On nesting of path designs*

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ABSTRACT. Let $h \geq 1$. For each admissible v, we exhibit a nested balanced path design $H(v, 2h+1, 1)$. For each admissible odd v, we exhibit a nested balanced path design $H(v, 2h, 1)$. For every $v \equiv 4 \pmod 6$, $v \geq 10$, we exhibit a nested balanced path design $H(v, 4, 1)$ except possibly if $v \in \{16, 52, 70\}$.

For each $v \equiv 0 \pmod{4h}$, $v \geq 4h$, we exhibit a nested path design $P(v, 2h+1, 1)$. For each $v \equiv 0 \pmod{4h-2}$, $v \geq 4h-2$, we exhibit a nested path design $P(v, 2h, 1)$. For every $v \equiv 3 \pmod 6$, $v \geq 9$, we exhibit a nested path design $P(v, 4, 1)$ except possibly if $v = 39$.

1 Introduction

Let $H = (V(H), E(H))$ be a graph. Denote by λH the graph H in which every edge has multiplicity λ. The multigraph λH is said to be G-*decomposable* if it is a union of edge disjoint subgraphs of K_v, each of them isomorphic to a fixed graph G. This situation is denoted by $\lambda H \to G$; λH is also said to admit a G-*decomposition*. A G-*design* is a G-decomposition of λK_v. A G-design is denoted by a pair $(V, \mathcal{B})$, where V is the vertex set of K_v and $\mathcal{B}$ is the edge-disjoint decomposition of λK_v into copies of G. Usually $\mathcal{B}$ is called the *block-set* of the G-design and any $B \in \mathcal{B}$ is said to be a *block*. A G-design $(W, \mathcal{A})$ is called to be a subdesign of $(V, \mathcal{B})$ if $W \subseteq V$ and $\mathcal{A} \subseteq \mathcal{B}$.

A *path design* $P(v, k, 1)$ of order v and block size k, is a P_k-design of K_v, where P_k is the simple path with $k-1$ edges (k vertices), $P_k = [a_1, a_2, \ldots, a_k] = \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$.

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M. Tarsi [8] proved that the necessary conditions for the existence of a $P(v,k,1)$, $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient.

We denote by $S(v,m+1,1)$ a decomposition of K_v into m-stars $S_m = [a; a_1, a_2, \ldots, a_m] = \{\{a,a_1\},\{a,a_2\},\ldots,\{a,a_m\}\}$. The vertex a of degree m in S_m is called the centre of the star and the vertices a_i of degree 1 are called the terminal vertices of the star. It is well-known [8] that the necessary conditions for the existence of a $S(v,m+1,1)$, $v \geq 2m$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2m}$, are also sufficient.

A *balanced G-design* is a G-design such that each vertex belongs to exactly r copies of G. Obviously not every G-design is balanced. A (balanced) G-design of λK_v is also called a *(balanced) G-design of order v*, block size $|V(G)|$ and *index* λ.

We denote by $H(v,k,1)$ a *balanced path design* $P(v,k,1)$. Clearly a $H(v,2,1)$ $(V,\mathcal{B})$ exists for every $v \geq 2$. S.H.Y. Hung and N.S. Mendelsohn [5] proved that a $H(v,2h+1,1)$, $(h \geq 1)$, exists if and only if $v \equiv 1 \pmod{4h}$, and a $H(v,2h,1)$, $(h \geq 2)$, exists if and only if $v \equiv 1 \pmod{2h-1}$.

An *m-cycle system* of order v (mCS) is a C_m-design of K_v, where C_m is an *m-cycle* (cycle of length m), $(a_1,a_2,\ldots,a_m) = \{\{a_1,a_2,\},\{a_2,a_3\},\ldots,\{a_{m-1},a_m\},\{a_1,a_m\}\}$. The obvious necessary conditions for the existence of an mCS of order v are: $v \geq m$ (if $v > 1$), v is odd and $v(v-1) \equiv 0 \pmod{2m}$. The sufficiency of these conditions has been proved in several classes, namely when $2m$ divides either v or $v-1$, and for all v when $m \leq 50$, but not in general, though no counter example has been found so far. For the history of the problem and detailed references, see [7].

A *nesting* of an m-cycle system $(V,\mathcal{C})$ of order v is a function $f : \mathcal{C} \to V$ such that $\{\{x, f(C)\} | x \in V(C), C \in \mathcal{C}\}$ is a partition of the edges of K_v. Notice that any nesting of $(V,\mathcal{C})$ maps each cycle $C \in \mathcal{C}$ to any m-star: The graph $S(C)$ with vertex set $V(C) \cup \{f(C)\}$ and edge set $\{\{x, f(C)\} | x \in C\}$ is an m-star centered on $f(C)$. Therefore, any nesting of an mCS of order v produces an edge-disjoint decomposition of K_v into m-stars. Also, notice that the graph $C \cup S(C)$ is obviously a wheel W_m. It is clear then that a nesting of an mCS of order v is equivalent to an edge-disjoint decomposition of $2K_v$ into wheels W_m having the additional property that for each pair of vertices a_1 and a_2, one of the edges joining a_1 to a_2 is on the rim of a wheel and the other is the spoke of a wheel.

Example 1. $(Z_9, \{C^i = (i, 1+i, 7+i, 2+i) | i \in Z_9\})$ is a 4-cycle system of order 9 that has a nesting defined by $f(C^i) = 3+i$, reducing all sums modulo 9.

The spectrum problem for mCS of order v that have a nesting was studied in many papers, see [7] for more details and references.

Theorem 1 *For all $m \geq 3$, there exists a nested mCS of order v for all $v = 2mx + 1$ except possibly if $x \in \{2, 3, 4, 6, 22, 23, 24, 26, 27, 28, 30, 34, 38\}$ when m is not a power of 2, and except possibly if $x \in \{2, 3, 4, 7, 8, 12, 14, 18, 19, 23, 24, 33, 34\}$ when m is a power of 2.*

The list of possible exceptions can be reduced when m is odd (see [7]).

It is natural to define a nesting of a G-design of K_v in a similar way as a nesting of an mCS of order v:

A nesting of a G-design of K_v $(V, \mathcal{B})$ is a pair $\{(V, \mathcal{S}), F\}$ where $(V, \mathcal{S})$ is a $S(v, m+1, 1)$ and $F : \mathcal{B} \to \mathcal{S}$ is a $1-1$ mapping such that:
(n_1) for every $B \in \mathcal{B}$ the centre of the m-star $S_m = F(B)$ is not in $V(B)$ and any terminal vertex of S_m is in $V(B)$;
(n_2) For every pair $B_1, B_2 \in \mathcal{B}$ the graphs $B_1 \cup F(B_1)$ and $B_2 \cup F(B_2)$ are isomorphic.

It is $|\mathcal{B}| = |\mathcal{S}|$ and $|V(S_m)| \leq |V(G)| + 1$. Then, for $v > 1$, a necessary condition for the existence of a nested G-design of K_v is

$$|E(G)| = m \leq |V(G)| \tag{1}$$

In this paper we study the case where G is a path P_k and either $v \equiv 0$ or $1 \pmod{2(k-1)}$ if k is odd or $v \equiv 0$ or $1 \pmod{k-1}$ if k is even. Let $(V, \mathcal{P})$ be a nested $P(v, k, 1)$. The necessary condition (1) implies $m = k - 1$, then every path $P \in \mathcal{P}$ contains exactly one vertex, say x, missing on the vertex set of $F(P)$. So, to satisfy the (n_2) it is necessary to decide the *position* of x into the path P.

Example 2. $(Z_9, \{P^i = [i, 1+i, 8+i, 2+i, 7+i] | i \in Z_9\})$ is a $P(9, 5, 1)$ that has a nesting defined by the $S(9, 5, 1)$ $(Z_9, \{S^i = [6+i; i, 8+i, 2+i, 7+i] | i \in Z_9\})$, reducing all sums modulo 9, and by the $1-1$ mapping F defined by $S^i = F(P^i)$.

Example 3. $(Z_9, \{[i, 1+i, 7+i], [i, 2+i, 7+i] | i \in Z_9\})$ is a $P(9, 3, 1)$ that has a nesting defined by the $S(9, 3, 1)$ $(Z_9, \{[3+i; i, 1+i], [3+i; 7+i, 2+i] | i \in Z_9\})$ and by the $1-1$ mapping F defined by $F([i, 1+i, 7+i]) = [3+i; i, 1+i]$ and $F([i, 2+i, 7+i]) = [3+i; 7+i, 2+i]$, reducing all sums modulo 9.

Theorem 2 *The existence of a nested $2m$-cycle system of order v implies the existence of a nested path design $P(v, m+1, 1)$.*

Proof: Let $(V, \mathcal{C})$ be a $2m$-cycle system of order v that has a nesting defined by $f(C^i) = b^i$ for $i = 1, 2, \ldots, v(v-1)/4m$. For every i split the $2m$-cycle $C^i = (a_1^i, a_2^i, \ldots, a_{2m}^i)$ into the two following paths: $P^i = [a_1^i, a_2^i, \ldots, a_{m+1}^i]$ and $\overline{P}^i = [a_{m+1}^i, a_{m+2}^i, \ldots, a_{2m}^i, a_1^i]$. Define $S^i = F(P^i) = [b^i; a_1^i, a_2^i, \ldots, a_m^i]$ and $\overline{S}^i = F(\overline{P}^i) = [b^i; a_{m+1}^i, a_{m+2}^i, \ldots, a_{2m}^i]$. It is easy to see that $(V, \mathcal{P}) =$

$\{P^i, \overline{P}^i | i = 1, 2, \ldots, v(v-1)/4m\})$ is a $P(v, m+1, 1)$ that has a nesting defined by $(V, \mathcal{S}) = \{S^i, \overline{S}^i | i = 1, 2, \ldots, v(v-1)/4m\})$ and F. □

Applying Theorem 2 to the nested $4CS$ of Example 1, we obtain the nested $P(9, 3, 1)$ of Example 3.

Corollary 1 *For all $2m \geq 4$, there exists a nested $P(v, m+1, 1)$ for all $v = 4mx + 1$ except possibly if x is defined as in Theorem 1.*

Let H be a subgraph of K_v and let $\widehat{G_k} = < a_1, a_2, \ldots, a_{k-1}, \widehat{a_k}; a > = (\{a, a_1, a_2, \ldots, a_k\}, [a_1, a_2, \ldots, a_k] \cup [a; a_1, a_2, \ldots, a_{k-1}])$. From now on we shell suppose that any edge disjoint decomposition $2H \rightarrow \widehat{G_k}$, $(V, \mathcal{B})$ satisfies the following properties:

(p_1) $(V(H), \{P_k(B) | B \in \mathcal{B}\})$ (where $P_k(B)$ is the subgraph of B isomorphic to the path $[a_1, a_2. \ldots, a_k]$) is a decomposition $H \rightarrow [a_1, a_2, \ldots, a_k]$;

(p_2) $(V(H), \{S_{k-1}(B) | B \in \mathcal{B}\})$ (where $S_{k-1}(B)$ is the subgraph of B isomorphic to the $(k-1)$-star $[a; a_1, a_2, \ldots, a_{k-1}]$) is a decomposition $H \rightarrow [a; a_1, a_2, \ldots, a_{k-1}]$.

When $H = K_v$, we say that a $2K_v \rightarrow \widehat{G_k}$ is a *$\widehat{G_k}$-design* $N(v, k+1, 2)$.

Let $(V, \mathcal{P})$ be a nested $P(v, m+1, 1)$ constructed as in Theorem 2 starting from a nested $2m$-cycle system. Let $P \in \mathcal{P}$, then it is easy to see that the vertex of P that is not a vertex of $S = F(P)$ has degree one in P. In this paper we ask that any nested path design satisfies this property. I.e. we look for a nesting of a (balanced) path design $P(v, k, 1)$, $(V, \mathcal{P})$, that is equivalent to a $N(v, k+1, 2)$, $(V, \mathcal{B})$, such that $\mathcal{P} = \{P_k(B) | B \in \mathcal{B}\}$, $\mathcal{S} = \{S_{k-1}(B) | B \in \mathcal{B}\}$ and $F : \mathcal{P} \rightarrow \mathcal{S}$ is defined by $F(P_k(B)) = S_{k-1}(B)$ for every $B \in \mathcal{B}$.

In this paper we exhibit a $\widehat{G_k}$-decomposition $N(v, k+1, 2)$ for the following values of v and k: each $v \equiv 0$ or $1 \pmod{4h}$, $v \geq 4h$ if $k = 2h+1$; each odd $v \equiv 1 \pmod{2h-1}$, $v \geq 2h$ if $k = 2h$; each $v \equiv 0 \pmod{4h-2}$, $v \geq 4h-2$ if $k = 2h$; each $v \equiv 3$ or $4 \pmod 6$, $v \geq 9$, except possibly if $v \in \{16, 39, 52, 70\}$ if $k = 4$. Moreover if either $v \equiv 1 \pmod{4h}$ or $v \equiv 1 \pmod{2h-1}$ or $v \equiv 4 \pmod 6$, the nested path design is balanced.

Generally, two well-known methods are used in construction: the difference method (see f.e. [3]) and the composition method (see f.e. [9] and [1]).

Usually, using the difference method, we will give only the *base blocks* of the decomposition since the rest of the blocks can be obtained by applying an automorphism of the group Z_v on the vertices of the base blocks, as illustrated in the following example.

Example 4. (4.1) The base blocks of the $N(9, 4, 2)$ given in Example 3 are $< 0, 1, \widehat{7}; 3 >$ and $< 7, 2, \widehat{0}; 3 >$ (mod 9).

(4.2) Let $V(K_{10}) = Z_5 \times Z_2$. For a $2K_{10} \to \widehat{G_5}$ take the base blocks (mod $(5,-)$):
$<(2,1),(4,1),(4,0),\widehat{(1,0)};(1,1)>$, $<(2,1),(1,1),(2,0),\widehat{(1,0)};(3,0)>$, $<(3,1),(0,0),(2,1),\widehat{(1,0)};(2,0)>$ (mod $(5,-)$). Hence the blocks of the decomposition are
$<(2+i,1),(4+i,1),(4+i,0),\widehat{(1+i,0)};(1+i,1)>$,
$<(2+i,1),(1+i,1),(2+i,0),\widehat{(1+i,0)};(3+i,0)>$,
$<(3+i,1),(i,0),(2+i,1),\widehat{(1+i,0)};(2+i,0)>$, for $i \in Z_5$.

(4.3) For a $N(16,6,2)$ we have $V(K_{16}) = Z_{15} \cup \{\infty\}$ and the base blocks are:
$<\infty,0,14,1,\widehat{13};4>$ and $<10,1,11,0,\widehat{7};2>$ (mod 15). Hence the blocks of the decomposition are
$<\infty,i,14+i,1+i,\widehat{13+i};4+i>$ and $<10+i,1+i,11+i,i,\widehat{7+i};2+i>$, for $i \in Z_{15}$.

(4.4) For a $2K_v \to \widehat{G_3}$ put:

$V(K_v) = Z_v$ and base blocks $<0,\widehat{\rho};v-\rho>$ (mod v), $\rho = 1,2,\dots,(v-1)/2$, if $v \geq 3$ is odd;

$V(K_v) = Z_{v-1} \cup \{\infty\}$ and base blocks $<\infty,\widehat{0};1>$, $<0,\widehat{\rho};v-1-\rho>$ (mod $v-1$), $\rho = 1,2,\dots,(v-2)/2$, if $v \geq 4$ is even.

Let Y be a finite set of *points*, $\mathcal{C}$ a family of distinct subsets of Y called *groups* which partition Y, and $\mathcal{A}$ a collection of subsets of Y called blocks. Let v be a positive integer and K and M sets of positive integers. The triple $(Y,\mathcal{C},\mathcal{A})$ is called a *group divisible design* (GDD) $GD[K,M;v]$ if:

(c_1) $|Y| = v$;

(c_2) $\{|C| \mid C \in \mathcal{C}\} \subseteq M$;

(c_3) $\{|B| \mid B \in \mathcal{A}\} \subseteq K$;

(c_4) $|C \cap B| \leq 1$ for every $C \in \mathcal{C}$ and every $B \in \mathcal{A}$;

(c_5) every pairset $\{x,y\} \subseteq Y$ such that x and y belong to distinct groups is contained in exactly one block of $\mathcal{A}$.

If $\mathcal{C}$ contains t_i groups of size m_i, for $i = 1,2,\dots,s$, we call $m_1^{t_1} m_2^{t_2} \dots m_s^{t_s}$ the group type of the GDD. When $K = \{k\}$ we will write $GD[k,M;v]$ instead of $GD[\{k\},M;v]$.

Let $2K_{n_1,n_2,\dots,n_h}$ be the complete multipartite multigraph on vertices $\cup_{i=1}^{h} X_i$ where $|X_i| = n_i$ with two edges joining each pair of vertices from different sets X_i, X_j, $i \neq j$. The composition method is based on the following four lemmas.

Lemma 1 *If* $2K_{n_i} \to \widehat{G_k}$ *for* $i = 1,2,\dots,h$ *and* $2K_{n_1,n_2,\dots,n_h} \to \widehat{G_k}$, *then* $2K_n \to \widehat{G_k}$ *where* $n = n_1 + n_2 + \dots + n_h$.

Lemma 2 *If $2K_{n_i} \to \widehat{G_k}$ for $i = 1, 2, \ldots, h$ and $2K_{n_1,n_2,\ldots,n_h} \to \widehat{G_k}$, then $2K_n \to \widehat{G_k}$ where $n = 1 + n_1 + n_2 + \ldots + n_h$.*

Lemma 3 *([1]) If $2K_{n,n,n,n} \to \widehat{G_k}$ then $2K_{pn,pn,pn,pn} \to \widehat{G_k}$ for every positive integer $p \neq 2, 6$.*

Lemma 4 *Suppose there exists a $GD[t, M; v]$, a $2K_{n_1,n_2,\ldots,n_t} \to \widehat{G_k}$ (with $n_1 = n_2 = \ldots = n_t = n$) and for any $m \in M$ a $\widehat{G_k}$-design $N(mn + w, k + 1, 2)$ containing a subdesign $N(w, k + 1, 2)$ (or $w = 0, 1$). Then there exists a $\widehat{G_k}$-design $N(nv + w, k + 1, 2)$.*

Example 5. Let $X_i = Z_3 \times \{i\}$ $i \in Z_3$ and $V(K_{3,3,3}) = \cup_{i=0}^{2} X_i$. For a $2K_{3,3,3} \to \widehat{G_5}$ take the base blocks:
$< (2,1), (1,0), (0,1), \widehat{(0,0)}; (0,2) >$, $< (1,0), (2,2), (0,0), \widehat{(0,2)}; (0,1) >$,
$< (0,2), (1,1), (1,2), \widehat{(0,1)}; (0,0) >$ (mod $(3, -)$).

Put $w = 1$ in Lemma 4. Since there exists a $GD[3, \{3\}; 9]$ (or a Kirkman triple system of order 9) and a $N(10, 5, 2)$ (see Example (4.2)), then Lemma 4 implies the existence of a $N(28, 5, 2)$.

At last we give the following notation that we will use in the next two sections. Let $x \in Z_v$. Define

$$|x| = \begin{cases} x & \text{if } 0 \leq x \leq \chi(v) \\ v - x & \text{if } \chi(v) + 1 \leq x \leq v - 1 \end{cases}$$

where

$$\chi(v) = \begin{cases} (v-1)/2 & \text{if } v \text{ is odd} \\ v/2 & \text{if } v \text{ is even} \end{cases}$$

2 Nesting of path designs of order v and block size odd

In this section we construct a nested path design $P(v, 2h+1, 1)$, $h \geq 1$, for any $v \equiv 0$ or $1 \pmod{4h}$. Moreover for $v \equiv 1 \pmod{4h}$, the nested path design is balanced.

Theorem 3 *Let $h \geq 1$. For every $v \equiv 1 \pmod{4h}$, $v \geq 4h + 1$, there is a nested balanced path design $H(v, 2h + 1, 1)$.*

Proof: Let $v = 1 + 4\alpha h$, $\alpha \geq 1$. For $\rho = 1, 2, \ldots, \alpha$ and $j = 0, 1, \ldots, h-1$, put
$a_j^\rho = 1 + j + 2h(\rho - 1)$, $b_j = 4\alpha h - j$, and $c_\rho = (\alpha + \rho - 1)h + 1$.

Let $V(K_v) = Z_v$. For a $2K_v \to \widehat{G_{2h+1}}$, $N(v, 2h + 2, 2)$, take the base blocks
$< 0, a_0^\rho, b_0, a_1^\rho, b_1, \ldots, a_{h-1}^\rho, \widehat{b_{h-1}}; c_\rho >$ (mod v).

Let $\mathcal{P}$ be the path set constructed by the base paths

$$[0, a_0^\rho, b_0, a_1^\rho, b_1, \dots, a_{h-1}^\rho, b_{h-1}] \pmod v.$$

It is well-known [6] that $(Z_v, \mathcal{P})$ is a balanced path design $H(v, 2h+1, 1)$.

Reducing all the sums $\pmod v$, it is easy to see that
$|c_\rho - 0| = h(\alpha + \rho - 1) + 1$,
$|c_\rho - a_j^\rho| = h(\alpha + 1 - \rho) - j$ for $j = 0, 1, \dots, h-1$, and, if $h \geq 2$,
$|c_\rho - b_j| = (\rho + \alpha - 1)h + j + 2$ for $j = 0, 1, \dots, h-2$.

Let
$D_1 = \{|c_\rho| \mid \rho = 1, 2, \dots, \alpha\}$,
$D_2 = \{|c_\rho - a_j^\rho| \mid \rho = 1, 2, \dots, \alpha,\ j = 1, 2, \dots, h-1\}$,
$D_3 = \{|c_\rho - b_j| \mid \rho = 1, 2, \dots, \alpha,\ j = 1, 2, \dots, h-2,\ h \geq 2\}$.
It is $D_1 \cup D_3 =$
$= \{(\alpha+\rho-1)h+1, (\alpha+\rho-1)h+2, \dots, (\alpha+\rho-1)h+h \mid \rho = 1, 2, \dots, \alpha\} =$
$= \{h\alpha+1, h\alpha+2, \dots, 2h\alpha\}$,
$D_2 = \{(\alpha-\rho)h+1, (\alpha-\rho)h+2, \dots, (\alpha-\rho)h+h \mid \rho = 1, 2, \dots, \alpha\} =$
$= \{1, 2, \dots, h\alpha\}$.

Since $\cup_{i=1}^3 D_i = \{1, 2, \dots, 2h\alpha\}$, then

$$[c_\rho; 0, a_0^\rho, b_0, a_1^\rho, b_1, \dots, a_{h-2}^\rho, b_{h-2}, a_{h-1}^\rho] \pmod v$$

are the base blocks of the $S(v, 2h+1, 1)$. □

Theorem 4 *Let $h \geq 1$. For each $v \equiv 0 \pmod{4h}$, $v \geq 4h$, there is a nested path design $P(v, 2h+1, 1)$.*

Proof: Let $v = 4h\alpha$, $\alpha \geq 1$. Put:
$a_{-1}^{\alpha-1} = 2h - 1$;
$a_j^0 = 4h\alpha - j - 2$, for $j = 0, 1, \dots, h-1$;
$a_j^\rho = (4\alpha - 2\rho)h - j - 1$, for $\rho = 1, 2, \dots, \alpha-1$ and $j = 0, 1, \dots, h-1$;
$c_\rho = (\alpha - \rho)h$, for $\rho = 0, 1, \dots, \alpha-1$.

Let $V(K_v) = Z_{v-1} \cup \{\infty\}$. For a $2K_v \to \widehat{G_{2h+1}}$, take the base blocks $\pmod{4h\alpha - 1}$:

$$< \infty, 0, a_0^0, 1, a_1^0, \dots, h-1, \widehat{a_{h-1}^0}; c_0 >,$$

and, if $\alpha \geq 2$, the followings ones

$$< 0, a_0^\rho, 1, a_1^\rho, \dots, h-1, a_{h-1}^\rho, \widehat{h}; c_\rho >, \text{for } \rho = 1, 2, \dots, \alpha-2,$$

$$< a_{h-1}^{\alpha-1}, h-1, a_{h-2}^{\alpha-1}, h-2, \dots, a_1^{\alpha-1}, 1, a_0^{\alpha-1}, 0, \widehat{a_{-1}^{\alpha-1}}; c_{\alpha-1} > .$$

It is easy to verify that the base paths $\pmod{4h\alpha - 1}$:

$$[\infty, 0, a_0^0, 1, a_1^0, \dots, h-1, a_{h-1}^0],$$

$$[0, a_0^\rho, 1, a_1^\rho, \dots, h-1, a_{h-1}^\rho, h],$$

$$[a_{h-1}^{\alpha-1}, h-1, a_{h-2}^{\alpha-1}, h-2, \dots, a_1^{\alpha-1}, 1, a_0^{\alpha-1}, 0, a_{-1}^{\alpha-1}],$$

give a $P(4h\alpha, 2h+1, 1)$.

Reducing all the sums $\pmod{4h\alpha - 1}$, it is:
$D_1 = \{|c_\rho - j| = (\alpha-\rho)h - j \mid \rho = 0, 1, \dots, \alpha-1,\ j = 0, 1, \dots, h-1\} =$
$= \{1, 2, \dots, h\alpha\}$; $D_2 = \{c_0 - a_j^0| = h\alpha + j + 1 \mid j = 0, 1, \dots, h-2\} =$
$= \{h\alpha+1, h\alpha+2, \dots, h(\alpha+1)-1\}$, and, if $\alpha \geq 2$
$D_3 = \{|c_\rho - a_j^\rho| = h(\alpha+\rho)+j \mid \rho = 1, 2, \dots, \alpha-1,\ j = 0, 1, \dots, h-1\} =$
$= \{h(\alpha+1), h(\alpha+1)+1, \dots, 2h\alpha - 1\}$.
Since $\cup_{i=1}^3 D_i = \{1, 2, \dots, 2h\alpha - 1\}$, then

$$[c_0; \infty, 0, a_0^0, 1, a_1^0, \dots, h-2, a_{h-2}^0, h-1],$$

$$[c_\rho; 0, a_0^\rho, 1, a_1^\rho, \dots, h-1, a_{h-1}^\rho],\ \rho = 1, 2, \dots, \alpha-2,$$

$$[c_{\alpha-1}; a_{h-1}^{\alpha-1}, h-1, a_{h-2}^{\alpha-1}, h-2, \dots, a_1^{\alpha-1}, 1, a_0^{\alpha-1}, 0],$$

are, $\pmod{4h\alpha - 1}$, the base blocks of the $S(4h\alpha, 2h+1, 1)$. □

3 Nesting of path designs of order v and block size even

In this section we deal with the problem of constructing a nested path design $P(v, 2h, 1)$, $h \geq 1$, when $v \equiv 0$ or $1 \pmod{2h-1}$. For $h = 1$ the problem is solved in Example (4.4). For $h = 2$ we solve the problem for any $v \equiv 0$ or $1 \pmod 3$, $v \geq 6$, except possibly if $v \in \{16, 39, 52, 70\}$. When $h \geq 2$, we construct a nested balanced path design of order v for each odd $v \equiv 1 \pmod{2h-1}$, $v \geq 4h+1$, and for each even $v \equiv 0 \pmod{2h-1}$, $v \geq 4h-2$.

Theorem 5 *Let $h \geq 2$. For every odd $v \equiv 1 \pmod{2h-1}$, $v \geq 4h-1$, there is a nested balanced path design $H(v, 2h, 1)$.*

Proof: Let $v = 1 + 2\alpha(2h-1)$, $\alpha \geq 1$. For $\rho = 0, 1, \dots, \alpha-1$ and $j = 0, 1, \dots, h-1$ put $a_j^\rho = (2\alpha - \rho)(2h-1) - j$, and $c_\rho = (\alpha-\rho)h$.

Let $V(K_v) = Z_v$. For a $2K_v \to \widehat{G_{2h}}$, $N(v, 2h+1, 2)$, take the base blocks

$$< 0, a_0^\rho, 1, a_1^\rho, \dots, \widehat{h-1, a_{h-1}^\rho}; c_\rho > \pmod v.$$

Let $\mathcal{P}$ be the path set constructed by the base paths

$$[0, a_0^\rho, 1, a_1^\rho, \dots, h-1, a_{h-1}^\rho] \pmod v.$$

It is well-known [6] that $(Z_v, \mathcal{P})$ is a $H(v, 2h, 1)$.

Reducing all the sums (mod v), it is easy to see that $\{|c_\rho - j| = |(\alpha-\rho)h - j| \mid \rho = 0,1,\dots,\alpha-1,\ j=0,1,\dots,h-1\} \cup$ $\{|c_\rho - a_j^\rho| = |\alpha h + j + \rho(h-1)+1| \mid \rho=0,1,\dots,\alpha-1,\ j=0,1,\dots,h-2\} =$ $\{1,2,\dots,\alpha(2h-1)\}$. Then

$$[c_\rho; 0, a_0^\rho, 1, a_1^\rho, \dots, h-2, a_{h-2}^\rho, h-1] \pmod v$$

are the base blocks of the $S(v, 2h, 1)$. □

Theorem 6 *For every $v \equiv 1 \pmod 3$, $v \geq 7$, there is a nested balanced path design $H(v,4,1)$ except possibly if $v \in \{16, 52, 70\}$.*

Proof: If $v \equiv 1 \pmod 6$ the result follows from Theorem 5. Let $v \equiv 4 \pmod 6$, $v \geq 10$. The case $v = 10$ is proved in Example 4.2.

Case $v = 22$. Let $V(K_{22}) = Z_{11} \times Z_2$. The base blocks, (mod $(22,-)$), are:
$< (6,1),(1,1),(1,0),\widehat{(2,0)};(7,1) >$, $< (3,1),(2,1),(1,0),\widehat{(3,0)};(6,1) >$,
$< (7,1),(3,1),(1,0),\widehat{(5,0)};(4,0) >$, $< (7,1),(4,1),(1,0),\widehat{(4,0)};(6,0) >$,
$< (7,1),(5,1),(1,0),\widehat{(6,0)};(5,0) >$, $< (8,0),(7,1),(1,0),\widehat{(6,1)};(5,1) >$,
$< (0,0),(9,1),(1,0),\widehat{(8,1)};(2,0) >$.

Case $v = 34$. Let $V(K_{34}) = Z_{17} \times Z_2$. The base blocks, (mod $(34,-)$), are:
$< (3,1),(1,1),(0,0),\widehat{(1,0)};(5,0) >$, $< (10,1),(2,1),(0,0),\widehat{(2,1)};(8,0) >$,
$< (7,1),(3,1),(0,0),\widehat{(3,1)};(6,0) >$, $< (13,1),(6,1),(0,0),\widehat{(6,0)};(7,0) >$,
$< (8,1),(7,1),(0,0),\widehat{(8,0)};(0,1) >$, $< (5,1),(0,1),(0,0),\widehat{(4,0)};(3,1) >$,
$< (16,1),(10,1),(0,0),\widehat{(5,0)};(4,1) >$, $< (8,1),(5,1),(0,0),\widehat{(7,0)};(9,1) >$,
$< (8,1),(0,0),(11,1),\widehat{(14,0)};(3,0) >$, $< (9,1),(0,0),(12,1),\widehat{(14,0)};(2,0) >$,
$< (14,0),(13,1),(0,0),\widehat{(4,1)};(1,0) >$.

Case $v = 40$. We give a decomposition $2K_{2,2,2,2} \to \widehat{G_5}$. Let $V(K_{2,2,2,2}) = \cup_{i=0}^{3} Z_2 \times \{i\}$. The base blocks (mod $(2,-)$) are:
$< (0,3),(0,2),(0,1),\widehat{(1,0)};(0,0) >$, $< (0,1),(0,3),(0,0),\widehat{(0,2)};(1,2) >$,
$< (0,0),(0,1),(1,2),\widehat{(0,3)};(1,3) >$, $< (1,2),(0,0),(1,3),\widehat{(0,1)};(1,1) >$.
By Lemma 3 (with $p = 5$) and the existence of a $N(10,5,2)$, it follows the existence of a $N(40,5,2)$.

Now we proceed as in Example 5. For $\alpha \geq 1$, the existence of a $GD(3,\{3\}; 3+6\alpha)$ (or Kirkman triple system) is well-known. The existence of a $GD(3,\{3,7\}; 7+6\alpha)$ of type $3^{2\alpha}7^1$ for any $\alpha \geq 2$ and of a $GD(3,\{3,11\}; 11+6\alpha)$ of type $3^{2\alpha}11^1$ for any $\alpha \geq 3$ is proved by Colbourn, Hoffman and Rees [2]. Then Lemma 4 (where we put $w = 1$), implies the existence of a $N(10+18\alpha,5,2)$ $\alpha \geq 1$, a $N(22+18\alpha,5,2)$ $\alpha \geq 2$ and a $N(34+18\alpha,5,2)$ $\alpha \geq 3$. At last note that all nested path designs constructed in this theorem are balanced. □

Theorem 7 *Let $h \geq 2$. For each $v \equiv 0 \pmod{4h-2}$, $v \geq 4h-2$, there is a nested path design $P(v, 2h, 1)$.*

Proof: Let $v = \alpha(4h-2)$, $\alpha \geq 1$. Define $a_j^0 = 2\alpha(2h-1) - j - 2$, for $j = 0, 1, \ldots, h-2$, and, if $\alpha \geq 2$, $a_j^\rho = (4\alpha - 2\rho)h - 2\alpha + \rho - j - 1$, for $j = 0, 1, \ldots, h-1$, $\rho = 1, 2, \ldots, \alpha - 1$.
If $\alpha = 1$, let $c_0 = 3(h-1)$.
If $\alpha \geq 2$, let
$c_{\alpha-1} = h$,
$c_{\rho-1} = a_0^\rho - c_\rho + h$, for $\rho = 2, 3, \ldots, \alpha-2, \alpha-1$, and
$c_0 = a_0^1 - c_1 + h - 1$.

Let $V(K_v) = Z_{v-1} \cup \{\infty\}$. Let $\mathcal{B}$ be the block set constructed from the following base blocks $\pmod{\alpha(4h-2)-1}$:

$$< \infty, 0, a_0^0, 1, a_1^0, \ldots, h-2, a_{h-2}^0, \widehat{h-1}; c_0 >,$$

and, if $\alpha \geq 2$,

$$< 0, a_0^\rho, 1, a_1^\rho, \ldots, h-1, \widehat{a_{h-1}^\rho}; c_\rho >, \text{for } \rho = 1, 2, \ldots, \alpha - 1.$$

To prove that $(V(K_v), \mathcal{B})$ is a $N(\alpha(4h-2), 2h+1, 2)$ suppose at first $\alpha = 1$. In this case it is easy to see that
$\{|c_0 - j| \mid j = 0, 1, \ldots, h-2\} \cup \{|c_0 - a_j^0| \mid j = 0, 1, \ldots, h-2\} =$
$= \{1, 2, \ldots, 2h-2\}$, and
$\{|a_j^0 - j|, |a_j^0 - j - 1| \mid j = 0, 1, \ldots, h-2\} =$
$= \{1, 2, \ldots, 2h-2\}$.
Hence $V(K_{4h-2}, \mathcal{B})$ is a $N(4h-2, 2h+1, 2)$.

Now suppose $\alpha \geq 2$. It is
$A_0 = \{|a_j^0 - j|, |a_j^0 - j - 1| \mid j = 0, 1, \ldots, h-2\} =$
$\{1, 2, \ldots, 2h-2\}$,
$A_\rho = \{|a_j^\rho - j| \mid j = 0, 1, \ldots, h-1\} \cup \{|a_j^\rho - j - 1| \mid j = 0, 1, \ldots, h-2\} =$
$= \{(2h-1)\rho + i \mid i = 0, 1, \ldots, 2h-2\}$.
Since

$$\left(\cup_{\rho=1}^{\alpha-1} A_\rho\right) \cup A_0 = \{1, 2, \ldots, (2h-1)\alpha - 1\},$$

then

$$[\infty, 0, a_0^0, 1, a_1^0, \ldots, h-2, a_{h-2}^0, h-1],$$

and, for $\rho = 1, 2, \ldots, \alpha - 1$,

$$[0, a_0^\rho, 1, a_1^\rho, \ldots, h-1, a_{h-1}^\rho]$$

are $\pmod{\alpha(4h-2)-1}$ the base blocks of a $P(\alpha(4h-2), 2h, 1)$. To prove that this path design is nested by $(V(K_v), \mathcal{B})$ we value the following difference sets:

$D_0^1 = \{|c_0 - j| \mid j = 0, 1, \ldots, h-2\}$, $D_0^2 = \{|c_0 - a_j^0| \mid j = 0, 1, \ldots, h-2\}$, $D_t^1 = \{|c_{\alpha-t} - j| \mid j = 0, 1, \ldots, h-1\}$, and $D_t^2 = \{|c_{\alpha-t} - a_j^{\alpha-t}| \mid j = 0, 1, \ldots, h-2\}$, for $t = 1, 2, \ldots, \alpha - 1$.

At first we prove that

$$c_{\alpha-t} = \begin{cases} [(2h-1)t+1]/2 & \text{if } t \text{ is odd} \\ [(2h-1)(2\alpha+t)-2]/2 & \text{if } t \text{ is even} \end{cases} \quad (2)$$

Suppose t odd. The (2) is true for $t = 1$. Suppose that (2) is true for $t = 2\mu - 1$. Then, for $t = 2\mu + 1$, it is
$c_{\alpha-t+2} = c_{\alpha-2\mu-1} = a_0^{\alpha-2\mu} - c_{\alpha-2\mu} + h = a_0^{\alpha-2\mu} - a_0^{\alpha-2\mu+1} + c_{\alpha-2\mu+1} =$
$= (2\mu+1)h - \mu = [(t+2)(2h-1)+1]/2$. Hence (2) holds for each odd $t \le \alpha - 1$.
Let t be even. The (2) is true for $t = 2$. Suppose that (2) is true for $t = 2\mu$. Then, for $t = 2\mu + 2$, it is
$c_{\alpha-t+2} = c_{\alpha-2\mu-2} = a_0^{\alpha-2\mu-1} - c_{\alpha-2\mu-1} + h = a_0^{\alpha-2\mu-1} - a_0^{\alpha-2\mu} + c_{\alpha-2\mu} =$
$= a_0^{\alpha-2\mu-1} - a_0^{\alpha-2\mu} + (2h-1)(\alpha+\mu+1) - 1 =$
$= [(2h-1)(2\alpha+t+2)-2]/2$. So (2) is completely proved for each $t \le \alpha - 1$.

By a simple calculation we obtain that:
$D_{2\mu-1}^1 = \{|2(\mu-1)h - \mu + 2 + j| \mid j = 0, 1, \ldots, h-1\}$,
$D_{2\mu-1}^2 = \{|h(2\alpha - 2\mu + 1) + \mu - \alpha + j| \mid j = 0, 1, \ldots, h-2\}$,
$D_{2\mu}^1 = \{|(2h-1)(\alpha-\mu) + j| \mid j = 0, 1, \ldots, h-1\}$,
$D_{2\mu}^2 = \{|(2h-1)\mu - j| \mid j = 0, 1, \ldots, h-2\}$,
$D_0^1 = \{|[(2h-1)\alpha + 2j + 2]/2| \mid j = 0, 1, \ldots, h-2\}$ if α is even,
$D_0^1 = \{|[(2h-1)\alpha - 2j - 1]/2| \mid j = 0, 1, \ldots, h-2\}$ if α is odd,
$D_0^2 = \{|[(2h-1)\alpha - 2j]/2| \mid j = 0, 1, \ldots, h-2\}$ if α is even, and
$D_0^2 = \{|[(2h-1)\alpha + 2j + 1]/2| \mid j = 0, 1, \ldots, h-2\}$ if α is odd.

It follows that:
$D_{2\mu-1}^1 \cup D_{2\mu}^2 = \{|2(\mu-1)h - \mu + 2 + i| \mid i = 0, 1, \ldots, 2h-2\}$,
$D_{2\mu}^1 \cup D_{2\mu-1}^2 = \{|(2h-1)(\alpha-\mu) + i| \mid i = 0, 1, \ldots, 2h-2\}$.

Suppose that α is odd. Then

$$\cup_{\mu=1}^{(\alpha-1)/2} (D_{2\mu-1}^1 \cup D_{2\mu}^2) = \{1, 2, \ldots, (\alpha-1)(2h-1)/2\},$$

$\cup_{\mu=1}^{(\alpha-1)/2} (D_{2\mu}^1 \cup D_{2\mu-1}^2) =$
$$= \{(\alpha+1)(2h-1)/2, (\alpha+1)(2h-1)/2 + 1, \ldots, (\alpha-1)(2h-1) + 2h - 2\},$$

and

$$D_0^1 \cup D_0^2 = \{[\alpha(2h-1) - 2h + 3 + 2i]/2, \quad | \ i = 0, 1, \ldots, 2h-3\}.$$

Let α be even. Then

$$\cup_{\mu=1}^{(\alpha-2)/2} (D_{2\mu-1}^1 \cup D_{2\mu}^2) = \{1, 2, \ldots, (\alpha-2)(2h-1)/2\},$$

$$\cup_{\mu=1}^{(\alpha-2)/2} (D_{2\mu}^1 \cup D_{2\mu-1}^2) = \{(\alpha+2)(2h-1)/2 + i \ | \ i = 0, 1, \ldots, 2h-2\},$$

$$D_0^1 \cup D_{\alpha-1}^2 = \{[\alpha(2h-1) + 2]/2 + i \ | \ i = 0, 1, \ldots, 2h-3\},$$

and

$$D_0^2 \cup D_{\alpha-1}^1 = \{[2h(\alpha-2) - \alpha + 4]/2 + i \ | \ i = 0, 1, \ldots, 2h-2\}.$$

Therefore for any $\alpha \geq 2$ it is

$$\cup_{\sigma=0}^{\alpha-1} (D_\sigma^1 \cup D_\sigma^2) = \{1, 2, \ldots, \alpha(2h-1) - 1\}$$

and the proof is completed. □

Theorem 8 *For every $v \equiv 0 \pmod 3$, $v \geq 6$, there is a nested path design $P(v, 4, 1)$ except possibly if $v = 39$.*

Proof: If $v \equiv 0 \pmod 6$ the result follows from Theorem 7. Let $v \equiv 3 \pmod 6$, $v \geq 9$.

Case $v = 9$. Let $V(K_9) = Z_3 \times Z_3$. The base blocks, $(\mathrm{mod}\ (3, -))$, are:
$< (1,0), (0,0), (2,1), \widehat{(1,2)}; (0,2) >$, $< (0,1), (1,1), (1,0), \widehat{(0,2)}; (0,0) >$,
$< (1,1), (0,0), (0,2), \widehat{(2,2)}; (1,2) >$, $< (2,0), (0,2), (0,1), \widehat{(1,2)}; (1,1) >$.

Case $v = 15$. Let $V(K_{15}) = Z_7 \times Z_2 \cup \{\infty\}$. The base blocks, $(\mathrm{mod}\ (7, -))$, are:
$< (1,1), (0,1), (1,0), \widehat{(0,0)}; (4,0) >$, $< (2,1), (0,1), (0,0), \widehat{(5,0)}; (2,0) >$,
$< (3,1), (0,1), (3,0), \widehat{(0,0)}; (2,1) >$, $< \infty, (2,0), (0,1), \widehat{(6,0)}; (3,1) >$,
$< \infty, (3,1), (0,0), \widehat{(2,1)}; (1,0) >$.

Case $v = 21$. We give a decomposition $2K_{7,7,7} \to \widehat{G_5}$. Let $V(K_{7,7,7}) = \cup_{i=0}^2 Z_7 \times \{i\}$. The base blocks $(\mathrm{mod}\ (7, -))$ are:
$< (3,0), (3,1), (6,0), \widehat{(6,2)}; (0,2) >$, $< (0,1), (0,2), (6,1), \widehat{(3,0)}; (5,0) >$,
$< (0,2), (6,0), (1,2), \widehat{(6,1)}; (5,1) >$, $< (0,1), (3,2), (6,1), \widehat{(0,0)}; (3,0) >$,
$< (4,2), (0,0), (5,2), \widehat{(6,0)}; (5,1) >$, $< (5,2), (0,1), (6,2), \widehat{(3,0)}; (0,0) >$,
$< (6,1), (1,0), (3,1), \widehat{(2,0)}; (4,0) >$.
Since there exists a nested $P(7, 4, 1)$ (see Theorem 5), then there is a nested $P(21, 4, 1)$.

Case $v = 33$. A decomposition of $2K_{2,2,2,2} \to \widehat{G_5}$ is given in the proof of Case $v = 40$ of Theorem 6. By Lemma 4 (with $w = 1$) and the existence of a $GD(4, \{4\}; 16)$, it follows the existence of a nested $P(33, 4, 1)$.

Now, proceeding as in Theorem 6, the proof follows from Lemma 4 (with $w = 0$) and the existence of the following GDDs ([2]): $GD(3, \{3, 5\}; 23)$, $GD(3, \{3\}; 3 + 6\alpha)$ for any $\alpha \geq 1$, $GD(3, \{3, 7\}; 7 + 6\alpha)$ for any $\alpha \geq 2$ and $GD(3, \{3, 7\}; 7 + 6\alpha)$ for any $\alpha \geq 2$. □

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