Queen Domination of Hexagonal Hives

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Abstract. A chess-like game board called a hive, consisting of hexagonal cells, and a board piece called a queen are defined. For queens on hexagonally shaped hives, values are obtained for the lower and independent domination numbers, the upper independence number and the diagonal domination number, as well as a lower bound for the upper domination number. The concept of a double column placement is introduced.

1. Introduction

Domination problems associated with the placement of various chess pieces on chessboards have been studied widely, on the one hand for their intrinsic interest but on the other hand because they can be also be formulated as domination problems in graph theory [1, 5]. In this paper a hexagonally shaped board with hexagonal cells is introduced, together with a board piece called a "queen" which can execute moves on lines through any of the six sides of a cell on which she is placed. Because of the similarity with the hexagonal structure of honeycombs, the board is called a hive. It could however also be seen as a playing board used in war games [4], or, trivially, as a variation of the board for Chinese checkers.

Various domination parameters of queens on hexagonally shaped hives are derived, similar to those derived for square hives in [6].

2. Definitions relating to hexagonal hives

For any positive integer n, a hive of order n has n rows consisting of hexagonal cells. A cell in row r and column c is denoted by $\langle r,c \rangle$.

The center of a hexagonal hive is taken as the origin, in cell $\langle 0,0\rangle$, with all rows in the upper (respectively lower) half of the hive having positive (negative) row numbers, as illustrated in fig. 1. The column numbers are taken as positive in the right hand side of the hive and negative in the left hand side. For rows with r even (respectively odd), the column numbers of the cells are even (odd).

A hexagonal hive of radius R or order $n = 2R+1$, denoted by H_n, is defined as follows : Start with a cell at the origin $\langle 0,0\rangle$. Surround this cell with the six neighbours, to form a circle with radius 1. Each additional circle of radius q ($1 \le q \le R$) contains $6q$ cells and creates a hive with $n = 2q+1$ rows. In total, a hive of radius R contains $1+6(1+2+...+R) = 1+3R(R+1) = n^2 - R(R+1)$ cells. A distinction is made between hives with even radii, $R = 2k$, and odd radii, $R = 2k+1$, $k \ge 0$. Fig. 1 illustrates a hexagonal hive of order 11 ($R = 5$).

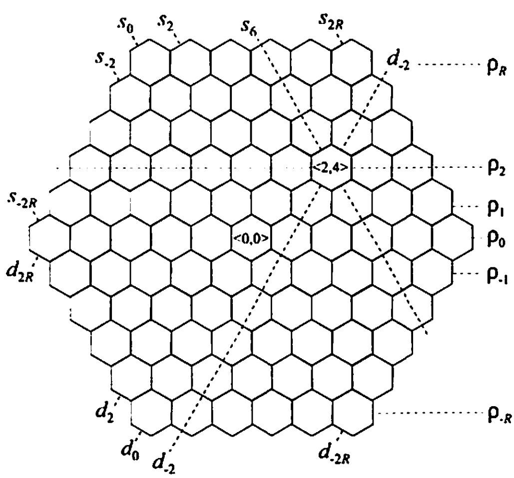

Fig. 1. Rows and diagonals on H_{11}.

Let ρ_r denote row number r, $r \in [-R;R]$, which is defined by the cells $\{\langle r,-(2R-|r|)+2j\rangle : 0 \le j \le (2R-|r|)\}$. Let s_i denote the diagonal consisting of all cells $\langle r, c\rangle$ of H_n such that $r + c = i$. Such a diagonal is called a *sum diagonal.* Similarly, let d_j denote the diagonal consisting of all cells $\langle r, c\rangle$ of H_n such that $r - c = j$. Such a diagonal is called a *difference diagonal.* Three *lines* intersect each cell, being the row and two diagonals through the cell. The three lines for cell $\langle 2\ ,\ 4 \rangle$ are shown by the dashed lines in fig. 1. The complete set of diagonals for H_n is given by $D(H_n) = \{s_{2i} : -R \le i \le R\} \cup \{d_{2j} : -R \le j \le R\}$, with cardinality $|D(H_n)| = 2n$.

The *main diagonals* are d_0 and s_0. The *outer diagonals* are the four diagonals s_{-2R}, s_{2R}, d_{-2R}, d_{2R} ; these outer diagonals plus row $-R$ and row $+R$ form the *border* on the *edge* of the hive.

In the definition above, a *circle* consists of six adjacent lines of equal length. We also define a *ring* as any set of six adjacent lines, not necesarily of equal length, but all with length at least 2 cells.

3. Definitions relating to queens and domination parameters

When a queen is placed on a particular cell, she *covers* (or *dominates*) that cell and all the cells in each of the three lines associated with that cell. Also, the queen *occupies* the three lines associated with her cell. A line is *empty* if there is no queen on that particular line. An empty row is *covered* if each cell in the row is covered by at least one diagonal from queens in other rows. Only one queen may be placed on a cell.

A *placement* on a hive is the set of cells which contain queens. Two queens *attack* each other if they are on the same line. A placement is *independent* (or the queens are *non-attacking*) if no queen in the set attacks another. An example of an independent placement is shown in fig. 2.

A cell which is covered by only one queen is called a *private neighbour* of that queen. A line from a queen is *essential* if it contains at least one private neighbour of that queen. A queen is *essential* if she has at least one private neighbour (which may be her own cell).

The *lower* (respectively *independent*) *domination number* γ_n (i_n) is the smallest number of (non-attacking) queens that can be placed on H_n so that each cell is covered. The *upper domination* (respectively *independence*) *number* Γ_n (β_n) is the largest number of (non-attacking) queens that can be placed on H_n so that each cell is covered and all the queens are essential. It is easy to see that

$$\gamma_n \le i_n \le \beta_n \le \Gamma_n, \tag{1}$$

which is similar to an inequality that was first noted in [3] for chessboards.

4. Double Column Placements

A placement of queens in a column will be termed a *column placement* if each cell of the column contains a queen within a certain range of rows, and no other cell in the column contains a queen. The number of queens in the column is the *size* of the column placement. A column placement C is defined by three parameters, namely the column c, the first row r_1 containing a queen and the size σ :

$$C(c, r_1, \sigma) = \{\langle r_1 + 2i , c\rangle : 0 \le i \le \sigma - 1\}.$$

For the sake of brevity some of the parameters will be suppressed occasionally and C_c will be used for $C(c, r_1, \sigma)$.

A fundamental property of a column placement is that if it covers any two parallel diagonals, then all possible diagonals between those two are also covered. We say that this set of diagonals is *associated* with the column placement; this set is denoted by $D(C_c)$, or simply D_c, where

$$D_c = D(C_c) = \{s_{r_1+c-2+2i} : 1 \le i \le \sigma\} \cup \{d_{r_1-c-2+2j} : 1 \le j \le \sigma\}.$$

Note that $|D_c| = 2\sigma$.

Let C_a and C_b denote column placements in different columns a and b respectively, defined by $C_a = C(a, r_1, \sigma_1)$ and $C_b = C(b, r_2, \sigma_2)$, with the associated diagonal sets D_a and D_b. Then a *Double Column Placement* (henceforth called a DCP) is defined by $C_a \cup C_b$, with the associated diagonal set $D_a \cup D_b$, where in general $|D_a \cup D_b| \le |D_a| + |D_b|$. The *size* of a DCP is $\sigma = |C_a| + |C_b| = \sigma_1 + \sigma_2$, and its *separation* is defined as $|a - b|$. Without loss of generality we take r_1 as the lowest numbered row in the DCP, and call the cell $\langle r_1 , a\rangle$ the *base* of the DCP.

A DCP is said to be *complete* if it has the property that if any two parallel lines (rows or diagonals) are occupied by two queens of the DCP, then all possible parallel lines between those two are also occupied. In order to occupy all the rows in a certain range it is necessary that $r_2 = r_1 + 1$; this implies that one column has to be even and the other odd. There will be no gap between the two sets of diagonals if the separation is small enough. A DCP $C_a \cup C_b$ is *independent* if $D_a \cap D_b = \emptyset$ and all the queens are on different rows. A

DCP $C_a \bigcup C_b$ is said to be *efficient* if it is complete as well as independent. Fig. 2 illustrates an efficient DCP of size 5.

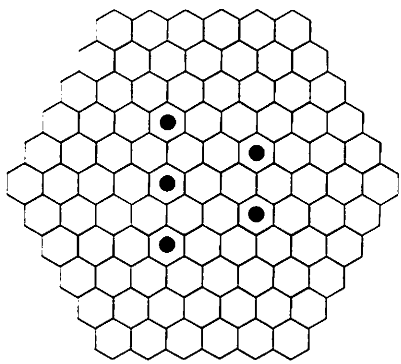

Fig. 2. A double column placement of five queens on H_{11}.

Using these definitions, the following lemma for hives follows :

Lemma 1 : *A double column placement with base* $\langle r_1, a\rangle$ *and size* $\sigma = \sigma_1 + \sigma_2$, *denoted by* $C(a, r_1, \sigma_1) \bigcup C(b, r_1 + 1, \sigma_2)$, *is efficient if and only if the following properties apply* : (1) σ is odd; (2) $\sigma_1 = \lceil \sigma / 2 \rceil$, $\sigma_2 = \lfloor \sigma / 2 \rfloor$; (3) $|a - b| = \sigma$.

Proof : First consider the case where $b = a + S$, for a positive integer *S*. The base has the associated difference diagonal d_{J_1}, where the index $J_1 = r_1 - a$. The top queen in column a is in cell $\langle r_1 + 2(\sigma_1 - 1), a\rangle$, and covers s_{I_1}, where $I_1 = r_1 + a + 2\sigma_1 - 2$. The lowest queen in column b is in cell $\langle r_1 + 1, a + S\rangle$ and covers s_{I_2}, where $I_2 = r_1 + 1 + a + S$, and the upper queen in this column is in cell $\langle r_1 + 1 + 2(\sigma_2 - 1), a + S\rangle$ and covers d_{J_2} where $J_2 = r_1 + 2\sigma_2 - 1 - a - S$.

Assume that the DCP is efficient. Then $I_2 = I_1 + 2$ and $J_2 = J_1 - 2$, from which it follows that $S = 2\sigma_1 - 1$ and $\sigma_2 = \sigma_1 - 1$. With $\sigma = \sigma_1 + \sigma_2$, the properties of the lemma follow. Conversely, if the properties of the lemma hold, it is easy to show that the DCP is efficient.

Similar arguments hold for the case where $b = a - S$. □

5. Lower and independent domination numbers

We prove a lower bound for γ_n and show that a DCP achieves this bound. In this proof we use the concept of the *largest empty ring* (or LER), which consists of the six empty lines closest to the border. For example, in fig. 2 the border forms the LER. However, if a queen were to be placed in row $-R$, the LER would consist of row $-(R-1)$ with the other lines as before.

Lemma 2. *For a hive with odd radius* $R = 2k+1$, $\gamma_{4k+3} \geq 2k+1$, for $k \geq 0$.

Proof. We prove the theorem by showing that the LER cannot be dominated by *2k* queens. Firstly, consider any placement of *2k* independent queens on the board. If the LER is the border, the LER cannot be dominated because there are $6(2k+1)$ cells on the border and each of the *2k* queens can only dominate 6 cells on the border.

If the LER is not the border, the LER can be constructed by starting with the border and then moving the sides of the ring one by one until an empty ring is obtained. For each move, the number of cells in the ring decreases by one - provided the ring had six sides before each move. This is easy to check. The queens at the corners cause two moves, and the other queens that end up outside the ring cause one move.

The queens that caused the ring to decrease by one cell (one move) dominate only four cells of the ring. Thus, in effect, one cell less is dominated. The queens in the corners, that caused the ring to decrease by two cells (two moves), dominate only two cells of the ring. Thus, in effect, two cells less are dominated.

Thus we see that the smaller the LER is, the more cells of the LER are not dominated. It is easy to see that if the *2k* queens are not independent, even more cells of the LER are not dominated. □

Corollary 1. *If a placement of R queens dominates* H_{2R+1}, $R \geq 1$, *(R odd or even) then the LER must be the border.*

Proof. H_{2R+1} has *6R* border cells. Therefore, by the same argumant as above, any LER which is smaller than the border cannot be dominated by the *R* queens. □

Lemma 3. *For a hive with even radius* $R = 2k$, $\gamma_{4k+1} \geq 2k+1$, for $k \geq 0$.

Proof. We show that *2k* queens cannot dominate H_{4k+1}, $k \geq 1$. Suppose we have a set of *2k* queens dominating H_{4k+1}; then from Corollary 1 the border must be empty. Also each border cell is covered exactly once.

The corner cells can only be dominated by a queen on a main diagonal. Consider any main diagonal. There must be one queen on it, and the remaining $2k-1$ queens are on the two sides. Any queen on one side of the diagonal covers four border cells on that side, and two border cells on the other side. To dominate the same number of border cells on the two sides therefore requires the same number of queens on either side of the diagonal. This is impossible, because $2k-1$ is odd.

The case $k=0$ is trivially true, with a single queen covering H_1. □

Lemma 4. *For* $n=4k+1$ *and* $n=4k+3$, $k \geq 0$, $i_n \leq 2k+1$.

Proof. We first consider an odd hive, with $R=2k+1$, $k \geq 1$, and show that an efficient DCP with base $\langle -k,-k \rangle$ and size $2k+1$, denoted by P^k, covers the hive. The associated set of diagonals is $D^k = \{s_{2i} : -k \leq i \leq k\} \cup \{d_{2j} : -k \leq j \leq k\}$. Such a placement is illustrated in fig. 2 for the case $k=2$.

All cells in the non-empty rows, $\{\rho_r : -k \leq r \leq k\}$, are covered, with the $k+1$ rows closest to each edge empty. The rightmost cell in row $(k+1)$, namely cell $\langle k+1, 3k+1 \rangle$, is covered by d_{-2k}, the rightmost difference diagonal in D^k. The leftmost difference diagonal in D^k, d_{2k}, covers cell $\langle R;1 \rangle$, with all the open cells on the diagonals between these two also covered because of the completeness of the DCP. Similarly, the sum diagonals in D^k cover all the cells on the diagonals between $\langle R;-1 \rangle$ and $\langle k+1;-(3k+1) \rangle$. All the cells in the top half of H_n are therefore covered. Due to the symmetry, similar arguments hold for the bottom half of the hive. The hive is therefore covered by P^k. Furthermore, P^k is per definition also independent, so that an upper bound for the independent domination number has been obtained for an odd hive.

This also applies for the case $k=0$, $R=1$, $n=3$, in which case the hive is covered by a single queen in cell $\langle 0,0 \rangle$.

In the case of an even hive, $R=2k$, $n=4k+1$, $k \geq 0$, the above placements can be used unchanged by simply removing one circle of cells from the odd hive, still leaving the hive covered. □

Theorem 1. *The lower and independent domination numbers of* H_n, *where* $n = 4k+1$ *or* $n = 4k+3$, *for any* $k \geq 0$, *are given by* $\gamma_n = i_n = 2k+1$.

Proof. From Lemmas 2 and 3 we have that $2k+1 \leq \gamma_n$. Using this with (1) and Lemma 4, the theorem follows. □

Note that for very large hives, this value tends to $n/2$. It is also noteworthy that the placement which was used in [6] to obtain similar results for square hives was in effect an efficient double row placement.

Corollary (Rectangular hives). *The placement* P^k *specified for Lemma 4 also covers a rectangular hive with dimensions* $n \times m$, *where* n *is the number of rows and the number of cells per row alternate between* m *and* $m-1$, *where the width* $m = 3(k+1)$.

Proof. Consider the upper right hand quarter of any H_n. The rightmost difference diagonal in D^k, namely d_{-2k}, covers cell $\langle k+1\ , 3k+1\rangle$, the rightmost cell in row $k+1$, as well as an additional cell $\langle k+2\ , 3k+2\rangle$ outside H_n in the next row. The remaining difference diagonals in D^k cover all the cells above and to the left of d_{-2k}.

Similarly, for the lower right quarter, the extreme sum diagonal s_{2k} covers cells $\langle -(k+1)\ , 3k+1\rangle$ and $\langle -(k+2)\ , 3k+2\rangle$. Therefore all cells $\{\langle r\ ; c\rangle : -R \leq r \leq R\ , 0 \leq c \leq 3k+2\}$ are covered by P^k. Similarly on the left hand side, s_{-2k} covers $\langle k+2\ , -(3k+2)\rangle$ and d_{2k} covers $\langle -(k+2),$ $-(3k+2)\rangle$, with all the cells $\{\langle r;c\rangle : -R \leq r \leq R\ , -(3k+2) \leq c \leq 0\}$ also covered. Therefore P^k covers a rectangular hive of width $m = \lceil (6k+5)/2 \rceil$. □

6. Diagonal domination

The *diagonal domination number diag(* H_n) [2] is the minimum number of queens which can be placed on a main diagonal in such a way that they completely cover H_n.

Theorem 2. *diag(* H_n *)* $= n-2$ *for* $n \geq 3$.

Proof : Without loss of generality we select d_0 as the main diagonal. In any placement of queens on d_0 in which three or more cells on d_0 are empty, it will be possible to identify two different empty cells on d_0 in rows r_1 and r_2 such that either both rows are non-negative or both are non-positive. Consider for example the cell at the intersection of row r_1 and the sum diagonal through cell $\langle r_2 , r_2 \rangle$. This cell can not be covered by any queen in the placement mentioned above. It is therefore not possible to cover H_n with a diagonal placement with less than $n-2$ queens.

Consider a placement with a queen on each cell of d_0 with the exception of the top and bottom rows : $P^d = \{ \langle i, i \rangle : -(R-1) \le i \le (R-1) \}$. The empty top row is covered by the diagonals $\{ s_{2i} : 0 \le i \le (R-1) \} \cup \{ d_0 \}$, and the empty bottom row by $\{ s_{-2i} : 0 \le i \le (R-1) \} \cup \{ d_0 \}$. Clearly P^d covers the empty rows, and clearly each sum diagonal is essential. Therefore this is a minimal covering diagonal placement and theorem 2 holds. □

7. Upper independence number

Theorem 3. $\beta_n = n$ *for* $n \ge 1$.

Proof : Any placement containing one queen in each row obviously covers the hive; clearly therefore the maximum possible number of non-attacking queens is *n*. For $n \ge 3$ $(R \ge 1)$, this can be achieved with a DCP $C_{-R} \cup C_{R+1}$ where $C_{-R} = C(-R, -R, R+1)$ and $C_{R+1} = C(R+1, -R+1, R)$. Noting that the size is $R+1+R=n$ and that the separation is $2R+1=n$ and that all the other requirements of Lemma 1 are met, it follows that this is in fact an efficient DCP and therefore also independent, so that theorem 3 applies for $n \ge 3$. For the case $n=1$, the hive is covered by a queen in cell $\langle 0, 0 \rangle$, and theorem 3 holds. □

8. Upper domination number

To obtain a lower bound for Γ_n we construct a placement in which as many queens as possible have only one essential line. This is achieved with an *outer diagonal placement* (henceforth, an ODP) which is defined as a placement with queens only on the four outer diagonals.

Odd radius : For hives with $R = 2k+1$, $(n = 4k+3)$, for any $k \geq 1$, the ODP as illustrated in fig. 3 for the case $k = 2$ is defined as $P^o = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$, where the four diagonal placements and associated sets of diagonals are specified as follows for the four quadrants :

$Q_1 = \{\langle i\ ,(2R-i)\rangle : 2 \leq i \leq 2k\}$, on s_{2R} with $D_1 = \{d_{-2R+2i} : 2 \leq i \leq 2k\}$;

$Q_2 = \{\langle i\ ,-(2R-i)\rangle : 2 \leq i \leq 2k\}$, on d_{2R} with $D_2 = \{s_{-2R+2i} : 2 \leq i \leq 2k\}$;

$Q_3 = \{\langle -(2i-1), -(2R-2i+1)\rangle : 1 \leq i \leq k+1\}$, on s_{-2R};

$$D_3 = \{d_{2R+2-4i} : 1 \leq i \leq k+1\};$$

$Q_4 = \{\langle -(2i-1), (2R-2i+1)\rangle : 1 \leq i \leq k+1\}$, on d_{-2R} ;

$$D_4 = \{s_{2R+2-4i} : 1 \leq i \leq k+1\}.$$

This ODP has the following properties :

- The size is $\sigma = 6k$.
- The empty rows, namely rows 0, 1, +R, -2, -4, .. -(R-1) are all covered by P^o, as can easily be verified.
- Each queen has a private neighbour and is therefore essential. The queens in Q_1 and Q_2 all have private neighbours in either row 0 or row 1, as indicated in fig. 3 by the small dots and dashed connecting lines. The queens in Q_3 (respectively Q_4) have their closest private neighbour on the adjacent diagonal s_{-2R+2} (d_{-2R+2}), also as indicated in fig. 3.

It can therefore be concluded that the upper domination number is bounded from below by

$$\Gamma_n \geq 6k \quad \text{for } n = 4k+3,\ k \geq 1. \qquad (4)$$

Even radius : For hives with $R = 2k$ $(n = 4k+1)$, for any $k \geq 1$, the ODP is defined similarly to the above in terms of the following four diagonal placements :

$Q_1 = \{\langle i\ ,(2R-i)\rangle : 2 \leq i \leq 2k\}$; $Q_2 = \{\langle i\ ,-(2R-i)\rangle : 2 \leq i \leq 2k-1\}$;
$Q_3 = \{\langle -(2i-1)\ , -(2R-2i+1)\rangle : 1 \leq i \leq k\}$;
$Q_4 = \{\langle -(2i-1)\ , (2R-2i+1)\rangle : 1 \leq i \leq k\}$.

Note that this placement differs from the previous ODP in that cell $\langle R, -R\rangle$ is empty because a queen in this cell does not have a private neighbour. Careful consideration shows that this ODP covers all the cells in H_n, and also that all the queens are essential. In this case the size is $\sigma = 6k - 3$, so that

$$\Gamma_n \geq 6k-3 \quad \text{for } n = 4k+1,\ k \geq 1. \tag{5}$$

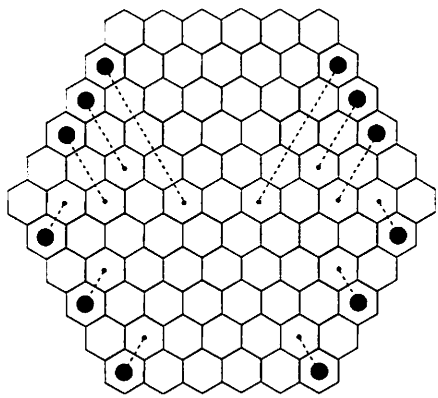

Fig. 3. An outer diagonal placement on H_{11}, also showing the private neighbours.

Small hives : The ODP defined above does not give the best solution for small hives, defined in this context as hives where $k < 2$. For hives defined for $k = 0$ and $k = 1$, i.e. hives for $n = 1, 3, 5, 7$, the independent DCP used to obtain β_n also gives the best value obtained here for Γ_n. In these cases therefore $\Gamma_n \geq n$.

This result together with (4) and (5) can be combined in the following theorem :

Theorem 4 : *The upper domination number of* H_n , *where* $n = 4k+1$ *or* $n = 4k+3$, *is bounded from below by the following : a) for* $k \leq 1$, $\Gamma_n \geq n$; *b) for* $k \geq 2$ *with even radius,* $n = 4k+1$, $\Gamma_n \geq 6k-3$ *c) for* $k \geq 2$ *with odd radius,* $n = 4k+3$, $\Gamma_n \geq 6k$.

Note that for very large hives, this value tends to $3n/2$.

9. Conclusion

The main results of this paper can be summarised in the table below, which is in agreement with the relationship (1). Unless otherwise specified, the values apply for $k \geq 0$.

Table 1 : Values and bounds for the domination numbers for H_n.

<table>
<tr><th></th><th>$\gamma_n = i_n$</th><th>$diag(H_n)$</th><th>β_n</th><th>Lower bound
Γ_n</th></tr>
<tr><td>Even radius
$n = 4k+1$</td><td rowspan="2">$2k+1$</td><td rowspan="2">$n-2$</td><td rowspan="2">n</td><td>$4k+1, \quad 0 \leq k \leq 1$
$6k-3, \quad k \geq 2$</td></tr>
<tr><td>Odd radius
$n = 4k+3$</td><td>$4k+3, \quad 0 \leq k \leq 1$
$6k, \quad k \geq 2$</td></tr>
<tr><td>$n \to \infty$</td><td>$n/2$</td><td colspan="2"></td><td>$3n/2$</td></tr>
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REFERENCES

[1] E.J. Cockayne, Chessboard domination problems, *Discrete Math.* **86**(1990) 13 - 20.

[2] E.J. Cockayne and S.T. Hedetniemi, On the diagonal queens domination problem, *J. of Combin. Theory Series A* **42**(1986) 137 - 139.

[3] E.J. Cockayne, S.T. Hedetniemi and D.J. Miller, Properties of hereditary hypergraphs and middle graphs, *Canad. Math. Bull.* **21**(1978) 461 - 468.

[4] J.F. Dunnigan, *The Complete Wargames Handbook,* William Morrow and Company, New York, 1980.

[5] G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, A.A. Macrae, C.K. Wallis, M.S. Jacobson, H.W. Martin and W.D. Weakly, Combinatorial problems on chessboards: a brief survey, *Proceedings of the Seventh International Conference on Graph Theory, Combinatorics, Algorithms and Applications*, Kalamazoo, Michigan, 1992.

[6] W.F.D. Theron, G. Geldenhuys, Domination by queens on a square beehive, accepted for publication by Discrete Math.