Note on the path covering number of a semicomplete multipartite digraph

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ABSTRACT. A digraph D is called is semicomplete c-partite if its vertex set $V(D)$ can be partitioned into c sets (partite sets) such that for any two vertices x and y in different partite sets at least one arc between x and y is in D and there are no arcs between vertices in the same partite set. The path covering nurnber of D is the minumum number of paths in D that are pairwise vertex disjoint and cover the vertices of D. Volkmann (1996) has proved two sufficient conditions on hamiltonian paths in semicomplete multipartite digraphs and conjectured two related sufficient conditions. In this paper, we derive sufficient conditions for a semicomplete multipartite digraph to have path covering number at most k and show that Volkmann's results and conjectures can be readily obtained from our conditions.

1 Introduction and terminology

A digraph D is called is a *semicomplete c-partite* (or, *multipartite*) *digraph* if its vertex set $V(D)$ can be partitioned into c sets (partite sets) such that for any two vertices x and y in different partite sets at least one arc between x and y is in D and there are no arcs between vertices in the same partite set, Clearly, the underlined graph of a semicomplete multipartite digraph is a complete multipartite graph. The *path covering number* of a digraph D ($pc(D)$) is the minimum number of paths in D that are pairwise vertex disjoint and cover the vertices of D (see [5, 7] for results and applications of the path covering number of directed and undirected graphs.) Clearly, $pc(D) = 1$ is equivalent to the existence of a hamiltonian path in D.

This note is motivated by two conjectures of L. Volkmann [11] on sufficient conditions for hamiltonian paths in semicomplete multipartite digraphs. Although a characterization of semicomplete multipartite digraphs containing hamiltonian paths is known (see Theorem 2.1), the condition of the characterization is not always easy to verify, thus readily checkable sufficient conditions for a semicomplete multipartite digraph to possess a hamiltonian path are of interest (this situation is similar to the situation with perfect matchings in graphs, where, in spite of the well-known Tutte's characterizations, numerous sufficient conditions were obtained, see e.g. [2, 3, 8, 10].) Volkmann [11] has proved two sufficient conditions on hamiltonian paths in semicomplete multipartite digraphs and conjectured two related sufficient conditions. In this paper, we derive sufficient conditions for a semicomplete multipartite digraph to have path covering number at most k and show that Volkmann's results and conjectures can be readily obtained from our conditions. We show that the conditions are best possible, in some sense.

A *factor* is a spanning subgraph of a digraph. A factor is *k-path-cycle* if it consists of a set of vertex disjoint paths and cycles, where k stands for the number of paths in the set. A *cycle* factor is a 0-path-cycle factor; a *k-path* factor is a k-path-cycle factor with no cycles. Clearly, $pc(D)$ is the least integer k such that D has a k-path factor.

For a digraph D and disjoint sets X and Y of its vertices, $a(X, Y)$ ($e(X, Y)$, respectively) is the number of arcs with first end-vertex in X and second end-vertex in Y (with one end-vertex in X and another end-vertex in Y, respectively); $I(D)$ is the maximum value of $|d^+(x) - d^-(x)|$ over all vertices x in D; $X \Rightarrow Y$ means that there is no arc from Y to X; $D\langle X\rangle$ is the subgraph of D induced by X; X is an *independent* set if there is no arc between the vertices in X.

2 Auxiliary Results

The following characterization was proved in [4] (see also [1, 6]).

Theorem 2.1. *A semicomplete multipartite digraph has a hamiltonian path if and only if it contains a 1-path-cycle factor.*

Corollary 2.2. *For a semicomplete multipartite digraph M, $pc(M) \leq k$ ($k \geq 1$) if and only if M contains a k-path-cycle factor.*

Proof: The result follows from the fact that, by Theorem 2.1, the vertices of the cycles of a k-path-cycle factor F in M and a path of F induce the subgraph of M containing a hamiltonian path. □

The next lemma was first proved in [13]. The original proof relies on a theorem of A.J. Hoffmann on circulations in networks. We give a shorter proof of this result based on Ore's theorem on cycle factors in digraphs [9].

Lemma 2.3. *A digraph D has no cycle factor if and only if its vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that*

$$R_1 \Rightarrow Y, (R_1 \cup Y) \Rightarrow R_2, Y \text{ is an independent set} \tag{1}$$

and $|Y| > |Z|$.

Proof: Let $D = (V, A)$ be a digraph. By Ore's theorem (see [9]), D has no cycle factor if and only if there is a subset X of V such that $|N^+(X)| < |X|$, where $N^+(X)$ is the set of the heads of the arcs whose tails are in X. Let $R_1 = V - (N^+(X) \cup X)$, $R_2 = N^+(X) \cap X$, $Y = X - R_2$, $Z = N^+(X) - R_2$. It is easy to verify that the sets Y, R_1 and R_2 satisfy (1) and $|Y| > |Z|$. □

Corollary 2.4. *A digraph D has no k-path-cycle factor ($k \geq 0$) if and only if its vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 that satisfy (1) and $|Y| > |Z| + k$.*

Proof: Assume that $k \geq 1$. Let D' be an auxiliary digraph obtained from D by adding k new vertices $u_1, \dots, u_k$ together with the arcs $\{u_i w, w u_i : w \in V(D), i = 1, 2, \dots, k\}$. Observe that D has no k-path-cycle factor iff D' has no cycle factor. By Lemma 2.3, the vertices of D' can be partitioned into sets Y, Z', R_1, R_2 that satisfy (1) and $|Y| > |Z'|$. By (1), the vertices $u_1, \dots, u_k$ are in Z'. Let $Z = Z' - \{u_1, \dots, u_k\}$. Clearly, the subsets Y, Z, R_1, R_2 satisfy (1) and $|Y| > |Z| + k$. □

In the rest of the paper, M stands for a semicomplete multipartite digraph with vertex set V, arc set A, partite sets $V_1, \dots, V_c$. We also denote $v = |V|$, $v_i = |V_i|$.

The following two lemmas are obtained in [13].

Lemma 2.5. *Let $X \subset Y \subseteq V$ and let $y_i = |Y \cap V_i|$ for all $i = 1, 2, \dots, c$. Then*

$$\frac{e(X, Y-X)}{|X|} + \frac{e(X, Y-X)}{|Y-X|} \geq |Y| - \max\{y_i : i = 1, 2, \dots, c\}.$$

Lemma 2.6. *Let $X \subset V$. Then*

$$I(M) \geq \max_{\emptyset \subset X \subseteq V} \left\{ \frac{a(X, V-X) - a(V-X, X)}{|X|} \right\}.$$

3 Main Results

Combining Corollaries 2.2 and 2.4, we obtain

Theorem 3.1. *For a semicomplete multipartite digraph M, $pc(M) > k$ ($k \geq 1$) if and only if V can be partitioned into subsets Y, Z, R_1, R_2 that satisfy (1) and $|Y| > |Z| + k$.*

Now we are ready to prove the following sufficient conditions:

Theorem 3.2. *Suppose that $v_1 \leq v_2 \leq \cdots \leq v_c$. If there exists a positive integer k such that $I(M) \leq \min\{v-3v_c+2k+1, (v-v_{c-1}-2v_c+3k+2)/2\}$, then $pc(M) \leq k$.*

Proof: Assume that $pc(M) > k$, then, by Theorem 3.1, V can be partitioned into sets Y, Z, R_1, R_2 that satisfy (1) and $|Y| \geq |Z|+k+1$. By the last inequality, Y is non-empty and since Y is independent there is a partite set V_i such that $Y \subseteq V_i$. Let $Q = V - V_i - Z$, $Y_1 = V_i \cap R_1$, $Y_2 = V_i \cap R_2$, $Q_1 = Q \cap R_1$ and $Q_2 = Q \cap R_2$. Note that $|Z| \leq |Y| - 1 - k \leq v_c - 1 - k$, $Q_1 \Rightarrow Y \Rightarrow Q_2$, $(Q_1 \cup Y_1) \Rightarrow (Q_2 \cup Y_2)$ and $Y_1 \cup Y_2 \cup Y \subseteq V_i$. If $i = c$ then let $j = c-1$ and if $i < c$ then let $j = c$. We now consider the following three cases.

Case 1. $Q_1 = \emptyset$: This implies that $a(Y, V-Y) - a(V-Y, Y) \geq |Y||Q_2| - |Y||Z| \geq |Y|(v-v_i-2|Z|) \geq |Y|(v-v_c-2(v_c-1-k)) = |Y|(v-3v_c+2+2k)$, which by Lemma 2.6 implies that $I(M) \geq v - 3v_c + 2 + 2k$. However this is a contradiction.

Case 2. $Q_2 = \emptyset$: We can arrive to a contradiction analogously to Case 1.

Case 3. $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$: Since $v_i + v_j \leq v_{c-1} + v_c$, we obtain that $|Q| - v_j \geq v - v_i - |Z| - v_j \geq v - v_{c-1} - v_c - (v_c - 1 - k)$. By Lemma 2.5,

$$\frac{a(Q_1,Q_2)}{|Q_1|} + \frac{a(Q_1,Q_2)}{|Q_2|} \geq |Q| - v_j \geq v - v_{c-1} - 2v_c + 1 + k.$$

Thus,

(i) $\frac{a(Q_1,Q_2)}{|Q_1|} \geq \frac{v-v_{c-1}-2v_c+1+k}{2} - |Y_2| + |Y_1|$ or

(ii) $\frac{a(Q_1,Q_2)}{|Q_2|} \geq \frac{v-v_{c-1}-2v_c+1+k}{2} + |Y_2| - |Y_1|$

Assume that (i) holds as the case when (ii) holds can be treated similarly.

By Lemma 2.6.

$$\begin{aligned}
I(M) &\geq \frac{a(Q_1, V-Q_1) - a(V-Q_1, Q_1)}{|Q_1|} \\
&= \frac{a(Q_1,Q_2)}{|Q_1|} + \frac{a(Q_1, Y\cup Y_2) - a(Y \cup Y_2, Q_1)}{|Q_1|} \\
&+ \frac{a(Q_1, Z\cup Y_1) - a(Z\cup Y_1, Q_1)}{|Q_1|} \\
&\geq \left(\frac{v - v_{c-1} - 2v_c + 1 + k}{2} - |Y_2| + |Y_1|\right) + (|Y| + |Y_2|) - (|Z| + |Y_1|) \\
&= \frac{v - v_{c-1} - 2v_c + 1 + k}{2} + |Y| - |Z| \\
&\geq \frac{v - v_{c-1} - 2v_c + 3 + 3k}{2},
\end{aligned}$$

which is impossible. □

The following result (in a slightly weaker form) was proved for $c \geq 7$ and conjectured for $5 \leq c \leq 6$ in [11].

Corollary 3.3. *Suppose that* $1 \leq r \leq v_1 \leq v_2 \leq \cdots \leq v_c = r+1$. *If* $c \geq 5$ *and* $I(M) \leq r+2$ *then* M *contains a hamiltonian path.*

Proof: As $I(M) \leq r+2$, $c \geq 5$, $v_c = r+1$, and $r \geq 1$ we obtain that $I(M) \leq r+2 \leq (cr+1) - 3r \leq v - 3v_c + 3$ and $I(M) \leq r + 2 \leq \frac{(c-3)r+4}{2} \leq \frac{((r+1)+v_{c-1}+(c-2)r)-v_{c-1}-2(r+1)+5}{2} \leq \frac{v-v_{c-1}-2v_c+5}{2}$. Now it follows from Theorem 3.2 that M has a hamiltonian path. □

Using Theorem 3.2, one can easily verify the next result.

Corollary 3.4. *Suppose that* $v_i = r$ *for all* $i = 1, 2, \ldots, c$. *If* $c \geq 3$ *and* $I(M) \leq \frac{(c-3)r+5}{2}$, *then* M *contains a hamiltonian path.*

The first part of the next claim was proved and the second was conjectured in [11]. The claim readily follows from the previous corollary.

Corollary 3.5. *Suppose that* $v_i = r \geq 2$ *for all* $i = 1, 2, \ldots, e$. M *has a hamiltonian path if at least one of the following conditions holds:*

1. $c \geq 5$ *and* $I(M) \leq 4$;
2. $c \geq 3$ *and* $I(M) \leq 2$.

4 Example

The following lemma is a reformulation of Exercise 2.4.12 in [12] and can be easily proved using the famous Euler theorem (for undirected graphs).

Lemma 4.1. *Every graph G has an orientation D such that $I(D) \leq 1$.*

The next result shows that the inequality for $I(M)$ in the conditions in Theorem 3.2 is best possible. A *multipartite tournament* is a semicomplete multipartite digraph without cycles of length two.

Theorem 4.2. *Let a, b, d and k be positive integers sueh that $2d(b-a) \geq k+1$. There exists a multipartite tournarnent M with $c = 2b+a$ partite sets $V_1, V_2, \ldots, V_c$, where $v_i = |V_i| = 2d$ for all $i = 1, 2, \ldots, c-1$ and $v_c = |V_c| = 2ad+k+1$, such that $I(M) = (v - v_{c-1} - 2v_c + 3k + 3)/2 \leq v - 3v_c + 2k + 2$, but $pc(M) > k$.*

Proof: Let $v_i = 2d$ for all $i = 1, 2, \ldots, c-1$ and let $v_c = 2ad + 1 + k$. Let $Z = V_{2b} \cup V_{2b+1} \cup \cdots \cup V_{c-1}$ and $Y = V_c$. Partition V_i into V_i' and V_i'' such that $|V_i'| = |V_i''| = d$ for all $i = 1, 2, \ldots, 2b-1$. Let $R_1 = \cup_{i=1}^{2b-1} V_i'$, $R_2 = \cup_{i=1}^{2b-1} V_i''$, $R_2 \Rightarrow Z \Rightarrow R_1 \Rightarrow Y \Rightarrow R_2$, and $R_1 \Rightarrow R_2$.

By Lemma 4.1, one can orient the rest of the arcs in M such that $I(M\langle Y \cup Z\rangle) \leq 1$, $I(M\langle R_1\rangle) \leq 1$ and $I(M\langle R_2\rangle) \leq 1$. Since $2b-2$ is even and $|V_i'| = |V_i''| = d$ ($i = 1, 2, \ldots, 2b-1$) we obtain that $I(M\langle R_1\rangle)$ and $I(M\langle R_2\rangle)$ must be even, which implies that $I(M\langle R_1\rangle) = 0$ and $I(M\langle R_2\rangle) = 0$.

By Theorem 3.1, $pc(M) > k$. Observe that $v = (c-1)v_1 + v_c = d(4b + 4a - 2) + k + 1$, $v - 3v_c + 2k + 2 = d(4b - 2a - 2) = d(2b-2) + 2d(b-a)$ and

$$\frac{v - v_{c-1} - 2v_c + 3k + 3}{2} = d(2b-2) + k + 1.$$

As $2d(b-a) \geq k+1$ we obtain that

$$v - 3v_c + 2k + 2 \geq \frac{v - v_{c-1} - 2v_c + 3k + 3}{2}.$$

If $x' \in R_1$, then $d^+(x') = a(x', V_c) + a(x', R_2) + a(x', R_1) = v_c + (2b-2)d + (2b-2)d/2 = d(3b+2a-3)+1+k$, and $d^-(x') = a(Z, x') + a(R_1, x') = 2ad + (2b-2)d/2 = d(2a+b-1)$. This implies that $|d^+(x') - d^-(x')| = d(2b-2)+k+1$. Analogously, if $x'' \in R_2$, then $|d^+(x'') - d^-(x'')| = d(2b-2)+k+1$. If $x \in Y \cup Z$, then $|d^+(x) - d^-(x)| \leq 1$. Therefore,

$$I(M) = \frac{v - v_{c-1} - 2v_c + 3k + 3}{2} \leq v - 3v_c + 2k + 2.$$

□

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