Fibonacci Numbers of Product Graphs

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ABSTRACT. The Fibonacci number of a graph is the number of independent sets of the graph. In this paper, we compute algorithmically the Fibonacci numbers of lattice product graphs.

The *Fibonacci number* $f(G)$ of a graph G, introduced by Prodinger and Tichy [4], is the number of independent sets of G, where an *independent set* of G is a subset of the vertex set of G such that no two of its vertices are adjacent. In [1] [2] [3] [4], the Fibonacci numbers of several graphs have been counted. They treated the graph of a path on $\{1, 2, \cdots, n\}$, 2-lattice graphs and tree graphs. It was shown in [2] and [4] that the Fibonacci number of the graph of a path on $\{1, 2, \cdots, n\}$ is equal to F_{n+2}, the $(n+2)$th Fibonacci number, which is defined by $F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2},\ n \geq 3$. The general solution is given by

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n.$$

The *product graph* (or ℓ-lattice graph) $P_{n,\ell}$ is a graph whose vertices are ordered pairs (i, j), and two vertices are adjacent whenever their distance in the Cartesian plane is 1, where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, \ell$. Prodinger and Tichy [4] obtained that the Fibonacci number of 2-lattice graph $P_{n,2}$ is

$$\frac{1}{2}(1+\sqrt{2})^{n+1} + \frac{1}{2}(1-\sqrt{2})^{n+1}.$$

A problem is asked there to find the Fibonacci numbers of ℓ-lattice graphs. In this note, we set a recurrence for the Fibonacci numbers of 3-lattice graphs, estimate ℓ-lattice graphs and give a computer algorithm for computing the general case of ℓ-lattice graphs.

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Let $P_{n,3}, B_n, C_n, D_n$ and E_n denote the graphs as in Figure 1, and a_n, b_n, c_n, d_n and e_n respectively be the Fibonacci numbers of the graphs $P_{n,3}, B_n, C_n, D_n$ and E_n.

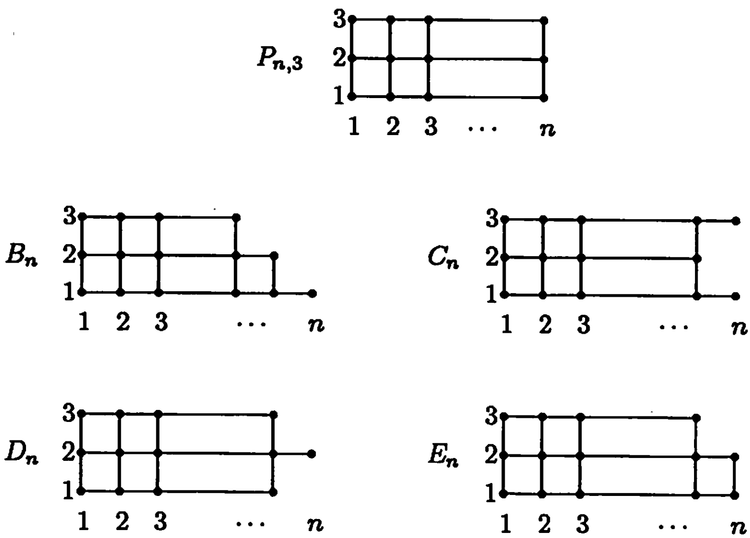

FIGURE 1.

Theorem 1. *The Fibonacci numbers a_n, b_n, c_n, d_n and e_n satisfy the recurrence relation*

$$x_{n+1} = 2x_n + 6x_{n-1} - x_{n-3}, \text{ and } a_n \leq min\{4^{n+1}/3, 17 \times 5^{n-2}\}. \quad (1)$$

Proof: The Fibonacci number for each graph of $P_{n,3}, B_n, C_n, D_n$ and E_n can be computed by counting those which have no vertex in the nth stalk and those which do have vertex in the nth stalk. We see that

$$a_n = a_{n-1} + 2b_n + c_{n-1} - d_{n-1}, \quad (2)$$

$$b_n = e_{n-1} + d_{n-1}, \quad (3)$$

$$c_n = a_{n-1} + 2b_n - d_{n-1}, \quad (4)$$

$$d_n = a_{n-1} + c_{n-1}, \quad (5)$$

$$e_n = a_{n-1} + c_{n-1} + e_{n-1}. \tag{6}$$

Substituting (4) in (2), and (3) into (4) yield respectively

$$a_n = c_n + c_{n-1}, \tag{7}$$

$$2e_{n-1} = c_n - a_{n-1} - d_{n-1}. \tag{8}$$

Replacing n by $n-1$ in (7) and (5) and then substituting the results into (8), we obtain

$$2e_{n-1} = c_n - (c_{n-1} + c_{n-2}) - (a_{n-2} + c_{n-2}) = c_n - c_{n-1} - 3c_{n-2} - c_{n-3}. \tag{9}$$

Substituting (9) and (7) into (6), we see that $c_{n+1} = 2c_n + 6c_{n-1} - c_{n-3}$, which is recursion (1). From (7), we see that a_n is a linear sum of c_n and c_{n-1} so a_n also satisfies (1). Similarly, from (5), d_n satisfies (1), from (4), b_n satisfies (1) and from (3), e_n satisfies (1).

The characteristic equation of recurrence relation (1) is $x^4 - 2x^3 - 6x^2 + 1 = 0$. It is not easy to compute the four real roots of this equation. However, we may substitute (3) into (2) and then express the relations (2), (4), (5) and (6) in the following matrix form:

Set $v_n = [a_n\ c_n\ d_n\ e_n]^T$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then $v_n = Av_{n-1}$. Hence,

$$v_n = A^{n-2} v_2 \text{ with } v_2 = [17\ 13\ 9\ 12]^T. \tag{10}$$

Taking the maximal row sum matrix norm $\|\ \|_\infty$, which is induced by the sup vector norm $\|\ \|_\infty$, we find an upper bound $a_n \le \|v_n\|_\infty \le \|A\|_\infty^{n-2} \|v_2\|_\infty = 17 \times 5^{n-2}$. Furthermore, we may estimate an upper bound of a_n by considering the recursion

$$z_{n+1} = 2z_n + 8z_{n-1} \tag{11}$$

with the same initials as a_n. Namely, $z_0 = 1$ and $z_1 = 5$. Then $a_n \le z_n$, and (11) has the characteristic equation $x^2 - 2x - 8 = (x-4)(x+2)$ so that general solution is $z_n = (7/6)4^n - 1/6(-2)^n$, whence $a_n \le 4^{n+1}/3$ and the proof is complete. □

We could use the recurrence relation (1) to compute the general terms of a_n, b_n, c_n, d_n and e_n. However, to do this we would need three initial values for each sequence. By using equation (10), it is easy to find the first four terms of a_n, c_n, d_n and e_n, and then from (3) find b_n. We compute the values shown in Table 1.

n	a_n	c_n	d_n	e_n	b_n
2	17	13	9	12	
3	63	50	30	42	21
4	227	177	113	155	72
5	827	650	404	559	268

TABLE 1.

Fix n, and let $a_{n,\ell}$ be the Fibonacci number of the ℓ-lattice $P_{n,\ell}$. It is clear that $a_{n,\ell} \leq F_{n+2}^{\ell}$. To obtain a lower bound of $a_{n,\ell}$, we count those which contain no vertex in the top ℓth path and those which contain a vertex in the top ℓth path. This gives rise to the inequality $a_{n,\ell} \geq a_{n,\ell-1} + F_{n+2}p_{\ell-2}$. Set $y_\ell = y_{\ell-1} + F_{n+2}y_{\ell-2}$ with $y_0 = a_{n,0}$, $y_1 = a_{n,1}$. Then $y_\ell \leq a_{n,\ell}$, and y_ℓ has characteristic equation $x^2 - x - F_{n+2} = 0$, which has roots $\frac{1+\sqrt{1+4F_{n+2}}}{2}$ and $\frac{1-\sqrt{1+4F_{n+2}}}{2}$. The initials $y_0 = 1$ and $y_1 = F_{n+2}$ imply that the general solution

$$y_\ell = \frac{\sqrt{1+4F_{n+2}}-1+2F_{n+2}}{2\sqrt{1+4F_{n+2}}}\left(\frac{1+\sqrt{1+4F_{n+2}}}{2}\right)^\ell + \frac{\sqrt{1+4F_{n+2}}+1-2F_{n+2}}{2\sqrt{1+4F_{n+2}}}\left(\frac{1-\sqrt{1+4F_{n+2}}}{2}\right)^\ell.$$

We have thus obtained the following inequality

Theorem 2.

$$\frac{\sqrt{1+4F_{n+2}}-1+2F_{n+2}}{2\sqrt{1+4F_{n+2}}}\left(\frac{1+\sqrt{1+4F_{n+2}}}{2}\right)^\ell + \frac{\sqrt{1+4F_{n+2}}+1-2F_{n+2}}{2\sqrt{1+4F_{n+2}}}\left(\frac{1-\sqrt{1+4F_{n+2}}}{2}\right)^\ell \leq f(P_{n,\ell}) \leq F_{n+2}^{\ell}.$$

The proof of Theorem 1 suggests that for large values $\ell \geq 4$, one may not expect to obtain a nice recurrence for $f(P_{n,\ell})$. Nevertheless, we describe in the following a computer algorithm that implements the computation of

$f(P_{n,\ell})$ for general cases. First we linearize the $n \times \ell$ nodes of $P_{n,\ell}$, and label the nodes consecutively as shown in Figure 2

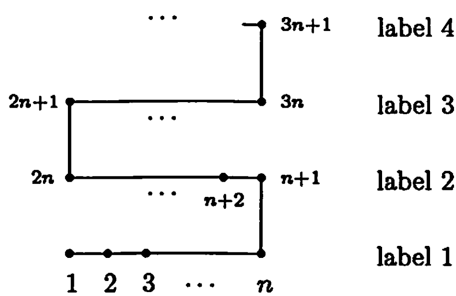

FIGURE 2.

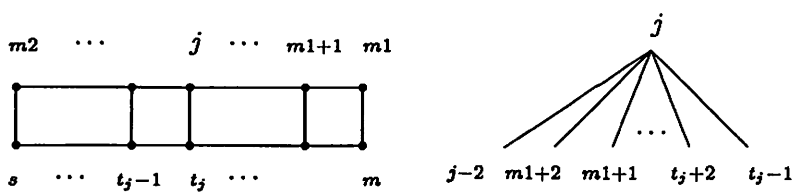

FIGURE 3.

For each labeled node j in Figure 2, the graph H_j is a subgraph of $P_{n,\ell}$ with vertices the labeled nodes $\{1, 2, \ldots, j\}$. First, we count *indep*[j] the number of independent sets of H_j containing the node j, and then determine $f(H_j)$ by

$$f(H_j) = indep[j] + f(H_{j-1}).$$

Algorithm

Step 1: Initialize level 1.

$$f(H_0) = 1;\ indep[1] = 1;\ f(H_1) = 2;$$
$$Do\ \Big[indep[j] = f(H_{j-2});$$
$$f(H_j) = indep[j] + f(H_{j-1}),$$
$$\{j, 2, n\}\Big]$$

Step 2: Next level.

$$
\begin{aligned}
&For\ k = 2, \ell\ (*\ level\ *)\\
&m = (k-1)*n;\ m1 = (k-1)*n+1;\ m2 = k*n;\\
&t = 2m+1;\ s = (k-2)*n+1;\\
&indep[m1] = f(H_{m1-2});\\
&f(H_{m1}) = indep[m1] + f(H_{m1-1});\\
&indep[m1+1] = indep[m1-1] + f(H_{m-2});\\
&f(H_{m1+1}) = indep[m1+1] + f(H_{m1});
\end{aligned}
$$

$$
\begin{aligned}
&Do\ \Big[t_j = t - j;\ (*\ see\ Figure\ 3\ for\ auxiliary\ *)\\
&\quad Initially\ set\ indep[j] = f(H_{t_j - 1});\\
&\quad Recount\ indep[q]\ by\ excluding\ independent\ sets\\
&\quad containing\ t_j;\ and\ add\ indep[q]\ to\ indep[j],\\
&\quad q = t_j + 2, \ldots, m, m1, m1+1;\\
&\quad Recurrsively\ recount\ indep[q]; and\\
&\quad add\ indep[q]\ to\ indep[j];\ q = m1+2, \ldots, j-2;\\
&\quad f(H_j) = f(H_{j-1}) + indep[j],\\
&\{j, m1+2, m2\}\Big]
\end{aligned}
$$

Here we present sample runs which executed Mathematica on Sun workstation for $f(P_{3,\ell})$ and $f(P_{10,\ell})$, $1 \leq \ell \leq 10$.

ℓ	$f(P_{3,\ell})$	$f(P_{10,\ell})$
1	5	144
2	17	8119
3	63	521721
4	227	32641916
5	827	2058249165
6	2999	129422885699
7	10897	8147000813446
8	39561	512615852515459
9	143677	32260116141540213
10	521721	2030049051145980050

Remark.

1. The algorithm can also compute $f(H_j)$, although H_j is not necessarily a lattice product graph.
2. The execution of $f(P_{n,\ell})$ runs fast if n is small, it will slow down if $n \geq 20$.
3. Our machine could work $f(P_{15,17})$, but it exited, out of memory, when running $f(P_{15,18})$.

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