On a Conjecture of Dénes, Mullen and Suchower

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ABSTRACT. In this note, we solve a conjecture of Dénes, Mullen and Suchower [2] on power sets of latin squares.

1 Introduction

A *latin square* of *side* n is an $n \times n$ array in which each cell contains a single element for $\{1, 2, \dots, n\}$, such that each element occurs exactly once in each row and exactly once in each column. Given two permutations α and β on $\{1, 2, \dots, n\}$, we define the composition permutation $\alpha\beta$ by $\alpha\beta(i) = \alpha(\beta(i))$ for $1 \leq i \leq n$. A square matrix is called a *row latin square* if each of its rows contains each of the elements $1, 2, \dots, n$ exactly once. Hence a row latin square R may be viewed as an n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of permutations where the ith row of R may be viewed as the image of $1, 2, \dots, n$ under the permutation α_i. Thus if $S = (\beta_1, \beta_2, \dots, \beta_n)$ is another row latin square of order n, then the product square $R \times S$ is given by $(\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_n\beta_n)$. In [4] Norton observed that all row latin squares of size $n \times n$ form a group Q_n under the group operatrion $\times$. Given a row latin square L, we define $L^1 = L$ and $L^p = L^{p-1} \times L$ where p is a positive integer. Two latin squares L and M of the same order are *orthogonal* if $L(a, b) = L(c, d)$ and $M(a, b) = M(c, d)$, implies $a = c$ and $b = d$. A set of latin squares $\{L_1, L_2, \dots, L_m\}$ is *mutually orhogonal*, or a set of MOLS, if $1 \leq i < j \leq m$, L_i and L_j are orthogonal.

Suppose L is a latin square of order n and that the order of L in Q_n If $L, L^2, \dots, L^{m-1}$, $m \leq p$ are latin squares, then the set $\{L, L^2, \dots, L^{m-1}\}$ is called a *latin power set* of *order* n. Our interest in looking for latin power sets is motivated by the fact that the squares $L, L^2, \dots, L^{m-1}$ are mutually orthogonal [4].

2 The Result

The following is conjectured in [2].

Conjecture 2.1 *If $n \neq 2, 6$, then there exists a latin power set containing at least two latin squares of order n.*

The validity of the conjecture is established in [2] for all n except when $n \equiv 2 \pmod 6$. The purpose of this note is to prove the conjecture. We need a definition. A latin square is *idempotent* if $L_{ii} = i$ for all i. It is obvious that if L is an idempotent latin square, then any power of L is also an idempotent latin square if it is a latin square.

When $n = 4, 7$, the construction in [2] gives an idempotent latin square, L, of order n such that L and L^2 are orthogonal.

Consider the following 5×5 latin square.

$$\begin{matrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \end{matrix}$$

It is easy to check that the above latin square is idempotent and is orthogonal to its square.

Theorem 2.2 *Suppose there exists a PBD(v, K). For each $k \in K$, suppose there exists a latin power set of order k having t idempotent MOLS. Then there exists a latin power set or order v having t idempotent MOLS.*

Proof: Apply the well known PBD construction for MOLS [1], we obtain an idempotent latin square of order v, say L. We claim that $\{L, L^2, \ldots, L^t\}$ is a set of t MOLS. Each row permutation of L consisting of disjoint cycles arising from the latin sub-squares in the PBD construction. Hence, $\{L, L^2, \ldots, L^t\}$ is a latin power set having t idempotent MOLS. We note that it is a construction for idempotent MOLS. □

Corollary 2.3 *If $n \equiv 2 \pmod 6$, then there exists a latin power set of order n containing at least two squares of order n.*

Proof: The result for $n \leq 50$ is established in [2]. If $n \geq 51$ and $n \equiv 2 \pmod 6$, then there exists a PBD$(n, \{4, 5, 7\})$ [3]. We know that there exists a latin power set on 4,5 and 7 points having 2 idempotent MOLS. We obtain the result by applying Theorem 2.2. □

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References

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