Permanents and Determinants of Graphs: A Cycle Polynomial Approach

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ABSTRACT. Various connections have been established between the permanent and the determinant of the adjacency matrix of a graph. Connections are also made between these scalars and the number of perfect matchings in a graph. We establish conditions for graphs to have determinant 0 or ± 1. Necessary conditions and sufficient conditions are obtained for graphs to have permanent equal to 0 or to 1.

1 Connections and Interconnections

We denote the permanent and determinant of the adjacency matrix of a graph G by per(G) and det(G), respectively.

A *cycle cover* C of a graph G is a cover (spanning subgraph) of G each component of which is isomorphic to a cycle graph C_j of order $j = 1, 2, \ldots$. It is convenient to treat a K_1 component of C as a cycle component of order 1 and we call C_1 a *trivial* component. Also we regard a K_2 component of C as a cycle component of order 2 and we sometimes call C_2 an *edge* component. Cycle components C_j for $j \geq 3$ are called *proper* cycle components. The cycle polynomial (sometimes called the circuit polynomial, see [1]) of G is denoted by $C(G; \underset{\sim}{w})$,

$$C(G; \underset{\sim}{w}) = \sum_{C} \prod_{j \geq 1} w_j^{c_j}$$

where $\underline{w}= (w_1, w_2, \ldots)$ is the cycle weight vector in which w_j is the weight associated with each cycle of length $j = 1, 2, 3\ldots$ in G. The sum is over all cycle covers $\mathcal{C}$ of G and C_j is the number of cycle components of order j in the cycle cover $\mathcal{C}$.

The following result was established in [2]. It gives the basic connections between the cycle polynomial, the permanent and the determinant of a graph.

Lemma 1. *For any graph G with adjacency matrix $\mathbf{A}(G)$,*

1) $per(G) = per(\mathbf{A}(G)) = C(G; (0, 1, 2, 2, 2, \ldots))$

2) $\det(G) = \det(\mathbf{A}(G)) = C(G; (0, -1, 2, -2, 2, \ldots))$

3) $\det(-G) = \det(-\mathbf{A}(G)) = C(G; (0, -1, -2, -2, -2, \ldots))$ □

These scalars associated with matrices have been widely studied (see, for example, [3]). Here we restrict our discussion to matrices that are adjacency matrices of graphs. The permanent and determinant of graph adjacency matrices are of importance in graph theory, in particular in the study of graph spectra (see [4]).

A cycle cover of a graph is called non-trivial if the cover contains no trivial component (isolated vertex). The following theorem gives a necessary and sufficient condition for the permanent and determinant scalars to be equal or to differ only in sign.

Theorem 1. *Let G be a graph. Then*

(1) *$per(G) = \det(G)$, if and only if G has no non-trivial cycle cover with an odd number of even cycles.*

(2) *$per(G) = -\det(G)$, if and only if G has no non-trivial cycle cover with an even number of even cycles.*

Proof: (1) Let H_i be a non-trivial cycle cover of G with a_i edge components, b_i even proper cycles and c odd proper cycles. Then, by Lemma 1, the contribution of H_i to per(G) is $2^{b_i+c_i} = w(H_i)$, while the contribution of H_i to $\det(G)$ is $(-1)^{a_i+b_i}2^{b_i+c_i} = (-1)^{a_i+b_i}w(H_i)$. Therefore

$$\text{per}(G) = \sum_i w(H_i) \text{ and } \det(G) = \sum_i (-1)^{a_i+b_i} w(H_i).$$

Clearly per$(G) = \det(G)$ if and only if, for each i, $a_i + b_i$ is even. Since $a_i + b_i$ is the number of even cycles in H_i, the result follows.

(2) In this case $a_i + b_i$ is odd for each i so that

$$\det(G) = \sum_i -w(H_i) = -\sum_i w(H_i) = -\mathrm{per}(G).$$

The converse is easily seen. That is, if

$$\sum_i (-1)^{a_i+b_i} w(H_i) = \sum_i -w(H_i)$$

then $a_i + b_i$ is odd for each i. □

A cycle cover C of a graph G is said to be *odd* (respectively, *even*) if the number of components in C is odd (respectively, even). The following theorem gives the connections between per(G) and det(G) based on the parities of the cycle covers of G.

Theorem 2. *Let G be a graph of order n.*

1) *If G has no odd non-trivial cycle cover, then* $\mathrm{per}(G) = (-1)^n \det(G)$.

2) *If G has no even non-trivial cycle cover, then* $\mathrm{per}(G) = (-1)^{n+1} \det(G)$.

Proof: (1) Since G has no odd non-trivial cycle cover, then either (i) G has no non-trivial cycle cover, or (ii) all the non-trivial cycle covers of G are even. In case (i), every cycle cover will contain an isolated vertex component and hence $\mathrm{per}(G) = \det(G) = 0$.

In case (ii), $\mathrm{per}(G) = \sum_i 2^{s_i}$, where s_i is the number of proper cycle components in the ith non-trivial cycle cover and the summation is over all non-trivial cycle covers of G. Thus, from Lemma 1(3), $\det(-G) = \sum_i (-2)^{s_i}(-1)^{t_i}$, where t_i is the number of edge components in the ith non-trivial cycle cover. However, for all i, $s_i + t_i$ is even since it is the number of components in the ith non-trivial cycle cover. Thus,

$$\det(-G) = \sum_i 2^{s_i} = \mathrm{per}(G).$$

Since $\det(-G) = (-1)^n \det(G)$, the result follows.

Part (2) is proved similarly. □

2 Permanents, Determinants, and Perfect Matchings

A *perfect matching* of a graph G is a cover of G each component of which is an edge component. In this section we examine relations between perfect matchings and the scalars per(G) and det(G).

Let G be a graph with $n = 2k$ vertices. If G has a perfect matching H, then the contribution of H to $\det(G)$ is

$$w(H) = (-1)^k.$$

Since every perfect matching in G contains the same number of edge components (k), it follows that $w(H)$ is the same for every perfect matching in G. Therefore, if γ is the number of perfect matchings in G, the contribution of all the perfect matchings to $\det(G)$ is $\gamma(-1)^k$. The contribution of the perfect matchings to $\text{per}(G)$ is γ. Hence the following result.

Theorem 3. *Let G be a graph with $n = 2k$ vertices. If the only non-trivial cycle covers in G are perfect matchings, then $\det(G) = \gamma(-1)^k$ and $per(G) = \gamma$, where is the number of perfect matchings in G.* □

Corollary 3.1. *If G is a forest with $n = 2k$ vertices and a perfect matching, then*

$$\det(G) = (-1)^k per(G) = (-1)^k.$$

□

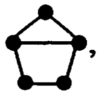
Corollary 3.2. *If G is a forest, then $\det(G) = per(G) = 0$ if and only if G has no perfect matching.* □

The following result shows the effect of the absence of perfect matchings.

Theorem 4. *If a graph G has no perfect matching, then $per(G)$ and $\det(G)$ are both even.*

Proof: If G has no perfect matching, then either (i) every cover of G is trivial, or (ii) G has non-trivial covers. In case (i), $\det(G) = \text{per}(G) = 0$. In case (ii), the contribution of every non-trivial cover to $\text{per}(G)$ is a non-zero power of 2, and the contribution to $\det(G)$ is plus or minus a non-zero power of 2, so that $\text{per}(G)$ and $\det(G)$ are sums of powers of 2. Hence they are both even. □

Remark: If G has no perfect matching but has non-trivial covers, then $\text{per}(G) \neq 0$. However, $\det(G)$ may be equal to zero. For example,

for G : , $\text{per}(G) = 2 + 2 = 4$
$\det(G) = 2 + (-1)2 = 0$

3 Graphs With Zero Permanent or Zero Determinant

We now investigate graphs G for which $\det(G) = 0$ or $\text{per}(G) = 0$. The following theorem characterizes graphs which have zero permanent.

Theorem 5. *Let G be a graph. Then per$(G) = 0$ if and only if G has no non-trivial cycle cover.*

Proof: This is an immediate consequence of Lemma 1(1). □

This theorem shows that graphs with permanent zero are easily characterized. This is certainly not true for graphs with zero determinant. Such graphs are called *singular*. It is not difficult to deduce the following result.

Lemma 2. *If G has no non-trivial cover, then G is singular.* □

Our interest in singular graphs stems from the fact that if a graph is singular, then 0 is an eigenvalue of G.

It is not difficult to see that if a graph is a cycle C_n with $n \geq 3$ vertices, then its only non-trivial covers are (i) two perfect matchings (if and only if n is even), and (ii) C_n itself. The contribution of each perfect matching to $\det(C_n)$ is $(-1)^{n/2}$, while the contribution of the cover consisting of C_n is $2(-1)^{n+1}$. Therefore,

$$\det(C_n) = 2(-1)^{n/2} + 2(-1)^{n+1} = \begin{cases} 2(-1)^{n/2} - 2 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}.$$

Hence we obtain the following result.

Lemma 3. *C_n is singular iff $n \equiv 0 \pmod 4$.* □

The following theorem gives a sufficient condition for a graph to be singular.

Theorem 6. *If a graph G contains a singular subgraph S, with no isolated vertex, such that every non-trivial cover of G contains a cover of S, then G is singular.*

Proof: Every cover H of G can be partitioned into a cover U of $G \setminus S$ and a cover V of S. Furthermore, the union of any cover U of $G \setminus S$ with any cover V of S is a cover of G. The contribution of a cover $H = U \cup V$ to $\det(G)$ is $w(H) = w(U)w(V)$. Thus,

$$\begin{aligned} \det(G) &= \sum_H w(H) = \sum_H w(U)w(V) \\ &= \sum_U w(U) \cdot \sum_V w(V) = \sum_U w(U) \cdot \det(S) = 0 \end{aligned}$$

since S is singular. □

From Lemma 1, we have that for any graph G

$$\det(G) = \sum_i w(H_i) = \sum_i (-1)^{r_i} 2^{s_i},$$

where H_i is a non-trivial cover of G containing r_i even cycles and s_i is the number of proper cycles in H_i. Thus, $s_i \geq 0$. Suppose that G is singular. Then $\sum_i(-1)^{r_i}2^{s_i} = 0$, so that some terms are positive and some are negative and this can be written:

$$2^{a_i} + 2^{a_2} + \cdots + 2^{a_m} = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_n}, \tag{1}$$

where a_i $(i = 1, 2, \ldots, m)$ and b_j $(j = 1, 2, \ldots, n)$ are non-negative integers.

Suppose that G has perfect matchings. Then some of the a_i or b_j are zero. Hence some of the 2^{a_i} or some of the 2^{b_j} are unity. Furthermore, all perfect matchings must have the same number of edge components and thus they will each contribute -1 or $+1$. Their contributions must appear on one side of Equation (1). However, 2 divides one side of Equation (1). Hence there must be an even number of terms equal to unity; that is, G has an even number of perfect matchings.

Suppose that G has no perfect matchings but has other non-trivial covers. Then G has a cover in which the number of proper cycles is minimal (we call these *minimal covers*). Let the number of proper cycles in a minimal cover be c. Then c is equal to some a_i or b_j in Equation (1). Also 2^c must divide each term in Equation (1). On dividing by 2^c, some terms will be equal to unity. The number of these terms is equal to the number of minimal covers.

If all of the unit terms appear on one side of the equation, then there must be an even number of unit terms. It they appear on both sides, then the numbers on each side must have the same parity in order for Equation (1) to hold.

Our discussion yields the following result.

Theorem 7. *If G is a singular graph, then*

(1) *G has an even number (greater than zero) of perfect matchings, or*

(2) *G has no perfect matching and*

  (i) *The number of minimal covers is even and they each have an even number of even cycles or they each have an odd number of even cycles, or*

  (ii) *The number of minimal covers with an even number of even cycles has the same parity as the number of minimal covers with an odd number of even cycles.* □

4 Graphs with Permanent Equal to 1

We have shown (Corollary 3.1) that for any forest G with $n = 2k$ vertices and a perfect matching,

$$\text{per}(G) = 1 \text{ and } \det(G) = (-1)^k.$$

We have also shown (Corollary 3.2) that if a forest G has no perfect matching, then $\text{per}(G) = \det(G) = 0$.

Suppose that for some graph G, $\text{per}(G) = 1$. Then, the sum of the weights of the non-trivial covers in G is unity. By Lemma 1, this implies that G contains no cycle and also that G has only one non-trivial cover which is a perfect matching. So G must be a forest with a perfect matching. We therefore have the following result which characterizes graphs with permanent equal to unity.

Theorem 8. *Let G be a graph. Then $\text{per}(G) = 1$ if and only if G is a forest with a perfect matching.* □

Obviously, the permanent of a graph cannot be negative.

5 Unit Graphs

Definition 1. *Let G be a graph. Then G is called* positive *(respectively,* negative*)* unit *provided* $\det(G) = 1$ *(respectively,* $\det(G) = -1$*).* □

We note that unit graphs are precisely those graphs with the property that the product of their eigenvalues is ± 1..

Theorem 9. *Let G be a unit graph, then G has an odd number of perfect matchings.*

Proof: In the case that G is a positive unit graph, then Equation (1) becomes

$$2^{a_i} + 2^{a_2} + \cdots + 2^{a_m} = 2^{b_i} + 2^{b_2} + \cdots + 2^{b_n} + 1. \tag{2}$$

However, in this case, each a_i is the number of proper cycles in a non-trivial cover with an even number of even cycles, and each b_j is the number of proper cycles in a non-trivial cover with an odd number of even cycles.

We consider two cases for Equation (2); (i) all the unit terms occur on one side of the equation, (ii) the unit terms occur on both sides of the equation.

Case (i). In this case, all the unit terms must occur on the right hand side of the equation. Since the left hand side is divisible by 2, then there must be an even number of unit terms on the right hand side. This implies that for an odd number of b_j, $2^{b_j} = 1$. But only perfect matchings can contribute 1 to $\det(G)$, thus G must have an odd number of perfect matchings.

Since G must have at least one perfect matching, then G must have an even number n of vertices. It also follows that each perfect matching of G contains an odd number of even cycles. In this case this means an odd number of edges. Thus, $n/2$ is odd, that is, $n \equiv 2 \pmod 4$.

Case (ii). In this case, the number of unit terms on each side of the equation must have the same parity.

Now, only perfect matchings can contribute ± 1 to $\det(G)$. Also every perfect matching has the same number of edges, that is $n/2$, where n is the order of G. It follows that all the contributions from perfect matchings must have the same sign. Hence each perfect matching contributes 1 or each perfect matching contributes -1. We can therefore conclude that there must be an odd number of unit terms resulting from perfect matchings and that these must be on the left hand side of Equation (2). Since these perfect matchings are covers with an even number of even cycles, this further implies that $n/2$ is even, or $n \equiv 0 \pmod 4$.

In the case that G is a negative unit graph, Equation (2) is essentially unchanged – we just reverse the interpretations of the a_i and b_j, and the proof is analagous. □

If G is a graph with $2k$ vertices and with exactly one non-trivial cover, which is a perfect matching, then $\det(G) = (-1)^k$. Hence we have the following result.

Theorem 10. *Let G be a graph with exactly one non-trivial cover which is a perfect matching. Then G is a unit graph.* □

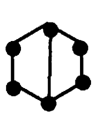
Let

$$D^-(G;x) = \det(x\mathbf{I}_n - \mathbf{A}(G)) = \sum_{j=0}^{n} c_j x^{n-j}$$

be the characteristic polynomial of a graph G of order n. We remark that in [5] we defined a graph to be characteristic palindromic provided that the coefficients c_j of the characteristic polynomial satisfy $|c_j| = |c_{n-j}|$ for each $j = 0, 1, \ldots, n$. It follows that every palindromic graph is unit.

Note that neither the converse of Theorem 9 nor the converse of Theorem 10 is true. For example, the complete graph K_4 has 3 perfect matchings, but since $\det(K_4) = -3$, K_4 is not unit.

Similarly, the theta graph,

Θ: 

is a unit graph, but has 3 perfect matchings and a C_6 cover. Thus, Θ is a counterexample to the converse of Theorem 10. We remark that Θ is characteristic palindromic. At this time we do not know of any counterexample to the converse of Theorem 10 that is not characteristic palindromic.

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