Traceability in Claw – Free Graphs Through Induced Bulls

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ABSTRACT. It is shown that if a graph G is connected claw – free such that the vertices of degree 1 of every induced bull have a common neighbor in G then G is traceable.

1 Introduction

We consider only finite undirected graphs without loops and multiple edges. For notation and terminology not defined here we refer to [2]. A graph G is said to be hamiltonian (resp. traceable) if G has a cycle (resp. path) containing $|V(G)|$ vertices. The claw is the three – edged star $K_{1,3}$ and the bull is the only graph B with degree sequence (3, 3, 2, 1, 1). An induced subgraph H of G that is isomorphic to the claw or bull will be called an induced claw or induced bull. A graph G is said to be claw – free if it contains no induced claw. The distance, $d(x, y)$, between vertex x and vertex y of a connected graph G is the least number of edges in a path with end – vertices x, y. The diameter, $d(G)$, of a connected graph G is the maximum distance between two vertices of G. If S is a subset of vertex set $V(G)$ of a graph G, the distance from the vertex x to S is defined to be $d(x, S) = \min(d(x, s) : s \in S)$. For any two distinct vertices x, y of a graph G, we say that they have a common neighbor if $N(x) \cap N(y) \neq \emptyset$. Let A, B be two disjoint subsets of $V(G)$, $E(A, B)$ is defined to be the set $\{ab \in E : a \in A, b \in B\}$.

The following result was obtained by Ryjáček which settles a conjecture in [3].

Theorem 1 [4] *Let G be a 2 – connected claw – free graph. If for every induced bull B in G the vertices of degree 1 in B have a common neighbor in G, then G is hamiltonian.*

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A corresponding theorem for the traceability of graphs is established in this paper.

Theorem 2 *Let G be a connected claw – free graph. If for every induced bull B in G the vertices of degree 1 in B have a common neighbor in G, then G is traceable.*

2 Proof of Theorem 2

We use a result in [1] as our lemma to prove Theorem 2.

Lemma 1 *Every connected claw – free graph of diameter at most 2 is traceable.*

Proof of Theorem 2: Suppose G is a graph satisfying the conditions in Theorem 2 and that it is not traceable. By Lemma 1, we can assume that $d(G) \geq 3$. Let $P = v_0 v_1 \cdots v_d$ be a path of length $d(G) = d(v_0, v_d)$. Obviously, we have the following facts: If $x \notin V(P)$ and $xv_i \in E$, then $xv_{i+t} \notin E$, where $0 \leq i \leq d-3$, $t \geq 3$; If $x \notin V(P)$ and $xv_i \in E$, then $xv_{i-t} \notin E$, where $3 \leq i \leq d$, $t \geq 3$. Set

$$
\begin{aligned}
X_i &= \{x \in V(G) \colon d(x, V(P)) = i\}.\\
A_0 &= \{x \notin V(P) \colon xv_0 \in E, xv_1 \notin E\}.\\
A_d &= \{x \notin V(P) \colon xv_d \in E, xv_{d-1} \notin E\}.\\
B_0 &= \{x \notin V(P) \colon xv_0, xv_1 \in E, xv_2 \notin E\}.\\
B_d &= \{x \notin V(P) \colon xv_d, xv_{d-1} \in E, xv_{d-2} \notin E\}.\\
C_0 &= \{x \notin V(P) \colon xv_0, xv_1, xv_2 \in E, xv_3 \notin E\}.\\
C_d &= \{x \notin V(P) \colon xv_d, xv_{d-1}, xv_{d-2} \in E, xv_{d-3} \notin E\}.\\
V_i &= \{x \notin V(P) \colon xv_i, xv_{i+1}, xv_{i+2} \in E, xv_{i-1}, xv_{i+3} \notin E\}.
\end{aligned}
$$

Then we have the following claims.

Claim 1. $G[A_0]$, $G[A_d]$, $G[B_0]$, $G[B_d]$, $G[C_0]$, $G[C_d]$, $G[V_i]$, *where* $1 \leq i \leq d-3$, *are complete.*

Since G is claw – free, it follows that Claim 1 is true.

Claim 2.

$$X_1 = A_0 \cup A_d \cup B_0 \cup B_d \cup C_0 \cup C_d \cup V_1 \cup V_2 \cup \cdots \cup V_{d-3}. \qquad (*)$$

Proof of Claim 2: Clearly, X_1 is not empty and the left – hand side of $(*)$ contains the right – hand side of $(*)$. Let x be a vertex in X_1.

Case 1. $xv_0 \in E$.

Subcase 1.1. $xv_1 \in E$, $xv_2 \in E$.

When $xv_1 \in E$, $xv_2 \in E$, and $xv_0 \in E$, then we must have that $x \in C_0$.

Subcase 1.2. $xv_1 \in E$, $xv_2 \notin E$.

When $xv_1 \in E$, $xv_2 \notin E$, and $xv_0 \in E$, then we must have that $x \in B_0$.

Subcase 1.3. $xv_1 \notin E$, $xv_2 \in E$.

This subcase can not occur; otherwise, $G[x, v_1, v_2, v_3]$ would be isomorphic to a claw.

Subcase 1.4. $xv_1 \notin E$, $xv_2 \notin E$.

When $xv_1 \notin E$, $xv_2 \notin E$, and $xv_0 \in E$, then we must have that $x \in A_0$.

Case 2. $xv_1 \in E$.

Subcase 2.1. $xv_0 \in E$, $xv_2 \in E$.

This subcase is the same as Subcase 1.1, namely, $x \in C_0$.

Subcase 2.2. $xv_0 \in E$, $xv_2 \notin E$.

This subcase is the same as Subcase 1.2, namely, $x \in B_0$.

Subcase 2.3. $xv_0 \notin E$, $xv_2 \in E$.

If $xv_0 \notin E$, $xv_2 \in E$, and $xv_1 \in E$, then $xv_3 \in E$. Otherwise, $G[x, v_0, v_1, v_2, v_3]$ would be isomorphic to a bull and $N(v_0) \cap N(v_3) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Thus, if $d(G) = 3$ then $x \in C_d$ and if $d(G) \geq 4$ then $x \in V_1$.

Subcase 2.4. $xv_0 \notin E$, $xv_2 \notin E$.

This subcase can not occur; otherwise, $G[x, v_0, v_1, v_3]$ would be isomorphic to a claw.

Case 3. $xv_2 \in E$.

Subcase 3.1. $xv_0 \in E$, $xv_1 \in E$.

This subcase is the same as Subcase 1.1, namely, $x \in C_0$.

Subcase 3.2. $xv_0 \in E$, $xv_1 \notin E$.

This subcase can not occur; otherwise, $G[x, v_1, v_2, v_3]$ would be isomorphic to a claw.

Subcase 3.3. $xv_0 \notin E$, $xv_1 \in E$.

This subcase is the same as Subcase 2.3.

Subcase 3.4. $xv_0 \notin E$, $xv_1 \notin E$.

If $xv_0 \notin E$, $xv_1 \notin E$, and $xv_2 \in E$, then $xv_3 \in E$. Otherwise, $G[x, v_1, v_2, v_3]$ would be isomorphic to a claw. Thus, $x \in B_d$ if $d(G) = 3$. If $d(G) \geq 4$, then $xv_4 \in E$; otherwise, $G[x, v_1, v_2, v_3, v_4]$ would be isomorphic to a bull and $N(v_1) \cap N(v_4) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Thus, $x \in V_2$.

Hence, x belongs to the right – hand side of (*) if x is adjacent to one of the vertices v_0, v_1, and v_2.

Similarly, x also belongs to the right – hand side of $(*)$ if x is adjacent to one of the vertices v_{d-2}, v_{d-1}, and v_d.

We now assume that $xv_i \notin E$, $xv_j \notin E$, where $0 \le i \le 2$, $d-2 \le j \le d$. Suppose that $xv_i \in E$, where $3 \le i \le d-3$. Since G is claw – free, we have $xv_{i-1} \in E$ or $xv_{i+1} \in E$. If both xv_{i-1} and xv_{i+1} belong to E, then $x \in V_{i-1}$. If $xv_{i-1} \notin E$ and $xv_{i+1} \in E$, then $xv_{i+2} \in E$; otherwise, $G[x, v_{i-1}, v_i, v_{i+1}, v_{i+2}]$ would be isomorphic to a bull and $N(x_{i-1}) \cap N(x_{i+2}) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Thus, $x \in V_i$. If $xv_{i-1} \in E$ and $xv_{i+1} \notin E$, then a similar argument shows that $x \in V_{i-2}$. Thus, $x \in V_1 \cup V_2 \cup \cdots \cup V_{d-3}$.

So, the right – hand side of $(*)$ contains the left – hand side of $(*)$, and Proof of Claim 2 is finished.

Note that $X_2 \neq \emptyset$; otherwise, G would have a hamiltonian path. The remainder of this paper is divided into two cases according to the size of $d(G)$.

Case A. $d(G) \ge 4$.

Claim A.1. *If $x \in X_2$, $y \in X_1$ and $xy \in E$, then $y \in A_0 \cup A_d$.*

Proof of Claim A.1: If $y \in B_0$, then $G[v_0, v_1, v_2, x, y]$ is isomorphic to a bull and there exists a vertex $z \in N(x) \cap N(v_2)$. Since $d(x, V(P)) = 2$, we have $z \notin V(P)$. Clearly, $zv_0 \notin E$; otherwise, $G[x, z, v_0, v_2]$ is isomorphic to a claw. Since $G[z, v_1, v_2, v_3]$ is not isomorphic to a claw, we have $zv_1 \in E$ or $zv_3 \in E$. We say zv_1 and zv_3 can not belong to E at the same time; otherwise, $G[x, z, v_1, v_3]$ would be isomorphic to a claw. If $zv_1 \in E$ and $zv_3 \notin E$, then $G[z, v_0, v_1, v_2, v_3]$ would be isomorphic to a bull and $N(v_0) \cap N(v_3) \neq \emptyset$ which leads a contradition to $d(G) = d(v_0, v_d)$. If $zv_1 \notin E$ and $zv_3 \in E$, then $zv_4 \notin E$; otherwise, $G[x, z, v_2, v_4]$ would be isomorphic to a claw. Moreover, $G[z, v_1, v_2, v_3, v_4]$ is isomorphic to a bull and $N(v_1) \cap N(v_4) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$ again. Thus, $y \notin B_0$. A similar argument shows that $y \notin B_d$.

Clearly, $y \notin C_0$; otherwise, $G[x, y, v_0, v_2]$ would be isomorphic to a claw. Similarly, $y \notin C_d$.

Furthermore, $y \notin V_i$, for all i, $1 \le i \le d-3$; otherwise, $G[x, y, v_i, v_{i+2}]$ would be isomorphic to a claw.

By Claim 2, we have $y \in A_0 \cup A_d$; and Proof of Claim A.1 is finished.

Set

$$S_0 = \{v_0\}, T_0 = \{v_d\}. S_1 = A_0, T_1 = A_d.$$

For $i \ge 2$, $S_i = \{s \in X_i :$ there exists a vertex $x \in S_{i-1}$ such that $sx \in E\}$.

For $j \ge 2$, $T_j = \{t \in X_j :$ there exists a vertex $y \in T_{j-1}$ such that $ty \in E\}$.

By Claim A.1 and the definitions of S_i and T_j, we have $X_i = S_i \cup T_i$, for $i \ge 2$.

Claim A.2.

(1) *If $S_i \neq \emptyset$, then $G[S_i]$ is complete, and for any vertex $x \in S_i$ and any vertex $y \in S_{i-1}$, $xy \in E$.*

(2) *If $T_j \neq \emptyset$, then $G[T_j]$ is complete, and for any vertex $x \in T_j$ and any vertex $y \in T_{j-1}$, $xy \in E$.*

Proof of Claim A.2: (1). Now, we perform induction on i. If $i = 0$, then $G[S_0]$ is complete. If $i = 1$, then Claim 1 implies that $G[S_1]$ is complete. It is obvious that for any vertex $x \in S_1$ and any vertex $y \in S_0$, $xy \in E$. If $i = 2$, suppose that there exist a vertex $x \in S_2$ and a vertex $y \in S_1$ such that $xy \notin E$. Let z be a vertex in S_1 such that $xz \in E(G)$, then $G[x, y, z, v_0, v_1]$ is isomorphic to a bull and there is a vertex $u \in N(x) \cap N(v_1)$. Since $x \in S_2 \subseteq X_2$, $u \notin V(P)$. By Claim A.1, $u \in A_0 \cup A_d$. Clearly, $u \notin A_0$. So, $u \in A_d$ which implies that $G[x, u, v_1, v_d]$ is isomorphic to a claw, which is a contradiction. Thus, for any vertex $x \in S_2$ and any vertex $y \in S_1$, $xy \in E$. Let p and q be any two distinct vertices in S_2, then $pq \in E$; otherwise, $G[v_0, w, p, q]$ would be isomorphic to a claw, where w is any vertex in S_1. Therefore, $G[S_2]$ is complete. Assume that $G[S_0], G[S_1], \cdots, G[S_k]$ ($k \geq 2$) are complete and that for any vertex $x \in S_i$ and any vertex $y \in S_{i-1}$, $xy \in E$, where $1 \leq i \leq k$. Suppose that there exist a vertex $x \in S_{k+1}$ and a vertex $y \in S_k$ such that $xy \notin E$. Let z be a vertex in S_k such that $xz \in E$, and u, v be two vertices in S_{k-1}, S_{k-2} respectively. Since $x \in S_{k+1}$, y and $z \in S_k \subseteq X_k$, $u \in S_{k-1} \subseteq X_{k-1}$, and $v \in S_{k-2} \subseteq X_{k-2}$, $G[x, y, z, u, v]$ is isomorphic to a bull and $N(x) \cap N(v) \neq \emptyset$ which leads a contradiction to $x \in X_{k+1}$. Hence, for any vertex $x \in S_{k+1}$ and any vertex $y \in S_k$, $xy \in E$. Let p and q be any two distinct vertices in S_{k+1}, then $pq \in E$; otherwise, $G[a, b, p, q]$ would be isomorphic to a claw, where $a \in S_k$, $b \in S_{k-1}$. Thus, $G[S_{k+1}]$ is complete.

(2) A symmetric argument shows that (2) in Claim A.2 is also true.

Thus, Proof of Claim A.2 is finished.

If $S_i \cap T_i = \emptyset$ holds for each i, then by Claim 1, Claim 2, Claim A.1, Claim A.2 and the connectedness of G, we can find a path in G containing all of the vertices of G so that G is traceable, which is a contradiction. We now assume that there exists an integer i such that $S_i \cap T_i \neq \emptyset$ and $j = \min\{i : S_i \cap T_i \neq \emptyset\}$. Clearly, $j \geq 2$.

Claim A.3. $S_k = T_k = \emptyset$ if $k \geq j + 1$.

Proof of Claim A.3: It suffices to show that $S_{j+1} = \emptyset$ and $T_{j+1} = \emptyset$. If $j = 2$, suppose that $S_3 \neq \emptyset$, then by Claim A.2 there exist vertices $x \in S_3$, $y \in S_2 \cap T_2$, $u \in S_1$, and $v \in T_1$ such that xy, yu, yv are in E. Since $G[y, x, u, v]$ is not isomorphic to a claw, $uv \in E$. Thus, $G[y, u, v, v_0, v_d]$ is isomorphic to a bull and $N(v_0) \cap N(v_d) \neq \emptyset$ which leads a contradiction to

$d(G) = d(v_0, v_d)$. So, $S_3 = \emptyset$. Similarly, $T_3 = \emptyset$. If $j \geq 3$, suppose that $S_{j+1} \neq \emptyset$, then by Claim A.3 there exist vertices $x \in S_{j+1}$, $y \in S_j \cap T_j$, $u \in S_{j-1}$, and $v \in T_{j-1}$ such that xy, yu, yv are in E. Clearly, $uv \in E$; otherwise, $G[y, x, u, v]$ would be isomorphic to a claw. Let $p \in S_{j-2}$, $q \in T_{j-2}$, and $w \in T_{j-3}$ be three vertices such that pu, qv, and qw are in E. Since $y \in S_j \cap T_j$, $p \in S_{j-2}$, $q \in T_{j-2}$, $v \in T_{j-1}$ and $w \in T_{j-3}$, we have that yp, yq, and vw are not in E. By the choice of j, we have pv, qu, and pw are not in E. Since $G[q, p, v, w]$ is not isomorphic to a claw, $pq \notin E$. Thus, $G[y, u, v, p, q]$ is isomorphic to a bull and there exists a vertex $r \in N(p) \cap N(q)$. We say $r \notin X_{j-1} \cup X_{j-3}$; otherwise, we would have a contradiction to the choice of j. So, r is in the set $X_{j-2} - \{p, q\}$. Note that the choice of j implies that $S_{j-2} - \{p\}$, $T_{j-2} - \{q\}$ is a partition of $X_{j-2} - \{p, q\}$. So, we have either $r \in S_{j-2} - \{p\}$ or $r \in T_{j-2} - \{q\}$ which imply that $G[q, r, w, v]$ or $G[p, u, r, z]$, where z is a vertex in S_{j-3}, are isomorphic to claws, which are contradictions. Thus, $S_{j+1} = \emptyset$. Similarly, $T_{j+1} = \emptyset$. So, Proof of Claim A.3 is finished.

By Claim 1, Claim 2, Claim A.1, and Claim A.2, Claim A.3 and the connectedness of G, we can find a path in G containing all of the vertices of G so that G is traceable, which is a contradiction.

Case B. $d(G) = 3$.

Set

$$D_0 = \{x \in X_2 : \text{ there exists a vertex } y \in B_0 \text{ such that } xy \in E\}.$$
$$D_d = \{x \in X_2 : \text{ there exists a vertex } y \in B_d \text{ such that } xy \in E\}.$$
$$F_0 = \{x \in X_3 : \text{ there exists a vertex } y \in D_0 \text{ such that } xy \in E\}.$$
$$F_d = \{x \in X_3 : \text{ there exists a vertex } y \in D_d \text{ such that } xy \in E\}.$$

Claim B.1. *If $x \in X_2$, $y \in X_1$ and $xy \in E$, then $y \in A_0 \cup A_d \cup B_0 \cup B_d$.*

Proof of Claim B.1: If $y \in C_0$ (resp. C_d), then $G[x, y, v_0, v_2]$ (resp. $G[x, y, v_{d-2}, v_d]$) is isomorphic to a claw, which is a contradiction. Thus, Claim B.1 follows from Claim 2.

Claim B.2. $D_0 = D_d$.

Proof of Claim B.2: Let x be any vertex in D_0 and y be a vertex in B_0 such that $xy \in E$, then $G[x, y, v_0, v_1, v_2]$ is isomorphic to a bull and there exists a vertex $z \in N(x) \cap N(v_2)$. Clearly, $z \notin V(P)$. We say $zv_1 \notin E$; otherwise, the facts that $G[x, z, v_1, v_3]$ and $G[x, z, v_0, v_2]$ are not isomorphic to claws imply that $zv_0 \notin E$ and $zv_3 \notin E$. Thus, $G[z, v_0, v_1, v_2, v_3]$ is isomorphic to a bull and $N(v_0) \cap N(v_3) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Since $G[z, v_1, v_2, v_3]$ is not isomorphic to a claw, we have $zv_3 \in E$. Thus, $z \in B_d$, $x \in D_d$, and $D_0 \subseteq D_d$. Symmetrically, $D_d \subseteq D_0$. Hence, $D_0 = D_d$ and Proof of Claim B.2 is finished.

Claim B.3. $F_0 = F_d = \emptyset$.

Proof of Claim B.3: Suppose that $F_0 \neq \emptyset$. Then, there exist a vertex $x \in X_3$ and a vertex $y \in D_0$ such that $xy \in E$. By Claim B.2, there exist vertices $u \in B_0$, $v \in B_d$ such that $uy \in E$, $vy \in E$. We say that $uv \in E$; otherwise, $G[x, y, u, v]$ would be isomorphic to a claw. So, $G[y, u, v, v_0, v_3]$ is isomorphic to a bull and $N(v_0) \cap N(v_3) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Thus $F_0 = \emptyset$. Similarly, $F_d = \emptyset$. So, Proof of Claim B.3 is finished.

Claim B.4. $G[D_0]$ *is complete.*

Proof of Claim B.4: Let x, y be two distinct vertices in D_0. If $N(x) \cap N(y) \cap B_0 \neq \emptyset$, then $xy \in E$; otherwise, $G[x, y, z, v_0]$ would be isomorphic to a claw, where $z \in N(x) \cap N(y) \cap B_0$. Similarly, if $N(x) \cap N(y) \cap B_d \neq \emptyset$, we have $xy \in E$. We now assume that $(N(x) \cap N(y)) \cap (B_0 \cup B_d) = \emptyset$. Let a and b be two distinct vertices in B_0 such that $ax \in E$ and $by \in E$. By Claim B.2 and the above assumption, there exist two distinct vertices p and q in B_d such that $px \in E$ and $qy \in E$. First, note that $ap \notin E$; otherwise, $G[x, a, p, v_0, v_d]$ would be isomorphic to a bull and $N(v_0) \cap N(v_d) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. If $xy \notin E$, then by the above assumption we have that $G[v_0, a, b, x, y]$ is isomorphic to a bull and there exists a vertex $w \in N(x) \cap N(y)$. Clearly, $wv_0 \notin E$; otherwise, $G[w, v_0, x, y]$ would be isomorphic to a claw. Since $G[a, p, x, w]$ is not isomorphic to a claw, at least one of wa and wp belongs to E. We say that both wa and wp are in E; otherwise, without loss of generality, we assume that $wa \in E$ and $wp \notin E$, then $G[w, a, x, v_0, p]$ is isomorphic to a bull and there exists a vertex $r \in N(v_0) \cap N(p)$. It is easy to verify that $r \notin V(P)$. We also note that $rv_d \notin E$; otherwise, we would have a contradiction to $d(G) = d(v_0, v_d)$. Since $G[x, p, r, v_d]$ is not isomorphic to a claw, we have $xr \in E$. So, $G[x, r, p, v_0, v_d]$ is isomorphic to a bull and $N(v_0) \cap N(v_d) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Therefore, wa and wp are in E. By the assumption that x and y have no common neighbor in $B_0 \cup B_d$, we have py and ay are not in E. Thus, $G[w, a, y, p]$ is isomorphic to a claw, which is a contradiction. Hence, $G[D_0]$ is complete and Proof of Claim B.4 is finished.

If $X_2 - D_0 = \emptyset$, by Claim 1, Claim 2, Claim B.1, Claim B.2, Claim B.3, Claim B.4 and the connectedness of G, we can find a path in G containing all vertices of G so that G is traceable, which is a contradiction. We now assume that $X_2 - D_0 \neq \emptyset$. Set

$$M_0 = \{x \in X_2 - D_0\colon \text{ there exists a vertex } u \in A_0 \text{ such that } xu \in E\}.$$

$$M_d = \{y \in X_2 - D_0\colon \text{ there exists a vertex } v \in A_d \text{ such that } yv \in E\}.$$

$$N_0 = \{x \in X_3\colon \text{ there exists a vertex } u \in M_0 \text{ such that } xu \in E\}.$$

$$N_d = \{y \in X_3\colon \text{ there exists a vertex } v \in M_d \text{ such that } yv \in E\}.$$

By Claim B.1, Claim B.2, and Claim B.3, we have $X_2 = M_0 \cup M_d \cup D_0$ and $X_3 = N_0 \cup N_d$.

Claim B.5. *$M_0 = M_d$ and $G[M_0]$ is complete.*

Proof of Claim B.5: Let x be any vertex in M_0, we claim that $x \in M_d$. Suppose, to the contrary, that $x \notin M_d$. Then, we will show that $d(x, v_2) \geq 4$ which leads a contradiction to $d(G) = 3$. First, we note that $x \notin M_d$ implies that $N(x) \cap A_d = \emptyset$. Thus, by Claim 2, Claim B.1 and the fact that $x \in M_0 \subseteq X_0 - D_0$, we have $N(x) \cap X_1 = N(x) \cap A_0$. We claim that $N(x) \cap A_0 = A_0$. Suppose, to the contrary, that $N(x) \cap A_0 \neq A_0$. Then, there exists a vertex $z \in A_0$ such that $xz \notin E$. Let y be a vertex in A_0 such that $xy \in E$, then $G[z, y, v_0, x, v_1]$ is isomorphic to a bull and there exists a vertex $r \in N(x) \cap N(v_1)$. Thus, $r \in N(x) \cap X_1 = N(x) \cap A_0 \subseteq A_0$, which is a contradiction. Moreover, we claim that $N(x) \cap D_0 = \emptyset$. Suppose, to the contrary, that $N(x) \cap D_0 \neq \emptyset$. Then, there exists a vertex $w \in N(x) \cap D_0$. By Claim B.2, there exist vertices $a \in B_0$ and $b \in B_d$ such that $wa \in E$ and $wb \in E$. Since $G[w, x, a, b]$ is not isomorphic to a claw, $ab \in E$. Thus, $G[w, a, b, v_0, v_d]$ is isomorphic to a bull and $N(v_0) \cap N(v_d) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. By the fact that $X_2 = M_0 \cup M_d \cup D_0$, we have $N(x) \cap X_2 = N(x) \cap ((M_0 - \{x\}) \cup M_d)$. Therefore, $N(x) = A_0 \cup (N(x) \cap ((M_0 - \{x\}) \cup M_d)) \cup (N(x) \cap X_3)$. The fact that $N(v_2) \subseteq B_d \cup C_d \cup C_0 \cup \{v_1\} \cup \{v_3\}$ is obvious. So, if we can prove that the sets $E(A_0, \{v_1, v_2, v_3\})$, $E(A_0, B_d)$, $E(A_0, C_d)$, and $E(A_0, C_0)$ are empty, then we have $d(x, v_2) \geq 4$ and a contradiction is reached. $E(A_0, \{v_1\})$ is empty because of the definition of A_0. $E(A_0, \{v_3\})$ is empty; otherwise, we would have a contradiction to $d(G) = d(v_0, v_d)$. Suppose $E(A_0, \{v_2\})$ is not empty. Then, there exists a vertex $a \in A_0$ such that $av_2 \in E$. Thus, $G[a, x, v_0, v_2]$ is isomorphic to a claw, which is a contradiction. So, the set $E(A_0, \{v_1, v_2, v_3\})$ is empty. Suppose that $E(A_0, B_d) \neq \emptyset$. Then, there exist vertices $a \in A_0$ and $b \in B_d$ such that $ab \in E$. Hence, $G[a, b, x, v_0]$ is isomorphic to a claw, which is a contradiction. Suppose that $E(A_0, C_d) \neq \emptyset$. Then, there exist vertices $a \in A_0$ and $c \in C_d$ such that $ac \in E$. So, $G[a, c, x, v_0]$ is isomorphic to a claw, which is a contradiction. Suppose that $E(A_0, C_0) \neq \emptyset$. Then, there exist vertices $a \in A_0$ and $c \in C_0$ such that $ac \in E$. Since $E(A_0, \{v_2\})$ is empty, $av_2 \notin E$. Thus, $G[v_0, a, c, x, v_2]$ is isomorphic to a bull and there exists a vertex $r \in N(x) \cap N(v_2)$. It is clear that $r \notin V(P)$; therefore, $r \in N(x) \cap X_1 = A_0$ and $rv_0 \in E$. So, $G[r, v_0, v_2, x]$ is isomorphic to a claw, which is a contradiction. In fact, we can show that $d(x, v_0) = 4$ by using the path $xav_0v_1v_2$, where a is a vertex in A_0. Thus, for any vertex $x \in M_0$, we have $x \in M_d$. Hence, $M_0 \subseteq M_d$. Symmetrically, $M_d \subseteq M_0$. So, $M_0 = M_d$. Now, we prove that $G[M_0]$ is complete. From the beginning of this proof, we know that for any vertex $x \in M_0$, x is adjacent to any vertex $z \in A_0$. Let y be any vertex in M_0 which is different from x, then $yz \in E$. Since $G[z, x, y, v_0]$ is not isomorphic to a claw, $xy \in E$. Thus, it follows that $G[M_0]$ is complete. So, Proof of Claim B.5 is finished.

Claim B.6. $N_0 = N_d = \emptyset$.

Proof of Claim B.6: Suppose that $N_0 \neq \emptyset$. Then, there exist a vertex $x \in X_3$ and a vertex $u \in M_0$ such that $xu \in E$. Since $M_0 = M_d$, there exist a vertex $y \in A_0$ and a vertex $z \in A_d$ such that uy, uz are in E. Since $G[u, x, y, z]$ is not isomorphic to a claw, we have $yz \in E$. Therefore, $G[u, y, z, v_0, v_d]$ is isomorphic to a bull and $N(v_0) \cap N(v_d) \neq \emptyset$ which leads a contradiction to $d(G) = d(v_0, v_d)$. Thus, $N_0 = \emptyset$. Similarly, $N_d = \emptyset$. So, Proof of Claim B.6 is finished.

By Claim 1, Claim 2, Claim B.1, Claim B.2, Claim B.3, Claim B.4, Claim B.5, Claim B.6 and the connectedness of G, we can find a path in G containing all vertices of G so that G is traceable, which is a contradiction.

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