A Note on Embedding Partial Extended Triple Systems of Even Index

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Abstract

Recently Raines and Rodger have proved that for all $\lambda \geq 1$, any partial extended triple system of order n and index λ can be embedded in a (complete) extended triple system of order v and index λ for any even $v \geq 4n+6$. In this note it is shown that if λ is even then this bound on v can be improved to all $v \geq 3n+5$, and under some conditions to all $v \geq 2n+1$.

1 Introduction

Let λK_n^+ be the complete graph on n vertices with λ edges joining each pair of vertices and with λ loops incident with each vertex. Define an *extended triple* to be a loop, a loop with an edge attached (known as a *lollipop*), or a copy of K_3 (known as a *triple*). We denote a loop incident with vertex a by $\{a,a,a\}$, a lollipop by $\{a,a,b\}$, $a \neq b$, when the loop of the lollipop is incident with vertex a, and a triple by $\{a,b,c\}$, where a, b, and c are distinct. A (partial) extended triple system of order n and index λ, (P)ETS(n,λ), also known as a (partial) totally symmetric quasigroup when $\lambda = 1$, is an ordered pair (V,B), where B is a set of extended triples defined on the vertex set V which partitions (a subset of) the edges of λK_n^+. A PETS$(n,\lambda)(V,B)$ is said to be embedded in an ETS$(v,\lambda)(V',B')$ if $V \subseteq V'$ and $B \subseteq B'$.

Recently Raines and Rodger have proved that for all $\lambda \geq 1$, any partial extended triple system of order n and index λ can be embedded in a (complete) extended triple system of order v and index λ for any even $v \geq 4n+6$ [8, 9, 10]. This follows earlier work by Lindner and Cruse [3], and a complete solution to the embedding problem for complete ETS$(n$, $1)$s by Hoffman and Rodger [5] (both results are in the guise of (partial) totally symmetric quasigroups, which are equivalent to (P)ETS$(n,1)$s). In this note it is shown that if λ is even then this bound on v can be improved

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to all $v \geq 3n + 5$ (see Theorem 2.4). For terms and notation not defined here, we refer the reader to [2].

These embeddings follow upon several well-known results in this area where partial triple systems of all indices were considered. Treash [12] obtained a finite embedding for partial Steiner triple systems. Lindner [7] reduced the size of the containing triple system to $v = 6n + 3$. Andersen, Hilton, and Mendelsohn [1] provided an embedding for admissible $v \geq 4n + 1$, and this is the best result to date. Rodger and Stubbs [11] found that a partial triple system of order n and index λ ($PTS(n, \lambda)$) can be embedded in a triple system of any odd λ-admissible order greater than $4n$. Subsequently, Hilton and Rodger [4] showed that if 4 divides λ, then any $PTS(n, \lambda)$ can be embedded in a $TS(v, \lambda)$ for any λ-admissible $v \geq 2n + 1$; this is the best possible lower bound on v. Recently, Johansson [6] showed that any $PTS(n, \lambda)$ can be embedded in a $TS(v, \lambda)$ where λ is even, whenever v is λ-admissible and $v \geq 2n + 1$.

2 The Small Embedding

Let λ be the multiplicity of some graph G. Define the *λ-sum*, $G +_\lambda H$, of the graphs G and H to be the multigraph obtained by joining every vertex of H to every vertex of G with λ edges. A *(partial) triangle decomposition* of a graph G is a decomposition of (a subset of) the edge set of G into triples. The following powerful theorem is the crucial ingredient in the proof of Theorem 2.4.

Theorem 2.1 ([6]) *Let G be an Eulerian multigraph on n vertices, $\lambda = 2\ell$, and $K = \lambda K_k$. Then $G +_\lambda K$ admits a triangle decomposition, no triangle of which is entirely contained in G, if and only if*

(i) $\Delta(G) \leq \lambda k$,

(ii) $\epsilon(G +_\lambda K) \equiv 0$ *(mod 3)*,

(iii) $2\epsilon(G) + \lambda k(k-1) \geq \lambda kn$, *with equality if* $k \leq 2$, *and*

(iv) for each connected component W of G,

$$\ell k \nu(W) - \epsilon(W) \neq 1 \text{ and is not odd if } k = 2.$$

For any PETS(n, λ), (V, B), define the deficiency graph, $G(B)$, of (V, B) to be the graph on n vertices whose edge set consists of all edges not found in any extended triple of B, and define $p(G(B)) \leq n$ to be the number of vertices of odd degree in $G(B)$.

The embedding process takes two steps. Let $u \leq n + (n/2) + 2$. We first use Proposition 2.2 to embed any maximal PETS(n, λ) in a PETS(u, λ) satisfying certain conditions. We then use Proposition 2.3 to embed this

PETS(u, λ) in an ETS(v, λ), for all $v \geq 2u+1$ and, thus, for all $v \geq 3n+5$.

We start with the following proposition.

Proposition 2.2 *Let λ be even, let (V, B) be any maximal PETS(n, λ), let $p = p(G(B))$, let $u = n + (p/2) + 2$, and let $k \geq 0$. Then (V, B) can be embedded in a PETS(u, λ) (V^*, B^*) such that $G(B^*)$ is an Eulerian multigraph with $\epsilon(G(B^*)) + \lambda ku + \lambda\binom{k}{2} \equiv 0$ (mod 3).*

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ be the set of odd degree vertices in $G(B)$. Let $V' = V \cup \{v_1, \ldots, v_{p/2}\}$. For $1 \leq i \leq p/2$ and for every pair $\{x_{2i-1}, x_{2i}\} \subseteq X$, let $\{v_i, v_i, x_{2i-1}\}, \{v_i, v_i, x_{2i}\} \in B'$. Clearly every vertex in $G(B')$ has even degree. Let $V^* = V' \cup \{\infty_1, \infty_2\}$, and suppose $\epsilon(G(B')) + \lambda ku + \lambda\binom{k}{2} \equiv i$ (mod 3). If $i = 1$, then add to B^* two copies of the lollipop $\{\infty_1, \infty_1, \alpha\}$, for some $\alpha \in V'$, and if $i = 2$, then add to B^* two copies of the lollipops $\{\infty_1, \infty_1, \alpha\}$ and $\{\infty_2, \infty_2, \alpha\}$, for some $\alpha \in V'$. Now $\epsilon(G(B^*)) + \lambda ku + \lambda\binom{k}{2} \equiv 0$ (mod 3), and $G(B^*)$ is clearly an Eulerian multigraph. □

Proposition 2.3 *Let λ be even, and let (V^*, B^*) be a PETS(u, λ) such that $G(B^*)$ is an Eulerian multigraph with $\epsilon(G(B^*)) + \lambda u(v-u) + \lambda\binom{v-u}{2} \equiv 0$ (mod 3). Then (V^*, B^*) can be embedded in an ETS(v, λ) $(\hat{V}, \hat{B})$ for all $v \geq 2u+1$.*

Proof. Let $K = \lambda K_{v-u}$ on the vertex set $\hat{V} \setminus V^*$. Since $\nu(K) > \nu(G(B^*))$ and since $G(B^*)$ is Eulerian, conditions (i), (iii), and (iv) of Theorem 2.1 are satisfied by $G(B^*)$ and K. Furthermore, since $\epsilon(G(B^*)) + \lambda u(v-u) + \lambda\binom{v-u}{2} \equiv 0$ (mod 3), condition (ii) of Theorem 2.1 is satisfied. Therefore, $G(B^*) +_\lambda K$ admits a triangle decomposition. Add all triangles (triples) from this triangle decomposition to $\hat{B}$. In addition, add to $\hat{B}$ any remaining loops incident with vertices in $\hat{V}$. Clearly, $(\hat{V}, \hat{B})$ is an ETS(v, λ). □

Theorem 2.4 *Let λ be even. Any partial extended triple system of order n and index λ can be embedded in an extended triple system of order v and index λ for all $v \geq 3n+5$.*

Proof. Let (V, B) be a PETS(n, λ). We can assume that (V, B) is maximal (by adding triples and loops, if necessary, but not lollipops). Let $u \leq n + (n/2) + 2$. By Proposition 2.2 (V, B) can be embedded in a PETS(u, λ) (V^*, B^*), satisfying the conditions of Proposition 2.3. Applying Proposition 2.3 to (V^*, B^*) embeds it in an ETS(v, λ), for any $v \geq 2u+1$ and, thus, for all $v \geq 3n+5$. □

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