Results about Graph Decomposition, Greatest Common Divisor Index for Graphs and for Digraphs

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ABSTRACT. A graph H is G-decomposable if H can be decomposed into subgraphs, each of which is isomorphic to G. A graph G is a greatest common divisor of two graphs G_1 and G_2 if G is a graph of maximum size such that both G_1 and G_2 are G-decomposable. The greatest common divisor index of a graph G of size q is the greatest positive integer n for which there exist graphs G_1 and G_2, both of size at least nq, such that G is the unique greatest common divisor of G_1 and G_2. The corresponding concepts are defined for digraphs. Relationships between greatest common divisor index for a digraph and for its underlying graph are studied. Several digraphs are shown to have infinite index, including matchings, short paths, union of stars, transitive tournaments, the oriented 4-cycle. It is shown that for $5 \leq p \leq 10$, if a graph F of sufficiently large size is C_p-decomposable, then F is also $(P_{p-1} \cup P_3)$-decomposable. From this it follows that the even cycles C_6, C_8 and C_{10} have finite greatest common divisor index.

JCMCC 32 (2000), pp. 173-184

1 Introduction

A nonempty graph H is *decomposable* into the subgraphs $G_1, G_2, \ldots, G_n$ of H if no graph G_i, $1 \leq i \leq n$, has isolated vertices and $E(H)$ can be partitioned into $E(G_1), E(G_2), \ldots, E(G_n)$. In such case we write $H \cong G_1 \oplus G_2 \oplus \ldots \oplus G_n$. If $G_i \cong G$ for each integer i, $1 \leq i \leq n$, then H is G-*decomposable*, in which case we say G *divides* H and write $G|H$. Similarly we define decomposition for digraphs. In general, we follow [4] for graph theory notation and terminology.

Let G_1 and G_2 be two nonempty graphs (digraphs). In [2] a graph (digraph) G without isolated vertices is defined to be a greatest common divisor of G_1 and G_2 if G is a graph (digraph) of maximum size such that $G|G_1$ and $G|G_2$. Since K_2 ($\vec{K}_2$) divides every nonempty graph (digraph), it is evident that every two nonempty graphs (digraphs) have a greatest common divisor. For the digraphs D_1 and D_2 of Figure 1, their unique greatest common divisor D is shown.

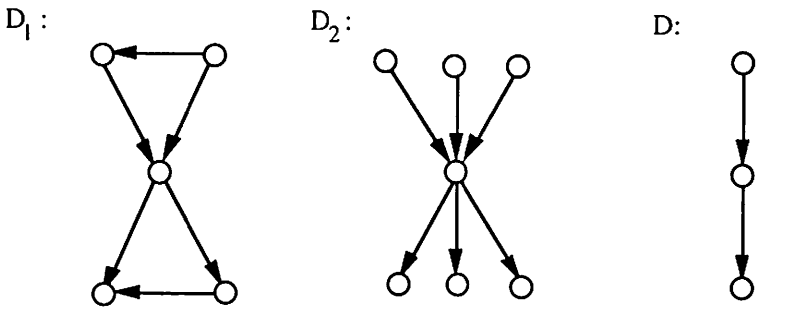

Figure 1.
Digraphs D_1 and D_2 and their greatest common divisor D

Although the two digraphs D_1 and D_2 of Figure 1 have a unique greatest common divisor, this is not always the case for graphs as well as for digraphs. Indeed it was shown in [3] that for every positive integer n, there exist graphs G_1 and G_2 having exactly n greatest common divisors. According to [2] we denote the set of greatest common divisors of G_1 and G_2 by $GCD(G_1, G_2)$ and write $GCD(G_1, G_2) = G$ if the greatest common divisor is uniquely G. Greatest common divisors of graphs and digraphs were investigated in detail in [5].

We say that $F \cong G_1 \cup G_2 \cup \cdots \cup G_r$, $r \geq 2$, is a *shatter* of G if $G \cong G_1 \oplus G_2 \oplus \ldots \oplus G_r$. For example, the graph $F \cong P_4 \cup P_3 \cup K_2$ is a shatter of $G \cong C_6$ (see Figure 2).

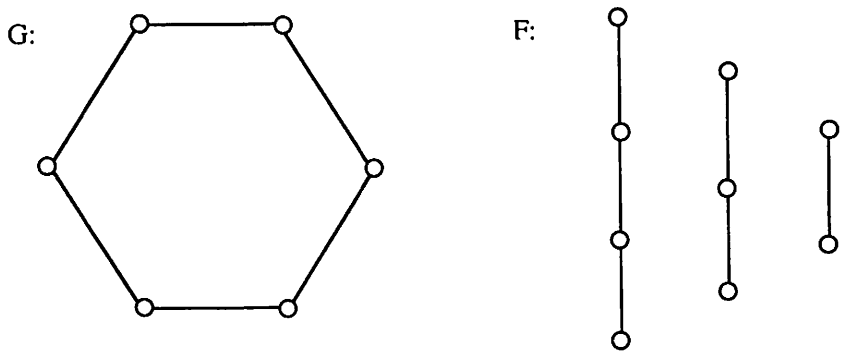

Figure 2. Graph G and a shatter F

In [1], for a given graph G of size q, *the greatest common divisor index* $i(G)$ was defined as the greatest integer n for which there exist graphs G_1 and G_2, both of size at least nq, such that $GCD(G_1, G_2) = G$. If one can find arbitrarily large n of such property, then the index was defined to be ∞. For a given graph G, we define *index* $i'(G)$ as the greatest integer n for which there exists a graph H which is decomposable into n copies of G but is not decomposable into n copies of any shatter of G.

Let us make the following observation.

Proposition 1. *For every graph G, we have $i'(G) \leq i(G)$.*

Proof: It is enough to show that, for every positive integer n, if $i'(G) \geq n$ then $i(G) \geq n$. Suppose $i'(G) \geq n$ is established by presenting a graph H which is decomposable into n copies of G but is not decomposable into n copies of any shatter of G. Let us construct two graphs $G_1 \cong H$ and $G_2 \cong pG$, where p is a prime number, $p > n$. Then $GCD(G_1, G_2) = G$, because the only divisors of G_2 of size $q(G)$ are shatters of G and G itself, but shatters of G do not divide G_1. Therefore, $i(G) \geq n$. □

Corollary 2. *$i(G) < \infty$ implies $i'(G) < \infty$.*

Corollary 3. *$i'(G) = \infty$ implies $i(G) = \infty$.*

We do not know any example of a graph G with $i(G) = \infty$ and $i'(G) < \infty$.

2 Greatest Common Divisor Index for Digraphs

The concepts of a shatter, the greatest common divisor index $i(D)$, and the index $i'(D)$ can be defined for a digraph D. The result analogous to Proposition 1 holds for digraphs.

Proposition 4. *For every digraph D, we have $i'(D) \leq i(D)$.*

We will established the relationship between the index of a digraph and the index of its underlying graph.

Proposition 5. *Let G be an arbitrary graph. If $\vec{G}$ is any orientation of G, then $i'(\vec{G}) \geq i'(G)$.*

Proof: It is enough to show that $i'(\vec{G}) \geq n$ whenever $i'(G) \geq n$. Suppose $i'(G) \geq n$ is established by presenting a graph H which is decomposable into n copies of G but is not decomposable into n copies of any shatter of G. Orient edges of H, yielding $\vec{H}$, such that $\vec{H}$ is $\vec{G}$-decomposable. We need to show that $\vec{H}$ is not decomposable into n copies of any shatter of $\vec{G}$. In fact, if $\vec{H}$ were decomposable into n copies of some shatter, say $\vec{F}$, of $\vec{G}$, then F, a shatter of G, would divide H; a contradiction. □

Corollary 6. *For a digraph $\vec{G}$ which is an orientation of a graph G, we have*

$$i'(G) = \infty \text{ implies } i'(\vec{G}) = \infty.$$

Proposition 7. *All digraphs D whose underlying graphs are kK_2, P_3, P_4, P_5, C_4, $K_4 - e$, $K_5 - e$, and union of stars have infinite index $i'(D)$ and, therefore, infinite index $i(D)$.*

Proof: Constructions in [1] show that the index $i'(G) = \infty$ for all graphs listed above. Therefore, from Corollary 6, $i'(D) = \infty$ for all digraphs D whose underlying graphs are graphs from this list. □

To underline some differences between greatest common divisors and index for graphs and digraphs, let us note first that it is easy to have $\vec{F}|\vec{H}$ and $\vec{G}|\vec{H}$, such that $\vec{F} \not\cong \vec{G}$ but $F \cong G$. Indeed, it is even possible to have $GCD(\vec{H}_1, \vec{H}_2) = \{\vec{F}, \vec{G}\}$, where $F \cong G$ is the unique greatest common divisor of H_1 and H_2. Let us consider the following digraphs $\vec{H}_1$ and $\vec{H}_2$ (see Figure 3) both of size 8.

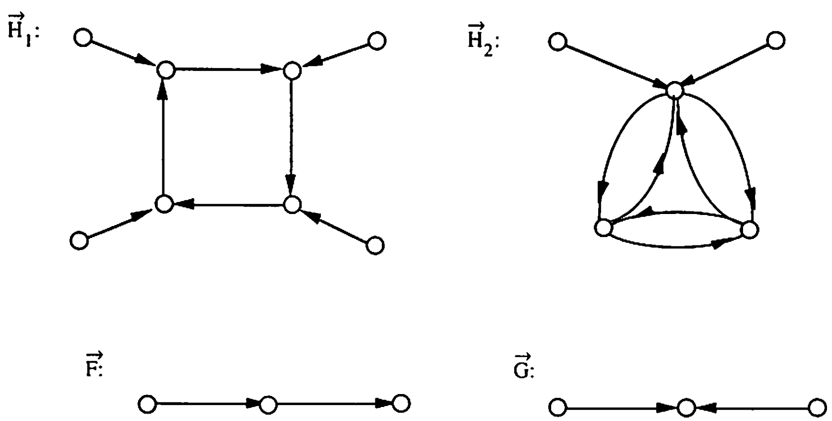

Figure 3

It is easy to verify that $GCD(\vec{H}_1, \vec{H}_2) = \{\vec{F}, \vec{G}\}$, but $GCD(H_1, H_2) = P_3$.

It was shown in [1] that $i(K_n) = 1$ for every complete graph K_n, $n \geq 3$. However, for tournaments the similar result is not true. We show that if T is the transitive tournament of order n, then $i'(T) = \infty$ and, therefore, $i(T) = \infty$.

Theorem 8. *For the transitive tournament T of order n, $n \geq 3$, $i'(T) = \infty$.*

Proof: Let $V(T) = \{v_1, v_2, \ldots, v_n\}$, where $v_i \to v_j$ for $i < j$. Let us construct a digraph H as follows. For n even, take $n^2/2$ copies $S_1, S_2, \ldots, S_{n^2/2}$ of T, together with $n-1$ copies $T_1, T_2, \ldots, T_{n-1}$ of T and identify them at a vertex z such that z corresponds to the vertex v_n from S_i, $i = 1, 2, \ldots, n^2/2$, and to the vertex v_i from T_i, $i = 1, 2, \ldots, n-1$ (see Figure 4).

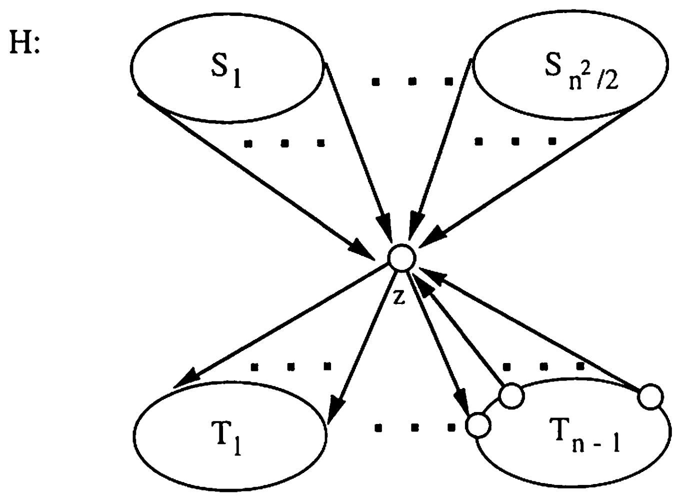

Figure 4

Of course, H is decomposable into $k = n^2/2 + n - 1$ copies of T. We will show that H is not decomposable into k copies of any shatter of T. Let $\vec{F}$ be a shatter of T and $\vec{F}|H$. Its underlying graph F has maximum degree at most $n-1$. Hence the vertex z must have degree exactly $n-1$ in every copy of F. Now, the outdegree of z divided by k is less than 1, so z is a sink in some copy of $\vec{F}$. But outdegree of z in H is positive, so z is not a sink in some copy of $\vec{F}$. Therefore, $\vec{F}$ has another vertex of outdegree $n-1$, which implies that F has another vertex of degree $n-1$. Now, if a copy of F has a vertex v of degree $n-1$ and v is distinct from z, then this copy of F uses the edge between the vertices v and z. Hence, all the vertices of degree $n-1$ lie in the same component F^* of F; in particular this component is 2-connected. Since z is a cut-vertex, this component F^* must lie completely inside a copy of S_i or T_i. Therefore, the copy $\vec{F}^*$ of

$\vec{F}$ has vertices of outdegree $0, 1, 2, \ldots, n-1$ which implies that $\vec{F}^* \cong T$; a contradiction.

For n odd, the analogous construction with $(n-1)^2/2$ copies $S_1, S_2, \ldots, S_{(n-1)^2/2}$ of T works. □

3 Digraphs of Finite Index

We say that a digraph D is arc-transitive if for every pair of arcs uv and xy of D there is an automorphism $\Phi\colon V(D) \to V(D)$ such that $\Phi u = x$ and $\Phi v = y$. The independence number of a digraph is the independence number of its underlying graph. A similar proof as those for graphs (see [1]) shows the existence of digraphs of finite index.

Theorem 9. *If D is an arc-transitive digraph of order p with the independence number $\beta(D) < p/2$ and H is a D-decomposable digraph of sufficiently large size,*[1] *then H is $(D - \vec{e}) \cup \vec{K}_2$-decomposable as well.*

Corollary 10. *The cyclic orientation $\vec{C}_{2k+1}$ of the odd cycle C_{2k+1}, $k \geq 2$, has finite index $i(\vec{C}_{2k+1}) \leq 4k(2k+1) - 4k - 1 = 8\ k^2 - 1$.*

Corollary 11. *The tournament T with*

$$V(T) = \{v_1, v_2, \ldots, v_{2k+1}\}$$

and

$$E(T) = \{v_i v_j \mid 1 \leq j - i \pmod{2k+1} \leq k\}$$

is arc-transitive with the independence number 1 and, therefore, $i(T) \leq (2k+1)2k + 1$.

Corollary 12. *For the complete digraph $\vec{K}_n$, $i(\vec{K}_n) = 1$.*

Corollary 10 does not hold for non-cyclic orientations of odd cycles. In particular, the following result is true.

Theorem 13. *If D is the digraph obtained by reversing two consecutive arcs in the cyclic orientation of C_5, then $i'(D) = \infty$.*

Proof: Let us construct a digraph H as follows. Take $n-1$ copies of D, identify the vertices of outdegree 2 (at the vertex x of H), and identify the vertices of outdegree 0 (at the vertex y of H). The last nth copy of D with the vertex set $V(D) = \{a, b, c, x, y\}$ has outdegree 1 at the vertices x, y, b, outdegree 2 at the vertex a, and outdegree 0 at the vertex c (see Figure 5).

Of course, H is decomposable into n copies of D. We will show that H is not decomposable into n copies of any shatter of D. Let F be a digraph which is a shatter of D and suppose that $F|H$. From the fact that

[1] "Sufficiently large" means its size is at least $2q\ [p(p-1) - q + 1]$.

$od_H(x) = 2n - 1 > n$, we conclude that F must have a vertex of outdegree 2. Similarly, because $id_H(y) = 2n - 1 > n$, the digraph F must have a vertex of indegree 2. Therefore, F must be one of the four digraphs F_i listed in Figure 6.

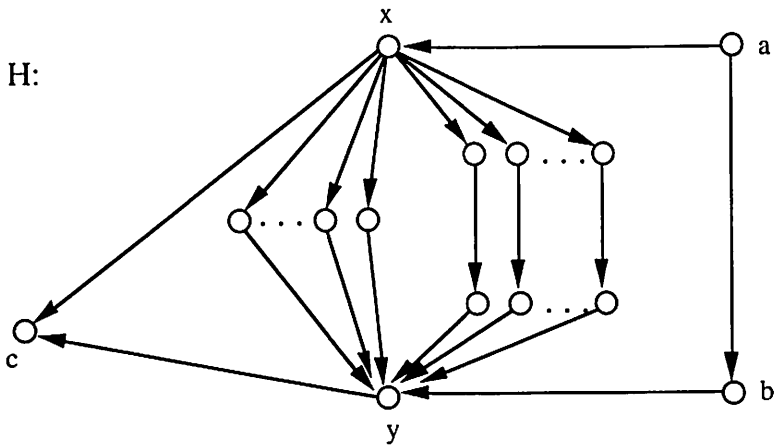

Figure 5. Digraph H from the proof of Theorem 13

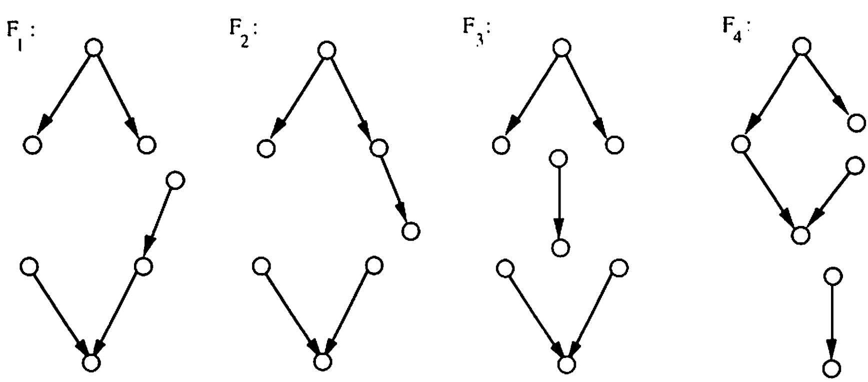

Figure 6

In the decomposition of H into copies of F_1, the arcs xc and yc must be used by a component of F_1 containing the vertex of indegree 2 (the "bottom" copy in the figure). However, $H - x - y - c \cong n\ \vec{K}_2$ and does not contain the other ("top") copy of F_1. Identical arguments show that F can be neither F_2 nor F_3. To eliminate the last possibility, namely F being F_4, let us notice that the arc ax cannot occur as $\vec{K}_2$ component of F_4, because $H - a - x$ does not contain the other component of F_4 as a subdigraph. Therefore, the arc ax must occur together with the arc ab

in the large component of F_4, and, moreover, this component uses both vertices x and y. These two vertices must be used by the large component of every copy of F_4. The last two observations imply that not n but only $n-1$ copies of $\vec{K}_2$ can be packed into H; a contradiction. □

4 Decomposition Results and Index for Even Cycles

In [1], it was shown that $i(C_3) = 1$ and $i(C_4) = \infty$. For the longer odd cycles we have $i(C_{2k+1}) < \infty$. The last result follows from the graph version of Corollary 10 via the decomposition theorem stating that if a graph of sufficiently large size is C_{2k+1}-decomposable, then it is also $(P_{2k+1} \cup K_2)$-decomposable. Whether the even cycles of length at least 6 have finite index as well has not been known. In the remaining part of this paper we prove that the even cycles C_6, C_8, and C_{10} also have a finite greatest common divisor index. This result will follow from another decomposition theorem stating that, for $5 \leq p \leq 10$, if a graph F of sufficiently large size is C_p-decomposable, then F is $(P_{p-1} \cup P_3)$-decomposable as well.

Let G_1 and G_2 be two copies of a graph G in the decomposition of some graph F. We say that G_1 *intrudes on* G_2 if the overlap $\Theta = V(G_1) \cap V(G_2)$ is dependent in G_1.

Lemma 14. *Let G be a graph of order p and size q. If a graph F is decomposable into k copies of G with k even and $k \geq 2\left[\binom{p}{2} - q\right]$, then there is a numbering of the copies such that G_{2i-1} does not intrude on G_{2i} for $1 \leq i \leq k/2$.*

Proof: Suppose that G_i intrudes on G_j. This means that the overlap $\Theta = V(G_i) \cap V(G_j)$ is dependent in G_i; that is, there is an edge of G_i that joins two vertices of G_j. Therefore, at most $\binom{p}{2} - q$ copies can intrude on G_j, or, equivalently, at least $k - (\binom{p}{2} - q)$ copies do not intrude on G_j and $k - (\binom{p}{2} - q) \geq k/2$. If we form a graph Γ whose vertices are copies of G with two vertices G_i and G_j adjacent if and only if G_i does not intrude on G_j, then the minimum degree of Γ is at least half of its order, so Γ has a perfect matching. This perfect matching establishes the required pairing and numbering of the copies of G. □

To handle an odd number of copies of G in F, we will find a suitable triple of copies together with a pairing described in Lemma 14. For three copies G_1, G_2, G_3 of G, we say that a trio $\{G_1, G_2, G_3\}$ forms a *good triple* if either the three copies are vertex disjoint, or they are mutually non-intrusive and there is a vertex x common to all three.

Lemma 15. *Let G be a graph of order p and size q. If a graph F is decomposable into k copies of G with k odd and sufficiently large, then there is a numbering of the copies such that G_1, G_2, G_3 form a good triple and G_{2i} does not intrude on G_{2i+1} for $2 \leq i \leq (k-1)/2$.*

Proof: Let B be the collection of those copies which intersect at least $(k-1)/2$ other copies. If $|B| < k/2$, then we can find three copies that are pairwise vertex disjoint; namely, take vertex disjoint $G_1, G_2 \notin B$, then there exists G_3 intersecting neither G_1 nor G_2.

So assume $|B| \geq k/2$. If we denote $A = \binom{p}{2} - q$, then there are at most kA intrusions in total. So there is a copy G_1 in B that intrudes on at most $2A$ copies. Thus, G_1 intersects, but does not intrude on at least $(k-1)/2 - 2A$ copies. At most A copies can intrude on G_1. If k is chosen such that $k \geq 4pA + 2p + 6A + 3$, then there are at least $(k-1)/2 - 3A \geq p(2A+1)+1$ copies which intersect G_1 but which do not intrude on G_1 and are not intruded on by G_1. Among them we can find at least $2A+2$ copies that intersect G_1 in some vertex, say x. Finally, from this collection we can find a pair G_2 and G_3 of non-intrusive copies. The reason is that among those $2A+2$ copies there are at most $(2A+2)A$ intrusions in total and $(2A+2)A \leq \binom{2A+2}{2}$. After selecting $\{G_1, G_2, G_3\}$ as a good triple, pairing of the remaining copies is guaranteed by Lemma 14. □

The next result takes care of copies which form a good triple.

Lemma 16. *Let G be a vertex transitive graph of degree of regularity r and let $H \cong (G-v) \cup K(1,r)$. If $\{G_1, G_2, G_3\}$ forms a good triple, then the graph F induced by the edges of the three copies G_1, G_2, G_3 of G is H-decomposable.*

Proof: If G_1, G_2, G_3 are vertex disjoint, then the result is obvious. Otherwise, we remove the vertex x of degree $3r$ from F. The remaining graph $F - x$ is decomposable into three copies of $G - v$. It is clear that we can partition the edges incident with x into three stars $K(1,r)$ to complete the H-decomposition. □

Notice that Lemma 16 holds in particular for $G \cong C_p$ and $H \cong P_{p-1} \cup P_3$.

Lemma 17. *Let $G \cong C_p$ and $H \cong P_{p-1} \cup P_3$, $5 \leq p \leq 10$. If G_1 and G_2 are two copies of G and G_1 does not intrude on G_2, then the graph induced by the edges of G_1 and G_2 is H-decomposable.*

Proof: We must examine all ways to put together two copies G_1 and G_2 of C_p such that their overlap Θ is an independent set when measured in G_1. Of course $|\Theta| \leq \lfloor \frac{p}{2} \rfloor$. Note that:

(1) If any vertex x of Θ has no neighbor in Θ, then we are done. For we make x the center of two P_3's.

(2) If both G_1 and G_2 have three consecutive private vertices (not in Θ), then we are done. For we make the edges joining the three private vertices into P_3's.

These observations allow simplification. Using the first observation and a counting argument, it follows that there must be a sequence of three consecutive vertices private to G_2 (except possibly when p is a multiple of 4 and $|\Theta| = p/2$). By the second observation, one can restrict the placement of the overlapping vertices in G_1. In particular, there must be at least $\lceil \frac{p}{3} \rceil$ vertices in Θ and the segments in G_1 between consecutive vertices of the overlap Θ must have length 2 or 3.

Using the simplification, we have one case for C_5 and two cases for C_6 and C_7. These are illustrated in Figures 7, 8, and 9, where the outer cycle is always G_1. One copy of H in the H-decomposition is represented by bold edges and the other copy by thin edges.

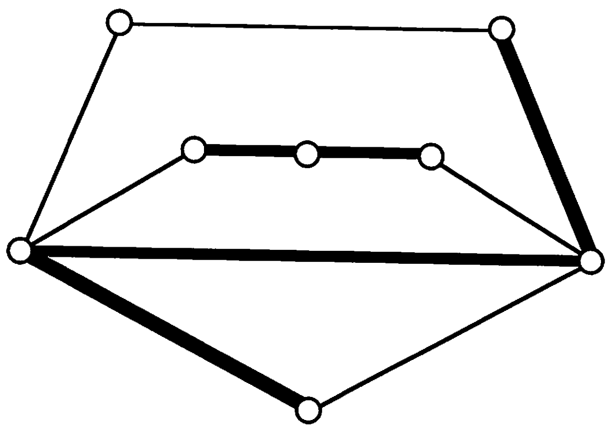

Figure 7. $(P_4 \cup P_3)$-decomposition for two copies of C_5

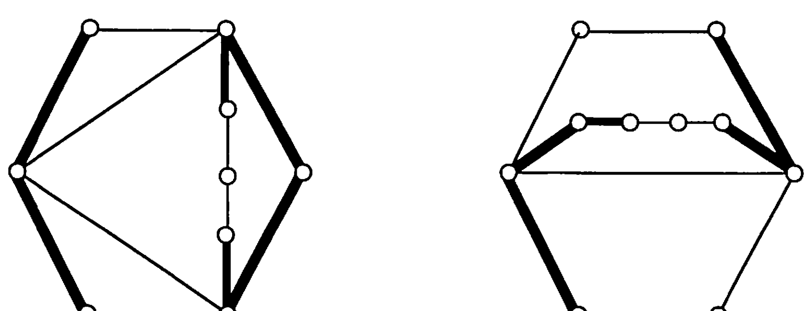

Figure 8.
Two cases with $(P_5 \cup P_3)$-decomposition of two copies of C_6

For cycles C_8, C_9 and C_{10}, there are many more non-isomorphic cases to consider, namely 11, 25, and 81, respectively. We fed all these cases into a computer and checked that the resultant graphs had the required decomposition. □

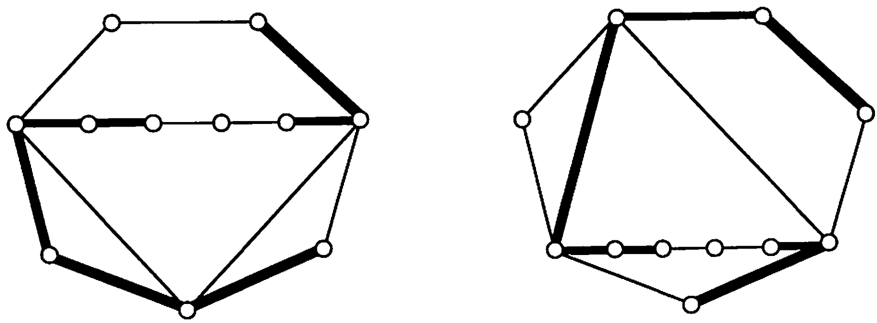

Figure 9.
Two cases with $(P_6 \cup P_3)$-decomposition of two copies of C_7

Theorem 18. *For $5 \leq p \leq 10$, if a graph F is decomposable into k copies of C_p for k sufficiently large, then F is also $(P_{p-1} \cup P_3)$-decomposable.*

Proof: If k is even, then Lemma 14 gives a numbering of the copies $G_1, G_2, \ldots, G_k$ of C_p such that G_{2i-1} and G_{2i} are non-intrusive. For every such pair G_{2i-1} and G_{2i}, $1 \leq i \leq k/2$, the graph induced by the edges of G_{2i-1} and G_{2i} is $(P_{p-1} \cup P_3)$-decomposable by Lemma 17. Combining these decompositions for each pair produces a $(P_{p-1} \cup P_3)$-decomposition of F.

If k is odd, then Lemma 15 guarantees an existence of a good triple $\{G_1, G_2, G_3\}$ and pairing G_{2i-1}, G_{2i} of the remaining copies of C_p such that G_{2i-1} and G_{2i} are non-intrusive. The $(P_{p-1} \cup P_3)$-decomposition of the graph F follows from application of Lemma 14 and Lemma 17. □

We conjecture that the above result is true always, not only for cycles of length between 5 and 10.

Conjecture 19. *For $p \geq 5$, if a graph F is decomposable into k copies of C_p for k sufficiently large, then F is also $(P_{p-1} \cup P_3)$-decomposable.*

Summarizing, we have the following results for the greatest common divisor index for cycles:

(1) $i(C_3) = 1$.

(2) $i(C_4) = \infty$.

(3) For $k \geq 2$, $i(C_{2k+1}) < \infty$; also the even cycles C_6, C_8, and C_{10} have finite index.

(4) The index of long even cycles C_p for $p \geq 12$ is unknown; however, we conjecture that it is also finite.

Acknowledgment. We are grateful to Gary Chartrand, Christina M. Mynhardt and Farrokh Saba for bringing this subject to our attention, and for many interesting discussions. We thank the referees for their valuable suggestions which led to an improved paper.

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