_3,u_3)\dots(y_3,u_{2n-1})][(y_3,u_{2n-1})(\dots)][\dots[\dots(y_{2m-1},u_{2n-1})][(y_{2m-1},u_{2n-1})(y_0,u_0)]$ and $\sigma_2 = ([(y_0,u_1)(y_1,u_2)(y_0,u_3)(y_1,u_4)\dots(y_1,u_0)][(y_1,u_0)(y_2,u_1)][(y_2,u_1)(y_3,u_2)(y_2,u_3)(y_3,u_4)\dots(y_3,u_0)][(y_3,u_0)(\dots)][\dots][\dots(y_{2m-1},u_0)][(y_{2m-1},u_0)]$ (Figure 8). This latter fact implies that α_1 and α_2 compose a 2-factor of $G \times C_{2n}$. It is easy to see that the edges of $L_3 \times C_{2n}$ not used in γ_1 and γ_2 are used in σ_1 and σ_2. We denote γ_2 by H'_2.

Choose now the edges (e,f,g,h) corresponding to C' in $G \times C_{2n}$ between two consecutive columns (or if necessary three consecutive columns) with $\{e,g\} \subset \gamma_1$, $f \in \alpha_1$ and $h \in \alpha_2$ such that $H'_1 = (\gamma_1 \setminus \{e,g\}) \cup \{f,h\}$ and $H'_3 = (\alpha_1 \cup \alpha_2 \setminus \{f,h\}) \cup \{e,g\}$ are connected. So (H'_1, H'_2, H'_3) is a h-decomposition of $G \times C_{2n}$ (Figures 6,7,8,9). □

3.3 Classes of Graphs

In this section we construct three classes of graphs satisfing the hypotheses of the last theorem

- $G = L_1 \cup L_2 \cup L_3$ is a non-bipartite cubic graph.
- $L_1 \cup L_2$ and $C = L_2 \cup L_3$ are two hamiltonian cycles.
- G_C is connected.
- $L_1 \cup L_3$ contains an alternating 4-cycle C', such that the edges of $L_3 \cap E(C')$ are one-parity edges.

3.3.1 The class $\mathcal{F}1$

Each graph of $\mathcal{F}1$ can be defined as follows

Let $H = L'_1 \cup L'_2 \cup L'_3$ be any strongly hamiltonian non-bipartite cubic graph, such that the dual graph $H_{L'_2 \cup L'_3}$ is connected. Choose one edge $[x,y]$ of L'_2 and replace it by G_{xy}.

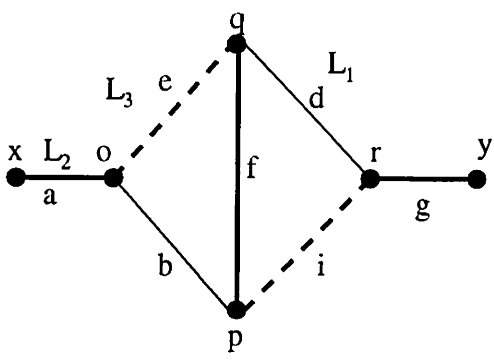

FIgure 10. G_{xy}

The new graph obtained $G = L_1 \cup L_2 \cup L_3$ is still a non-bipartite cubic graph. The cycles $L_1 \cup L_2 = (L'_1 \cup \{b, d\}) \cup (L'_2 \cup \{a, f, g\})$ and $C = L_2 \cup L_3 = (L'_2 \cup \{a, f, g\}) \cup (L'_3 \cup \{e, i\})$ are hamiltonian. The dual graph G_C is given from $H_{L'_2 \cup L'_3}$ by replacing the edge $[(x, g(x))(y, g(y))]$ by a path as follows:

$G_C = ([H_{L'_2 \cup L'_3} \setminus \{[(x, g(x))(y, g(y))]\}) \cup [(x, g(x))(o, g(x))(p, g(x))(r, \overline{g(x)})(q, \overline{g(x)})(p, \overline{g(x)})(o, \overline{g(x)})(q, g(x))(r, g(x))(y, g(x))))]$, with $V(G_{xy}) = \{p, q, r, x, y\}$. G_C is connected. The cycle $C' = \{e, d, i, b\}$ is an alternating 4-cycle such that the edges e and i are one-parity edges.

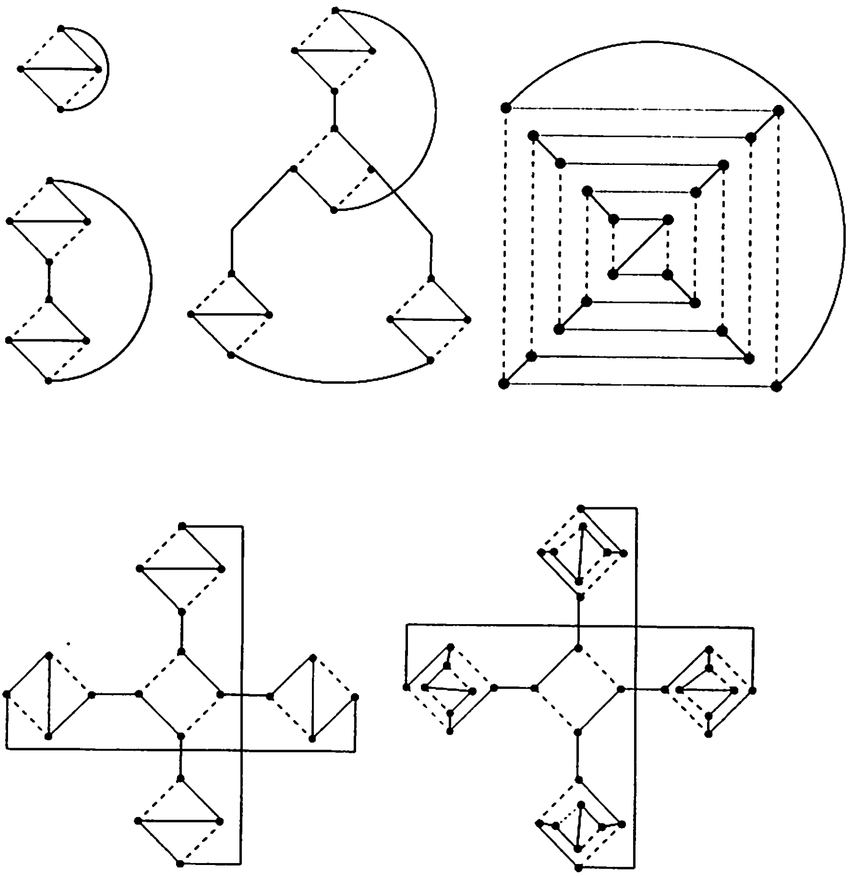

Figure 11. Some graphs of class $\mathcal{F}2$

3.3.2 The class $\mathcal{F}2$

Each graph G' of this class $\mathcal{F}2$ is obtained recursively from K_4, by replacing the edges of L_2 by G_{xy}:

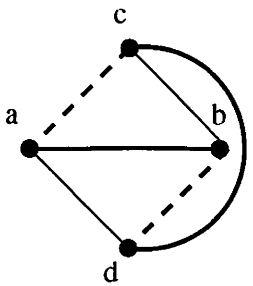

Figure 12. K_4

Let $\Upsilon^0 = \{K_4\}$.

We assume that Υ^{i-1} is defined and for each cubic graph of Υ^{i-1}, there exists a 1-factorization $(L_1(G), L_2(G), L_3(G))$. We put $\Upsilon^i = \{G' = (G \setminus \{[x, y]\}) \cup G_{xy} \ : \ [x, y] \in L_2(G), \text{ for some } G \in \Upsilon^{i-1}\}$. G' satisfies the hypotheses of last theorem (Figure 11), the verification is left to the reader.

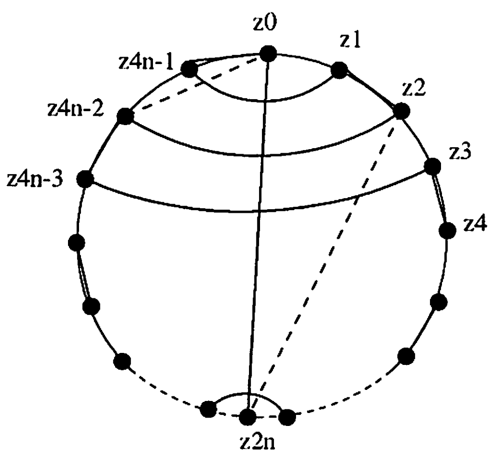

Figure 13.

3.3.3 The class $\mathcal{F}3$

Let H be the strongly hamiltonian non-bipartite cubic graph of Figure 13, where $L_3 = \{[z_0, z_{2n}], [z_1, z_{4n-1}], [z_2, z_{4n-2}], [z_3, z_{4n-3}], \cdots, [z_{2n-1}, z_{2n+1}]\}$. We create an alternating 4-cycle in $L_1 \cup L_3$, by removing the edges $[z_0, z_{2n}]$ and $[z_{4n-2}, z_2]$ and replacing them by the edges $[z_0, z_{4n-2}]$ and $[z_2, z_{2n}]$.

So in the obtained graph H', the cycle $L_1 \cup L_2$ remains hamiltonian because its edges are not changed, and $L_2 \cup L_3$ is still a hamiltonian cycle because $[z_0, z_{4n-2}]$ and $[z_2, z_{2n}]$ are two crossing chords of the cycle. The dual graph $H'_{L_2 \cup L_3}$ is still connected, the verification is left to the reader.

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