Group Divisible Designs With Block Size 5 and Index Odd

Ahmed M. Assaf
Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48859

ABSTRACT. In this paper we show that group divisible designs with block size five, group-type and index odd exist with a few possible exceptions.

1 Introduction

Let K be a set of positive integers each of which is at least 2 and let λ be a positive integer. A (K, λ)-group divisible design ($(k, \lambda) - GDD$) is a triple $(X, \mathbf{G}, \mathbf{B})$ where X is a finite set of points, $\mathbf{G}$ is a partition of X into subsets called groups and $\mathbf{B}$ is a set of subsets of X, called blocks, such that

1) $|B_i \cap G_j| \leq 1$ for all $B_i \in \mathbf{B}$ and $G_j \in \mathbf{G}$;

2) $|B_i| \in K$;

3) every pair of points from distinct groups occurs in λ blocks.

The group-type (type) of the GDD is a listing of the group sizes using "exponential" notations, i.e. $1^a 2^b 3^c$ denotes a groups of size 1, b groups of size 2, etc.. If all groups have the same size, say, g then the type of the group divisible design is called uniform of type g^n. When $K = \{k\}$, we simply write k for K. In this case a (k, λ)-GDD of type g^n denotes a GDD with block size k, group size g and index λ. Simple counting arguments show the following.

Lemma 1.1. *The necessary conditions for the existence of a (k, λ)-GDD of type g^n are $n \geq k$, $\lambda(n-1)g \equiv 0 \pmod{k-1}$ and $\lambda n(n-1) \equiv 0$ mod $k(k-1)$.*

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In the case $k = 3, 4$ these necessary conditions have been proven to be sufficient [8, 10] with the exception of (4,1)-GDD of type 2^4 and 6^4. In the case $k = 5$, the first result due to Hanani.

Lemma 1.2. [10] *Let $q \equiv 1 \pmod 4$ be a prime power. Then there exists a (5,1)-GDD of type 5^q.*

Capitalizing on this result, Hanani's student D. Avidan [3] showed that a (5,1)-GDD of type 5^n exists for all $n \equiv 1 \pmod 4$, $n \geq 5$ with the possible exceptions of $n = 33, 57, 93, 133, 177, 213, 413, 437, 473, 489, 493, 497$. Later on, Assaf [2] showed that a (5,1)-GDD of type 5^n exists for all $n \geq 5$ with the possible exception of $n = 33$. Recently, Yin, Abel, Colbourn, and Ling, proved the following:

Theorem 1.1. [11] *The necessary conditions for the existence of a (5,1)-GDD of type g^n are also sufficient except for $g^n \in \{2^5, 2^{11}, 3^5, 6^5\}$ and the possible exceptions of:*

1. $g^u = 3^{45}, 3^{65}$;
2. $g \equiv 2, 6, 14, 18 \pmod{20}$ *and*
   - *(a)* $g = 2$ *and* $u \in \{15, 35, 55, 65, 71, 75, 85, 95, 111, 115, 135, 195, 215, 315, 335, 435, 515, 575\}$;
   - *(b)* $g = 6$ *and* $u \in \{15, 35, 45, 75, 95, 115, 135\}$;
   - *(c)* $g = 18$ *and* $u \in \{11, 15, 35, 71, 111, 115, 135, 195\}$;
   - *(d)* $g = 2\alpha$ *for* $\alpha > 1$ *and* $(\alpha, 30) = 1$, *and* $u \in \{11, 15, 35, 71, 75, 111, 115, 135, 195\}$;
   - *(e)* $g = 6\gamma$, $\gamma \not\equiv 0 \pmod 5$, $\gamma \neq 3$ *odd, and* $u = 15$;
3. $g \equiv 10 \pmod{20}$ *and*
   - *(a)* $g = 10$ *and* $u \in \{5, 7, 15, 23, 27, 33, 35, 39, 47, 63\}$.
   - *(b)* $g = 30$ *and* $u \in \{9, 15\}$;
   - *(c)* $g = 90$ *and* $u \in \{7, 23, 27, 39, 47\}$;
   - *(d)* $g = 10\alpha$, $\alpha \equiv 1, 5 \pmod 6$, *and* $u \in \{7, 15, 23, 27, 35, 39, 47\}$.
   - *(e)* $g = 30\gamma$, $\gamma \geq 5$ *odd*, $\gamma \not\equiv 0 \pmod 3$ *or* $u = 15$.

In the case λ is even, F. Bennett, N. Shalaby, and J. Yin have shown the following:

Theorem 1.2. [5] *The necessary conditions for the existence of a $(5, \lambda)$-GDD of type g^n where λ even are also sufficient with the possible exception of g^{15} where $g \equiv 1$ or $5 \pmod 6$, g is not divisible by 5 and $(g, n, \lambda) = (9, 15, 2)$.*

Since $\lambda = 1$ and λ even have been treated with few possible exceptions, our interest here is the case λ odd. In this paper we prove the following:

Theorem 1.3. *The necessary conditions for the existence of a $(5,\lambda)$-GDD of type g^n, where λ is odd, are also sufficient with the possible exceptions of:*

$$\lambda = 3, g = (10\alpha)^n \text{ where } \alpha \equiv 1 \text{ or } 5 \pmod 6), n \in \{23, 27, 39\}.$$

$$\lambda = 5, g = 6 \text{ and } n \in \{19, 23, 27\}.$$

2 Recursive Constructions

To describe our recursive constructions, we need the notion of pairwise balanced designs (PBD), transversal designs, and modified group divisible designs. For the definition of these designs, we refer the reader to [2]. We shall adopt the following notations. A PBD (v, k, λ) denotes a pairwise balanced design on v points, block sizes from K and index λ. When $K = \{k\}$, the PBD is called balanced incomplete block design and is denoted by $B[v, k, \lambda]$. A $T[k, \lambda, m]$ denotes a transversal design with block size k, group size m and index λ. A $MGD[k, \lambda, m, n]$ denotes a modified group divisible design with block size k, group size m, row size n, and index λ.

The following theorem is our first recursive construction.

Theorem 2.1. *If there exists a $PBD(n, K, \lambda)$ and for every $k \in K$ there exists a $(5,\mu)$-GDD of type g^k then there exists a $(5, \lambda\mu)$-GDD of type g^n.*

The application of the above theorem requires the existence of PBD. The following result is most useful for us.

Theorem 2.2. 1. [10] *There exists a $B[v, 5, \lambda]$ for all $v \geq 5$, $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$.*

2. [9] *There exists a $PBD(v, \{5, k^*\}, 1)$ where $*$ means there is exactly one block of size k where $k = 9$ if $v \equiv 9$ or $17 \pmod{20}$, $v \geq 37$, $v \neq 49$, and $k = 13$ if $v \equiv 13 \pmod{20}$, $v \geq 53$.*

3. [4] *Let v be a positive odd integer $v \geq 5$, $v \notin \{11, 13, 15, 17, 19, 23, 27, 29, 31, 33, 39\}$ then there exists a $PBD(v, \{5, 7, 9\}, 1)$ with the possible exceptions of $v \in \{43, 51, 59, 71, 75, 83, 87, 93, 95, 99, 107, 111, 113, 115, 119, 123, 131, 133, 135, 139, 143, 153, 163, 167, 173, 179, 183, 191, 193, 195, 243, 283, 347, 411, 459, 563\}$.*

Another way of constructing PBD is the following lemma:

Lemma 2.1. *If there exist a $T[k, l, m]$, then there exist a $PBD(km, \{k, m\}, 1)$ and a $PBD(km + 1, \{k, m + 1\}, 1)$.*

Lemma 2.2. [1] (i) *There exists a $T[5, 1, m]$ for all $m \geq 4$, $m \neq 6, 10$.*
(ii) *There exist a $T[6, 1, m]$ for all $m \geq 5$, $m \neq 6, 10, 14, 18, 22$.*

Using Wilson's Fundamental Construction [12], one can construct GDD with large groups from GDD with small groups. A simple version of this construction is the following:

Theorem 2.3. *Assume there exist a* $(5, \lambda)$*-GDD of the type* g^n *and a* $T[5, \mu, m]$*. Then there exists a* $(5, \lambda\mu)$*-GDD of type* $(gm)^n$*.*

And by breaking the groups, one can construct a GDD with large number of small groups from a GDD with a small number of large groups as the following theorem explains

Theorem 2.4. *Suppose that there exists a* (k, λ)*-GDD of type* $\{m_i \mid 1 \leq i \leq r\}$*. Let* $h \geq 0$ *be an integer. If for each* i *there exists a* (k, λ)*-GDD of type* $\{m_{ij} \mid 1 \leq j \leq k(i)\} \cup \{h\}$ *where* $m_i = \sum_{1 \leq j \leq k(i)} m_{ij}$ *then there exists a* (k, λ)*-GDD of type* $\{m_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq k(i)\} \cup \{h\}$*.*

A modified group divisible design is called resolvable and denoted by $RMGD[k, \lambda, m, n]$ if its blocks can be partitioned into parallel classes. Notice that a $RMGD[k, 1, k, m]$ is the same as $RT[k, 1, m]$ with one parallel class of blocks singled out and since $RT[k, 1, m]$ is equivalent to $T[k+1, 1, m]$ we have the following:

Theorem 2.5. [1] *There exists a* $RMGD[5, 1, 5, m]$ *for all positive integers* m*,* $m \notin \{2, 3, 4, 6\}$ *with the possible exceptions of* $m \in \{10, 14, 18, 22\}$*.*

Theorem 2.6. *Assume there exists*

1) a $RMGD[k, 1, k, m]$,

2) a $(5, \lambda)$*-GDD of type* t^k *and* t^{k+1}*,*

3) a $(5, \lambda)$*-GDD of type* $t^m s^1$*,*

4) a $(5, \lambda)$*-GDD of type* $r^{\frac{kt+r}{r}}$*,*

5) a $(5, \lambda)$*-GDD of type* $r^{\frac{ut+r+s}{r}}$*,*

Then there exists a $(5, \lambda)$*-GDD of type* r^j *where* $j = (kmt+ut+r+s)/r$*.*

Proof: Take a $RMGD[k, 1, k, m]$ and inflate this design by a factor of t. To each of u parallel classes, $0 \leq u \leq m - 1$, we adjoin t new points and on each block we construct a $(5, \lambda)$-GDD of type t^{k+1}. On the remaining parallel classes we construct a $(5, \lambda)$-GDD of type t^k. To the parallel class of block size m we adjoin s new points and construct a $(5, \lambda)$-GDD of type $t^m s^1$.

To the groups adjoin r new points and construct a $(5, \lambda)$-GDD of type $r^{\frac{kt+r}{r}}$ such that these r points form a group. Notice that the total number of points added to the original design is $ut+r+s$ so on these points construct a $(5,\lambda)$-GDD of type $r^{\frac{ut+r+s}{r}}$. Now it is clear that this construction yields a $(5,\lambda)$-GDD of type r^j where $j = (kmt + ut + r + s)/r$.

The application of the above theorem requires the existence of a $(5,\lambda)$-GDD of type $t^m s^1$. In the case $t = 4$ we have the following result

Lemma 2.3. (i) [9] *There exists a $(5,1)$-GDD of type $4^m 8^1$ for all $m \equiv 0$ or $2 \pmod 5$, $m \geq 7$ with the possible exception of $m = 10$.*

(ii) [2] *There exists a $(5,1)$-GDD of type $4^m s^1$ where $s = 0$ when $m \equiv 0$ or $1 \pmod 5$ and $s = 4$ when $m \equiv 0$ or $4 \pmod 5$.*

Lemma 2.4. *If there exists a $(5,\lambda)$-GDD of type g^n and a $(5,\mu)$-GDD of type g^n then there exists a $(5,\lambda+\mu)$-GDD of type g^n.*

Another notion that we require to prove our result is group divisible designs with a hole. A group divisible design with a hole H and index λ is a quadruple $(X, H, \mathbf{G}, \mathbf{B})$ which satisfies the following:

1) X is a finite set of points and $H \subset X$.

2) $\mathbf{G} = \{G_1, \ldots, G_n\}$ is a partition of X into n sets called groups.

3) There is a positive integer $m < n$ such that $\{G_m, \ldots, G_n\}$ is a partition of H.

4) $|B \cap G_i| \leq 1$ for every $B \in \mathbf{B}$ and $G_i \in \mathbf{G}$

5) No block contains two points of H.

6) Every pair $\{x, y\}$ from distinct groups such that at least one of x, y is in $X \backslash H$ belongs to exactly λ blocks.

Theorem 2.7. *If there exists a $(5,\lambda)$-GDD of type m^r on $X \backslash H$ and a $(5,\lambda)$-GDD of type m^t on H then there exists a $(5,\lambda)$-GDD of type m^{r+t} on X.*

Finally, about the notation in this paper, a block , $\langle k\ k{+}m\ k{+}n\ k{+}j\ f(k)\rangle \pmod \nu$ where $f(k) = a$ if k is even and $f(k) = b$ if k is odd is denoted by $\langle 0\ m\ n\ j\rangle \cup \{a, b\} \pmod \nu$. Similarly, a block $\langle (0,k)\ (0,k+m)\ (1,k+n)\ (1,k+j)\ f(k)\rangle \pmod{-,\nu}$ where $f(k) = a$ if k is even and $f(k) = b$ if k is odd is denoted by $\langle (0,0)\ (0,m)\ (1,n)\ (1,j)\rangle \cup \{a,b\}$ mod $(-,\nu)$.

3 GDDs With Index Odd Not Multiple of 5

The necessary conditions for the existence of $(5,\lambda)$-GDDs for $\lambda > 1$ odd not multiple of 5 are the same necessary conditions of the case $\lambda = 1$. Therefore, we need only construct $(5,\lambda)$-GDDs for the exceptional cases of Theorem 1.1. We treat the case $\lambda = 3$, other values of λ are obtained by means of Lemma 4.2.

Lemma 3.1. *There exists a* $(5,3)$*-GDD of type* 2^n *for all* $n \equiv 1$ *or* $5 \pmod{10}$, $n \geq 5$.

Proof: For $n = 11, 15$ see the following table.

In general, the construction as follows. Let $X = Z_{v-m} \cup H_m$ or $X = Z_2 \times Z_{\frac{v-m}{2}} \cup H_m$ where $H_m = \{h_1, \ldots, h_m\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks. The notation **xm** following a base block indicates that the entire orbit is to be taken m times. Except where otherwise stated the groups are $\{(0,i)(1,i)\}_{i=0}^{m}$ if $X = Z_2 \times Z_m$ and m is odd, and the groups are $\{(i,j),(i,j+\frac{r}{2})\}_{j=0}^{\frac{r}{2}}$, $i = 0, 1$ together with the groups on the hole if $X = Z_2 \times Z_r \cup H_m$ and r is an even integer.

For $n = 5$ See Theorem 3.11 of [10].

For $n = 35$ take a $(5,3)$-GDD of type 10^7, Lemma 3.3, and then apply Theorem 2.4 with $\lambda = 3$, $h = 0$ and $m_{ij} = 2$.

For $n = 55, 71, 75$ applying Lemma 2.1 we obtain a $PBD(n, K, 1)$ where $k \in \{5, 11, 15\}$. Now apply Theorem 2.1.

For $n = 65$ take a $(6,3)$-GDD of type 5^7 [10]. Delete all but two points from last group and inflate the design by a factor of four. That is, replace the blocks of size five and six by the blocks of a $(5,1)$-GDD of type 4^5 and 4^6 respectively. The resultant design is a $(5,3)$-GDD of type $20^6 8^1$. Now apply Theorem 2.4 with $h = 2$, $k = 5$, $\lambda = 3$ and $m_{ij} = 2$.

For $n = 85, 95, 111, 115, 135, 195, 215, 315, 335, 435, 515, 575$ apply Theorem 2.6 with $k = 5$, $t = 4$, $r = 2$, $s = 0, 4$ or 8, m and u as follows:

n	85	95	111	115	135	195	215	315	335	435	515	575
m	7	9	11	11	12	19	20	30	31	41	50	55
u	5	1	0	2	5	1	7	7	12	12	12	12

	Point Set	Base Blocks
11	$Z_2 \times Z_{11}$	On $\{0\} \times Z_{11}$ construct a $B[11,5,2]$. Further, take the following blocks $\langle(0,0)(0,1)(1,2)(1,3)(1,5)\rangle$ $\langle(0,0)(0,2)(1,1)(1,7)(1,8)\rangle$ $\langle(0,0)(0,3)(1,6)(1,9)(1,10)\rangle$ $\langle(0,0)(0,4)(1,1)(1,7)(1,9)\rangle$ $\langle(0,0)(0,5)(1,2)(1,4)(1,9)\rangle$
15	$Z_2 \times Z_{15}$	$\langle(i,0)(i,3)(i,6)(i,9)(i,12)\rangle$, $i = 0,1$ (orbit length three) $\langle(1,0)(1,1)(1,2)(1,4)(1,8)\rangle$ $\langle(0,0)(0,1)(0,2)(1,3)(1,5)\rangle$ $\langle(0,0)(0,1)(0,3)(1,11)(1,12)\rangle$ $\langle(0,0)(0,3)(0,7)(1,8)(1,14)\rangle$ $\langle(0,0)(0,2)(0,7)(1,9)(1,14)\rangle$ $\langle(0,0)(0,7)(1,5)(1,10)(1,13)\rangle$ $\langle(0,0)(0,4)(0,9)(1,8)(1,13)\rangle$ $\langle(0,0)(0,4)(0,9)(1,6)(1,10)\rangle$

Lemma 3.2. *There exists a* $(5,3)$*-GDD of type* 6^n *for all* $n \equiv 1$ *or* $5 \pmod{10}$, $n \geq 5$.

Proof: For $n = 5$ See Theorem 3.11 of [10].

For $n = 15$ let $X = Z_2 \times Z_{42} \cup H_6$. Groups are $\{(i,0)(i,7)(i,14)(i,21)(i,28)(i,35)\} \cup H_6$, $i = 0,1$. Blocks are the following mod $(\_, 42)$.

$\langle(i,0)(i,2)(i,5)(i,24)(i,32)\rangle$ $i = 0,1$
$\{\langle(0,0)(0,19)(1,0)(1,4)(1,15)\rangle$
$\langle(0,0)(0,6)(0,16)(1,17)(1,30)\rangle$
$\langle(0,0)(0,9)(0,17)(1,35)(1,37)\rangle$
$\langle(0,0)(0,4)(0,22)(1,13)(1,16)\rangle$
$\langle(0,0)(0,4)(1,16)(1,26)(1,36)\rangle$
$\langle(0,0)(0,5)(1,29)(1,30)\rangle \cup \{h_3, h_4\}$
$\langle(0,0)(0,5)(1,1)(1,13)h_1\rangle$
$\langle(0,0)(0,9)(1,4)(1,20)h_3\rangle$
$\langle(0,0)(0,16)(1,9)(1,33)h_5\rangle$
$\langle(1,0)(1,1)(1,5)(1,6)(1,18)\rangle$
$\langle(0,0)(0,1)(1,7)(1,32)(1,40)\rangle$
$\langle(0,0)(0,2)(0,13)(1,5)(1,21)\rangle$
$\langle(0,0)(0,3)(0,15)(1,2)(1,25)\rangle\} \times 2$
$\langle(0,0)(0,4)(0,22)(1,13)(1,19)\rangle$
$\langle(0,0)(0,1)(1,0)(1,3)\rangle \cup \{h_1, h_2\}$
$\langle(0,0)(0,13)(1,7)(1,18)\rangle \cup \{h_5, h_6\}$
$\langle(0,0)(0,6)(1,12)(1,34)h_2\rangle$
$\langle(0,0)(0,11)(1,21)(1,27)h_4\rangle$
$\langle(0,0)(0,17)(1,31)(1,40)h_6\rangle$

For $n = 35$ take three copies of a $(5,1)$-GDD of type 30^7 and then apply Theorem 2.4 with $h = 0$, $k(i) = 5$ and $m_{ij} = 6$.

For $n = 45, 75$ notice that $45 \in PBD(\{5\}, 1)$ and $75 \in PBD(\{5, 15\}, 1)$, Lemma 2.1, and hence, the result follows from Theorem 2.1.

For $n = 95, 115$ apply Theorem 2.6 with $k = 5$, $t = 12$, $r = 6$ and $(m, u, s) = (9, 1, 12), (11, 2, 0)$ respectively.

For $n = 135$ take a $(5,1)$-RGD of type 5^{13} and inflate the design by a factor of 12, [7]. To each of two parallel classes we adjoin 12 new points and construct on each block a $(5,3)$-GDD of type 12^6. On the remaining parallel classes we construct a $(5,3)$-GDD of type 12^5. To the groups we adjoin six new points $\{h_1, \ldots, h_6\}$ and on each group construct a $(5,3)$-GDD of type 6^{11} such that $\{h_1, \ldots, h_6\}$ is a group. Finally, take the set $\{h_1, \ldots, h_6\}$ with the 24 points we added and construct a $(5,3)$-GDD of type 6^5 such that $\{h_1, \ldots, h_6\}$ is a group.

Lemma 3.3. *There exists a* $(5,3)$*-GDD of type* 10^n *for* n *odd,* $n \geq 5$ *with the possible exceptions of* $n = 23, 27, 39$.

Proof: For $n = 5$ see Theorem 3.11 of [10].

For $n = 15$ take a $(5,3)$-GDD of type 2^{15} and then apply Theorem 2.3 with $\mu = 1$ and $m = 5$.

For $n = 33$ take a $(5,1)$-GDD of type 5^{33} and then apply Theorem 2.3 with $\mu = 3$ and $m = 2$.

For $n = 35, 63$ apply Theorem 2.1 with $K = \{5, 7, 9\}$, $\lambda = 1$, $g = 10$ and $\mu = 3$.

For $n = 47$ apply Theorem 2.4 to a $(5,1)$-GDD of type $80^5 60^1$ with $h = 10$.

Notice that a $(5,1)$-GDD of type $80^5 60^1$ can be constructed by deleting five points from a $T[6,1,20]$ and then replace each block of size five and six by the blocks of a $(5,1)$-GDD of type 4^5 and 4^6 respectively.

For $n = 7$. Let $X = Z_2 \times Z_{30} \cup H_{10}$. Groups $\{(i,j)(i,j+3)\dots(i,j+27)\} \cup H_{10}$, $i = 0,1$, $j \in Z_3$.

Blocks are the following mod $(\_, 30)$

$\langle(0,0)(0,4)(0,14)(1,0)(1,1)\rangle$	$\langle(0,0)(0,10)(1,9)(1,22)(1,29)\rangle$
$\langle(0,0)(0,2)(0,10)(1,8)(1,24)\rangle$	$\langle(0,0)(0,2)(1,23)(1,25)(1,27)\rangle$
$\langle(0,0)(0,4)(0,11)(1,7)(1,15)\rangle$	$\langle(0,0)(0,8)(1,2)(1,18)(1,28)\rangle$
$\langle(0,0)(0,11)(0,13)(1,13)\rangle \cup \{h_1,h_2\}$	$\langle(0,0)(1,3)(1,8)(1,16)\rangle \cup \{h_1,h_2\}$
$\langle(0,0)(0,1)(0,17)(1,5)\rangle \cup \{h_3,h_4\}$	$\langle(0,0)(1,6)(1,10)(1,17)\rangle \cup \{h_3,h_4\}$
$\langle(0,0)(0,5)(0,19)(1,20)\rangle \cup \{h_5,h_6\}$	$\langle(0,0)(1,7)(1,9)(1,14)\rangle \cup \{h_5,h_6\}$
$\langle(0,0)(0,1)(1,0)(1,5)\rangle \cup \{h_1,h_2\}$	$\langle(0,0)(0,5)(1,1)(1,2)\rangle \cup \{h_3,h_4\}$
$\langle(0,0)(0,5)(1,12)(1,23)\rangle \cup \{h_5,h_6\}$	$\langle(0,0)(0,7)(1,15)(1,28)\rangle \cup \{h_7,h_8\}$
$\langle(0,0)(0,13)(1,5)(1,24)\rangle \cup \{h_9,h_{10}\}$	$\langle(0,0)(0,4)(1,16)(1,17)h_7\rangle$
$\langle(0,0)(0,7)(1,13)(1,21)h_8\rangle$	$\langle(0,0)(0,8)(1,3)(1,19)h_9\rangle$
$\langle(0,0)(0,1)(1,10)(1,20)h_{10}\rangle$	

Theorem 3.1. *Let $\lambda > 1$ be an odd integer, not multiple of 5. Then there exists a $(5,\lambda)$-GDD of type g^n for all $g \geq 0$ and $n \geq 5$ satisfying the necessary conditions with the possible exceptions of $g = (10\alpha)^n$ where $\alpha \equiv 1$ or $5 \pmod 6$, $n \in \{23, 27, 39\}$ and $\lambda = 3$.*

Proof: we first prove the theorem for $\lambda = 3$, and it is clear that we only need to consider the exceptional cases of Theorem 1.1. By Lemma 3.1 a $(5,3)$-GDD of type 2^n exists for all $n \equiv 1, 5 \pmod{10}$, $n \geq 5$. Therefore applying Theorem 2.3, a $(5,3)$-GDD of type g^n exists for all $n \equiv 1$ or $5 \pmod{10}$, $n \geq 5$, $g \equiv 2, 6, 14, 18 \pmod{20}$, $g \neq 6$. For $g = 6$ the result follows from Lemma 3.2.

For a $(5,3)$-GDD of type 3^5 see [10], and of type $3^{45}, 3^{65}$ apply Theorem 2.1 with $K = \{5\}$, $\lambda = 1$, $\mu = 3$, $g = 3$ and $n = 45, 65$ respectively.

For $g = 10$ the result follows from Lemma 3.3 with the possible exceptions of $n = 23, 27, 39$.

For $(5,3)$-GDD of type 30^9 apply Theorem 2.3 with $\lambda = 1$, $g = 5$, $n = 9$, $\mu = 3$ and $m = 6$ and for type 30^{15} apply Theorem 2.3 with $\lambda = 3$, $g = 2$, $n = 15$, $\mu = 1$, and $m = 15$.

For a $(5,3)$-GDD of type 90^n, $n \in \{7, 23, 27, 39, 47\}$ apply Theorem 2.3 with $\lambda = 1$, $g = 30$, $\mu = 3$ and $m = 3$.

For a $(5,3)$-GDD of type g^n where $g = 30\gamma$, $\gamma \geq 5$ odd, $\gamma \not\equiv 0 \pmod 3$ or $\gamma = 9$, $n = 15$ take a $(5,3)$-GDD of type 2^{15} and then apply Theorem

2.3 with $m = 15\gamma$ and $\mu = 1$.

For a (5,3)-GDD of type $(10\alpha)^n$ where $\alpha \equiv 1, 5 \pmod 6$ and $n \in \{7, 47\}$ take a (5,3)-GDD of type 10^n, $n = 7, 47$ and then apply Theorem 2.3 with $m = \alpha$ and $\mu = 1$.

For a (5,3)-GDD of type $(10\alpha)^n$ where $\alpha \equiv 1$ or $5 \pmod 6$ and $n = 15, 35$ take a (5,3)-GDD of type 2^n, $n = 15, 35$ and then apply Theorem 2.3 with $m = 5\alpha$ and $\mu = 1$.

For λ odd, $\lambda > 3$ not a multiple of five write $\lambda = 2\alpha + 5\beta$ and notice that a (5,2)-GDD of type g^n exists for all g and n satisfying the necessary conditions with the possible exception of type 9^{15} and g^{15} where $g \equiv 1$ or $5 \pmod 6$, g is not divisible by 5 and a (5,5)-GDD of type g^n exists for all g and n satisfying the necessary conditions with the possible exceptions of $(g, n) = (6,19)(6,23)(6,27)$ (See Theorem 4.5 of the next section). Now the results follows from Lemma 4.2.

4 GDDs With Index Odd Multiple of 5

In this section we construct $(5,\lambda)$-GDDs of type g^n where λ is a positive odd integer which is a multiple of 5. The necessary conditions are:

1) If $g \equiv 1 \pmod 2$ then $n \equiv 1 \pmod 4$.

2) If $g \equiv 2 \pmod 4$ then $n \equiv 1 \pmod 2$.

3) If $g \equiv 0 \pmod 4$ then $n \geq 5$.

Theorem 4.1. *Let $g \equiv 1 \pmod 2$ be a positive integer, then there exists a (5,5)-GDD of type g^n for all $n \equiv 1 \pmod 4$.*

Proof: For $g \neq 3$, since a $B[n,5,5]$ and a (5,1)-GDD of type g^5 exists for all such n and g, the result follows from Theorem 2.1.

For $g = 3$ notice that if $n \equiv 1 \pmod 4$ then:

a) There exists a $B[n,5,1]$ if $n \equiv 1$ or $5 \pmod{20}$.

b) There exists a $PBD(n, \{5, 9^*\}, 1)$ for $n \equiv 9$ or $17 \pmod{20}$, $n \geq 37$, $n \neq 49$, where $*$ means the block of size 9 is unique.

c) There exists a $PBD(n, \{5, 13^*\}, 1)$ for $n \equiv 13 \pmod{20}$, $n \geq 53$. Hence, by Theorem 2.1 we need only to construct a (5,5)-GDD of type 3^n for $n = 5, 9, 13, 17, 29, 33, 49$.

For $n = 5$ the result follows from Theorem 3.11 of [10].

For $n = 9, 13, 17, 29, 33$ see table 4.1.

For $n = 49$ adjoin a point to the groups of a (5,1)-GDD of type 8^6 to obtain a $PBD(49, \{5, 9\}, 1)$. Now apply Theorem 2.1 to get the result.

Table 4.1

	Point Set	Base Blocks
9	Z_{27}	$\langle 0\ 1\ 2\ 4\ 15\rangle\ \langle 0\ 2\ 7\ 13\ 19\rangle\ \langle 0\ 3\ 6\ 10\ 22\rangle\ \langle 0\ 1\ 2\ 5\ 12\rangle$ $\langle 0\ 1\ 5\ 8\ 22\rangle\ \langle 0\ 2\ 8\ 15\ 19\rangle$
13	$Z_{38} \cup H_3$	$\langle i\ i+9\ i+18\ i+27\ h_j\rangle\ i, 0, \ldots, 8, j = 1, 2, 3$ $\langle 0\ 1\ 2\ 4\ 30\rangle\ \langle 0\ 3\ 14\ 19\ 23\rangle\ \langle 0\ 5\ 11\ 19\ 26\rangle\ \langle 0\ 1\ 2\ 3\ 5\rangle$ $\langle 0\ 3\ 7\ 13\ 23\rangle\ \langle 0\ 5\ 13\ 19\ 27\rangle\ \langle 0\ 4\ 15\ 21\rangle\ \langle 0\ 5\ 16\ 23\ h_2\rangle$ $\langle 0\ 7\ 15\ 26\ h_3\rangle$
17	Z_{51}	$\{\langle 0\ 1\ 2\ 4\ 13\rangle\ \langle 0\ 3\ 8\ 22\ 29\rangle\ \langle 0\ 4\ 10\ 28\ 35\rangle\ \langle 0\ 5\ 11\ 20\ 32\rangle\} \times 2$ $\langle 0\ 1\ 3\ 13\ 38\rangle\ \langle 0\ 4\ 9\ 19\ 37\rangle\ \langle 0\ 6\ 14\ 24\ 35\rangle\ \langle 0\ 7\ 15\ 23\ 38\rangle$
29	$Z_{72} \cup H_{15}$	$\langle i\ i+18\ i+36\ i+54\ h_j\rangle\ i \in Z_{18},\ j = 13, 14, 15$ $\langle 0\ 1\ 3\ 7\ 17\rangle \times 2\ \langle 0\ 8\ 19\ 42\ 51\rangle \times 2\ \langle 0\ 1\ 7\ 16\ 29\rangle$ $\langle 0\ 4\ 14\ 34\ 49\rangle\ \langle 0\ 5\ 31\ 44\ h_1\rangle\ \langle 0\ 5\ 31\ 44\ h_2\rangle\ \langle 0\ 12\ 27\ 47\ h_3\rangle$ $\langle 0\ 12\ 27\ 47\ h_4\rangle\ \langle 0\ 11\ 26\ 51\ h_5\rangle\ \langle 0\ 1\ 3\ 7\ h_6\rangle\ \langle 0\ 5\ 13\ 21\ h_7\rangle$ $\langle 0\ 9\ 19\ 44\ h_8\rangle\ \langle 0\ 11\ 23\ 45\ h_9\rangle\ \langle 0\ 12\ 30\ 52\ h_{10}\rangle\ \langle 0\ 5\ 31\ 60\ h_{11}\rangle$ $\langle 0\ 4\ 11\ 33\ h_{12}\rangle\ \langle 0\ 8\ 28\ 46\ h_{13}\rangle\ \langle 0\ 9\ 25\ 39\ h_{14}\rangle$ $\langle 0\ 13\ 32\ 49\ h_{15}\rangle\ \langle 0\ 2\ 19\ 41\rangle \cup \{h_i\}_{i=1}^{4}\ \langle 0\ 1\ 3\ 6\rangle \cup \{h_i\}_{i=5}^{8}$ $\langle 0\ 10\ 31\ 45\rangle \cup \{h_i\}_{i=9}^{12}$
33	Z_{99}	$\{\langle 0\ 1\ 3\ 11\ 28\rangle\ \langle 0\ 5\ 21\ 43\ 67\rangle\ \langle 0\ 6\ 35\ 55\ 69\rangle\ \langle 0\ 13\ 31\ 54\ 73\rangle$ $\langle 0\ 1\ 4\ 10\ 49\rangle\ \langle 0\ 5\ 22\ 41\ 81\rangle\ \langle 0\ 7\ 28\ 62\ 75\rangle\ \langle 0\ 12\ 26\ 64\ 79\rangle\} \times 2$ $\langle 0\ 1\ 8\ 12\ 46\rangle\ \langle 0\ 5\ 30\ 56\ 72\rangle\ \langle 0\ 9\ 28\ 57\ 69\rangle\ \langle 0\ 2\ 4\ 11\ 15\rangle$ $\langle 0\ 2\ 16\ 31\ 78\rangle\ \langle 0\ 3\ 10\ 59\ 84\rangle\ \langle 0\ 6\ 22\ 30\ 57\rangle\ \langle 0\ 8\ 17\ 44\ 87\rangle$

Theorem 4.2. *There exists a* $(5,5)$*-GDD of type* 2^n *for all* n *odd,* $n \geq 5$.

Proof: For $n \equiv 1$ or $5 \pmod{20}$ apply Theorem 2.1 and notice that a $(5,5)$-GDD of type 2^5 exists by Theorem 3.11 of [10].

For $n \equiv 11$ or $15 \pmod{20}$ apply Lemma 2.4 with $\lambda = 3$, $\mu = 2$ and $g = 2$.

For $n \equiv 9, 13, 17 \pmod{20}$ there exists a $PBD(n, \{5, k^*\}, 1)$ where $k = 9$ if $n \equiv 9$ or $17 \pmod{20}$, $n \geq 37$, $n \neq 49$ and $k = 13$ if $n \equiv 13 \pmod{20}$, $n \geq 53$. Therefore, by Theorem 2.1 we only need to construct a $(5,5)$-GDD of type 2^n for $n = 9, 13, 17, 29, 33, 49$.

For $n = 49$ we have shown that $49 \in PBD(\{5,9\}, 1)$ and hence the result follows from Theorem 2.1.

For the remaining values of n see table 4.2.

For $n \equiv 3, 7$ or $19 \pmod{20}$, we first construct a $(5,5)$-GDD of type 2^n for $n = 7, 19, 23, 27, 39, 43, 47, 63, 67, 71$.

For $n = 7, 19, 23, 27, 39, 43$ see table 4.2.

For $n = 47, 63, 67$ there exists a $PBD(n, \{5,7,9\}, 1)$, Theorem 2.2, so the result follows from Theorem 2.1.

For $n = 71$ notice that $71 \in PBD(\{5, 15\}, 1)$ by Lemma 2.1 so the result follows from Theorem 2.1.

For all other values of $n \equiv 3, 7$ or $19 \pmod{20}$ apply Theorem 2.5 with $k = 5$, $t = 4$, $r = 2$, $\lambda = 5$, $s = 0, 4$ or 8, as described in Lemma 2.3, and $0 \leq u \leq m - 1$.

Table 4.2

	Point Set	Base Blocks
7	$Z_2 \times Z_6 \cup H_2$	$\langle(0,0)(0,1)(0,2)(1,0)(1,1)\rangle$ $\langle(0,0)(0,2)(1,0)(1,1)(1,2)\rangle$ $\langle(0,0)(0,1)(0,2)(1,3)h_1\rangle$ $\langle(0,0)(1,0)(1,2)(1,4)h_1\rangle$ $\langle(0,0)(0,2)(1,1)(1,3)h_2\rangle$ $\langle(0,0)(0,2)(1,4)(1,5)h_2\rangle$ $\langle(0,0)(0,1)(1,3)(1,4)\rangle \cup \{h_1, h_2\}$.
9	Z_{18}	$\langle 0\ 1\ 2\ 3\ 5\rangle$ $\langle 0\ 1\ 3\ 8\ 13\rangle$ $\langle 0\ 1\ 6\ 8\ 12\rangle$ $\langle 0\ 3\ 6\ 10\ 14\rangle$.
13	Z_{26}	$\langle 0\ 1\ 2\ 6\ 16\rangle$ $\langle 0\ 2\ 8\ 11\ 20\rangle$ $\langle 0\ 3\ 8\ 12\ 19\rangle$ $\langle 0\ 1\ 2\ 3\ 6\rangle$ $\langle 0\ 2\ 6\ 11\ 18\rangle$ $\langle 0\ 3\ 8\ 12\ 19\rangle$.
17	Z_{34}	$\langle 0\ 1\ 2\ 4\ 10\rangle \times 2$ $\langle 0\ 3\ 8\ 16\ 23\rangle$ $\langle 0\ 4\ 11\ 20\ 25\rangle$ $\langle 0\ 5\ 12\ 18\ 24\rangle$ $\langle 0\ 3\ 4\ 15\ 25\rangle$ $\langle 0\ 2\ 5\ 20\ 26\rangle$ $\langle 0\ 4\ 11\ 18\ 23\rangle$.
19	$Z_2 \times Z_{19}$	$\langle(i,0)(i,1)(i,3)(i,5)(i,11)\rangle, i = 0, 1$ $\langle(0,0)(0,4)(1,9)(1,14)(1,17)\rangle$ $\langle(0,0)(0,1)(0,2)(0,5)(1,3)\rangle$ $\langle(0,0)(1,1)(1,2)(1,3)(1,8)\rangle$ $\langle(0,0)(0,1)(0,8)(0,11)(1,7)\rangle$ $\langle(0,0)(0,1)(1,6)(1,12)(1,16)\rangle$ $\langle(0,0)(0,5)(0,12)(1,8)(1,14)\rangle$ $\langle(0,0)(0,2)(1,9)(1,12)(1,16)\rangle$ $\langle(0,0)(0,3)(0,10)(1,1)(1,2)\rangle$ $\langle(0,0)(0,7)(1,6)(1,11)(1,18)\rangle$ $\langle(0,0)(0,6)(1,4)(1,12)(1,13)\rangle$ $\langle(0,0)(0,4)(0,13)(1,8)(1,17)\rangle$ $\langle(0,0)(0,2)(1,1)(1,3)(1,7)\rangle$ $\langle(0,0)(0,4)(0,13)(1,8)(1,16)\rangle$ $\langle(0,0)(0,5)(1,10)(1,13)(1,15)\rangle$ $\langle(0,0)(0,3)(1,9)(1,12)(1,16)\rangle$ $\langle(0,0)(0,5)(0,11)(1,7)(1,16)\rangle$
29	58	$\langle 0\ 1\ 3\ 8\ 27\rangle \times 2$ $\langle 0\ 4\ 18\ 30\ 41\rangle \times 2$ $\langle 0\ 6\ 15\ 28\ 48\rangle \times 2$ $\langle 0\ 1\ 3\ 5\ 11\rangle$ $\langle 0\ 3\ 7\ 20\ 43\rangle$ $\langle 0\ 5\ 14\ 24\ 47\rangle$ $\langle 0\ 6\ 18\ 27\ 42\rangle$ $\langle 0\ 7\ 19\ 27\ 44\rangle$ $\langle 0\ 1\ 2\ 6\ 17\rangle$ $\langle 0\ 3\ 13\ 27\ 36\rangle$ $\langle 0\ 7\ 19\ 37\ 45\rangle$
33	66	$\{\langle 0\ 1\ 3\ 7\ 21\rangle$ $\langle 0\ 5\ 15\ 40\ 49\rangle$ $\langle 0\ 8\ 24\ 36\ 47\rangle$ $\langle 0\ 1\ 6\ 14\ 35\rangle$ $\langle 0\ 2\ 19\ 26\ 48\rangle$ $\langle 0\ 3\ 16\ 28\ 39\rangle\} \times 2$ $\langle 0\ 1\ 3\ 7\ 58\rangle$ $\langle 0\ 4\ 13\ 29\ 56\rangle$ $\langle 0\ 9\ 21\ 31\ 49\rangle$ $\langle 0\ 4\ 19\ 24\ 34\rangle$.

23	$Z_2 \times Z_{22} \cup H_2$	On $\{0\} \times Z_{22}$ construct a $(5,2)$-GDD of type 2^{11}.
		$\{\langle(0,0)(0,2)(1,0)(1,1)(1,7)\rangle\ \langle(0,0)(0,4)(1,6)(1,14)(1,19)\rangle$
		$\langle(0,0)(0,6)(1,9)(1,17)(1,19)\rangle\ \langle(0,0)(0,8)(1,4)(1,16)(1,20)\rangle\} \times 2$
		$\langle(1,0)(1,1)(1,2)(1,5)(1,15)\rangle\ \langle(0,0)(0,10)(1,0)(1,3)(1,18)\rangle$
		$\langle(0,0)(0,6)(1,11)(1,13)(1,16)\rangle\ \langle(0,0)(0,3)(0,10)(1,15)(1,17)\rangle$
		$\langle(0,0)(0,1)(0,10)(1,0)(1,1)\rangle\ \langle(0,0)(0,2)(0,9)(1,1)(1,4)\rangle$
		$\langle(0,0)(0,3)(0,8)(1,9)(1,18)\rangle\ \langle(0,0)(0,1)(0,5)(1,3)(1,7)\rangle$
		$\langle(0,0)(0,3)(1,14)(1,19)h_1\rangle\ \langle(0,0)(0,5)(1,4)(1,13)h_1\rangle$
		$\langle(0,0)(0,7)(1,11)(1,17)h_2\rangle\ \langle(0,0)(0,9)(1,3)(1,18)h_2\rangle.$
		$\langle(0,0)(0,1)(1,6)(1,9)\rangle \cup \{h_1, h_2\}$
27	$Z_2 \times Z_{22} \cup H_{10}$	$\{\langle(i,0)(i,1)(i,3)(i,5)(i,9)\rangle$
		$\langle(i,0)(i,1)(i,6)(i,9)(i,16)\rangle\}\ i = 0, 1.$
		$\langle(0,0)(0,2)(0,7)(1,0)(1,9)\rangle\ \langle(0,0)(0,4)(1,3)(1,5)(1,10)\rangle$
		$\langle(0,0)(0,1)(0,9)(1,13)h_1\rangle\ \langle(0,0)(1,14)(1,17)(1,18)h_1\rangle$
		$\{\langle(0,0)(0,2)(1,0)(1,3)h_2\rangle\ \langle(0,0)(0,4)(1,2)(1,10)h_3\rangle$
		$\langle(0,0)(0,6)(1,14)(1,18)h_4\rangle\ \langle(0,0)(0,8)(1,19)(1,21)h_5\rangle$
		$\langle(0,0)(0,10)(1,9)(1,15)h_6\rangle\ \langle(0,0)(0,9)(1,4)(1,16)h_7\rangle$
		$\langle(0,0)(0,5)(1,8)(1,18)h_8\rangle\ \langle(0,0)(0,10)(1,15)(1,16)h_9\rangle\} \times 2$
		$\langle(0,0)(0,3)(1,11)(1,19)h_{10}\rangle\ \langle(0,0)(0,3)(1,7)(1,14)h_{10}\rangle$
		$\langle(0,0)(0,3)(1,7)(1,14)\rangle \cup \{h_1, h_2\}$
		$\langle(0,0)(0,7)(1,2)(1,9)\rangle \cup \{h_i, h_j\}\ (i,j) = (3,4), (5,6)$
		$\langle(0,0)(0,1)(1,1)(1,10)\rangle \cup \{h_i, h_j\},\ (i,j) = (7,8), (9,10).$
39	$Z_2 \times Z_{39}$	$\langle(i,0)(i,4)(i,9)(i,17)(i,33)\rangle \times 2,\ i = 0, 1.$
		$\langle(i,0)(i,1)(i,3)(i,21)(i,28)\rangle,\ i = 0, 1$
		$\{\langle(0,0)(0,9)(1,10)(1,17)(1,37)\rangle\ \langle(0,0)(0,19)(1,15)(1,18)(1,24)\rangle$
		$\langle(0,0)(0,2)(0,14)(1,9)(1,33)\rangle\ \langle(0,0)(0,13)(1,4)(1,6)(1,29)\rangle$
		$\langle(0,0)(0,5)(0,15)(1,26)(1,27)\rangle\ \langle(0,0)(0,1)(0,17)(1,3)(1,14)\rangle\} \times 3$
		$\langle(0,0)(0,1)(0,12)(0,19)(1,36)\rangle\ \langle(0,0)(0,3)(1,25)(1,30)(1,35)\rangle$
		$\langle(0,0)(0,6)(1,11)(1,15)(1,19)\rangle\ \langle(0,0)(0,14)(1,18)(1,20)(1,21)\rangle$
		$\langle(0,0)(0,3)(0,7)(1,1)(1,26)\rangle\ \langle(0,0)(0,6)(1,15)(1,23)(1,36)\rangle$
		$\langle(0,0)(0,3)(0,11)(1,13)(1,31)\rangle\ \langle(0,0)(0,18)(1,4)(1,11)(1,21)\rangle$
		$\langle(0,0)(0,7)(0,11)(1,29)(1,34)\rangle\ \langle(0,0)(0,8)(1,7)(1,20)(1,24)\rangle$
		$\langle(0,0)(0,2)(0,6)(1,1)(1,14)\rangle\ \langle(0,0)(0,7)(1,5)(1,23)(1,33)\rangle$
		$\langle(0,0)(0,8)(0,11)(1,14)(1,31)\rangle\ \langle(0,0)(0,18)(1,8)(1,20)(1,28)\rangle$
43	$Z_2 \times Z_{34} \cup H_{18}$	$\{\langle(i,0)(i,1)(i,3)(i,7)(i,15)\rangle \times 2\ \langle(i,0)(i,1)(i,8)(i,11)(i,20)\rangle\}\ i = 0, 1$
		$\{\langle(0,0)(0,5)(0,18)(1,33)h_i\rangle\ \langle(0,0)(1,0)(1,5)(1,18)h_i\rangle\}\ i = 1, \ldots, 5.$
		$\langle(0,0)(0,11)(1,4)(1,13)\rangle \cup \{h_i, h_j\},\ (i,j) = (1,2)(3,4)(5,6).$
		$\langle(0,0)(0,3)(1,6)(1,25)\rangle \cup \{h_i, h_j\},\ (i,j) = (7,8)(9,10)$
		$\langle(0,0)(0,7)(1,8)(1,11)\rangle \cup \{h_i, h_j\},\ (i,j) = (11,12)(13,14)$
		$\langle(0,0)(0,1)(1,20)(1,27)\rangle \cup \{h_{15}, h_{16}\}$
		$\langle(0,0)(0,11)(1,7)(1,14)\rangle \cup \{h_{17}, h_{18}\}$
		$\{\langle(0,0)(0,2)(1,8)(1,19)h_{16}\rangle\ \langle(0,0)(0,10)(1,1)(1,21)h_7\rangle$
		$\langle(0,0)(0,14)(1,3)(1,9)h_8\rangle\ \langle(0,0)(0,8)(1,17)(1,29)h_9\rangle$
		$\langle(0,0)(0,4)(1,14)(1,16)h_{10}\rangle\ \langle(0,0)(0,6)(1,26)(1,30)h_{11}\rangle$
		$\langle(0,0)(0,9)(1,7)(1,31)h_{12}\rangle\} \times 2\ \langle(0,0)(0,6)(1,26)(1,30)h_{13}\rangle$
		$\langle(0,0)(0,4)(1,14)(1,16)h_{13}\rangle\ \langle(0,0)(0,9)(1,7)(1,31)h_{14}\rangle$
		$\langle(0,0)(0,1)(1,10)(1,20)h_{14}\rangle\ \langle(0,0)(0,2)(1,23)(1,32)h_{15}\rangle$
		$\langle(0,0)(0,9)(1,11)(1,19)h_{15}\rangle\ \langle(0,0)(0,10)(1,16)(1,24)h_{16}\rangle$
		$\langle(0,0)(0,10)(1,26)(1,27)h_{16}\rangle\ \langle(0,0)(0,12)(1,2)(1,25)h_{17}\rangle$
		$\langle(0,0)(0,12)(1,1)(1,7)h_{17}\rangle\ \langle(0,0)(0,15)(1,12)(1,13)h_{18}\rangle$
		$\langle(0,0)(0,15)(1,12)(1,23)h_{18}\rangle$

Theorem 4.3. *There exists a* $(5,5)$*-GDD of type* 6^n *for all odd* $n \geq 5$ *with the possible exceptions of* $n = 19, 23, 27$.

Proof: The proof of this theorem is similar to the previous one. For $n \equiv 1$ or 5 (mod 20) apply Theorem 2.1 and notice that a $(5,5)$-GDD of type 6^5 exists by Theorem 3.11 of [10].

For $n \equiv 11$ or 15 (mod 20) take a $(5,3)$ and $(5,2)$-GDD of type 6^n.

For $n \equiv 9, 13$ or 17 (mod 20) by the same argument of the previous theorem we only need to construct a $(5,5)$-GDD of type 6^n for $n = 9, 13, 17, 29, 33$.

For $n = 29, 33$ apply Theorem 2.6 with $k = r = 6$, $t = 4$, $m = 7$, and $(u, s) = (0,0), (4,8)$ respectively.

For $n = 9, 13, 17$ see table 4.4.

For $n \equiv 3, 7$ or 19 (mod 20) if $n \notin \{7, 19, 23, 27, 39, 43, 59, 83, 87, 99, 119, 123, 139, 143, 163, 167, 183, 243, 283, 347, 459, 563\} = \mathbf{A}$ then $n \in PDB$ $(\{5, 7, 9\}, 1)$ and hence the results follows from Theorem 2.1.

For $n = 7$ see table 4.4.

For $n \in \mathbf{A} \backslash \{19, 23, 27, 39, 99, 167\}$ apply Theorem 2.6 with $k = r = 6$, $t = 4$ and m, u, and s as described in table 4.3.

For $n = 99, 167$ apply Theorem 2.6 with $k = 5$, $r = 6$, $t = 12$, and $(m, u, s) = (9, 3, 12), (16, 3, 0)$ respectively.

For $n = 39$ take a $RMGD[6, 1, 6, 8]$ and inflate this design by a factor of 4. To each of the 7 parallel classes of block size 6 we adjoin 4 new points and construct a $(5,5)$-GDD of type 4^7. On the parallel class of block size 8 we construct a $RMGD[4, 1, 4, 8]$. There are 7 parallel classes of quadruples. To each parallel class we adjoin a new point and we take 5 copies of each parallel class. To all the blocks of size 8 we adjoin a new point and construct a $B[9, 5, 5]$. Finally, to the groups we adjoin 6 new points $\{h_1, \ldots, h_6\}$ and on each group we construct a $(5,5)$-GDD of type 6^5 such that $\{h_1, \ldots, h_6\}$ is a group. The total number of points we added is 42. On these 42 points we construct a $(5,5)$-GDD of type 6^7 such that $\{h_1, \ldots, h_6\}$ is a group. It is easy to check that the above construction yields a $(5,5)$-GDD of type 6^{39}.

n	m	u	s	n	m	u	s
43	9	8	4	163	37	19	8
59	13	5	16	183	43	1	56
83	19	8	4	243	57	19	8
87	19	14	4	283	67	19	8
119	27	13	8	347	85	9	0
123	29	8	4	459	111	21	0
139	32	13	8	563	139	8	4
143	32	19	8				

	Point Set	Base Blocks
7	$Z_2 \times Z_{21}$	Groups $\{(0,0)(0,7)(0,14)(1,0)(1,7)(1,14)\} + i$, $i \in Z_7$. Blocks: $\langle(i,0)(i,1)(i,3)(i,5)(i,11)\rangle$ $i = 0,1$, $\langle(0,0)(0,1)(0,4)(0,9)(0,13)\rangle$ $\{\langle(0,0)(0,1)(0,6)(1,11)(1,16)\rangle$ $\langle(0,0)(0,10)(1,2)(1,18)(1,19)\rangle$ $\langle(0,0)(0,3)(1,4)(1,6)(1,15)\rangle\} \times 2$, $\langle(0,0)(0,2)(1,5)(1,8)(1,20)\rangle$ $\langle(0,0)(0,2)(1,1)(1,13)(1,19)\rangle$ $\langle(0,0)(0,1)(1,2)(1,3)(1,6)\rangle$ $\langle(0,0)(0,3)(1,12)(1,18)(1,20)\rangle$ $\langle(0,0)(0,8)(1,6)(1,10)(1,16)\rangle$ $\langle(0,0)(0,9)(1,4)(1,12)(1,13)\rangle$ $\langle(0,0)(0,4)(0,10)(1,1)(1,9)\rangle$ $\langle(0,0)(0,4)(0,9)(1,17)(1,20)\rangle$ $\langle(0,0)(0,6)(0,8)(1,4)(1,17)\rangle$
9	Z_{54}	$\{\langle 0\ 1\ 3\ 24\ 32\rangle$ $\langle 0\ 4\ 17\ 28\ 38\rangle$ $\langle 0\ 5\ 12\ 34\ 40\rangle$ $\langle 0\ 3\ 11\ 15\ 17\rangle\} \times 2$ $\langle 0\ 1\ 3\ 16\ 47\rangle$ $\langle 0\ 1\ 5\ 15\ 22\rangle$ $\langle 0\ 1\ 13\ 20\ 39\rangle$ $\langle 0\ 5\ 10\ 16\ 29\rangle$
13	Z_{78}	$\{\langle 0\ 1\ 3\ 7\ 41\rangle$ $\langle 0\ 5\ 15\ 35\ 51\rangle$ $\langle 0\ 8\ 19\ 37\ 61\rangle$ $\langle 0\ 9\ 23\ 44\ 56\rangle$ $\langle 0\ 1\ 5\ 22\ 55\rangle$ $\langle 0\ 2\ 10\ 16\ 69\rangle$ $\langle 0\ 3\ 18\ 30\ 50\rangle\} \times 2$ $\langle 0\ 1\ 3\ 7\ 30\rangle$ $\langle 0\ 5\ 14\ 34\ 45\rangle$ $\langle 0\ 7\ 15\ 25\ 61\rangle$ $\langle 0\ 7\ 19\ 35\ 56\rangle$.
17	Z_{102}	$\{\langle 0\ 1\ 3\ 6\ 42\rangle$ $\langle 0\ 5\ 38\ 48\ 62\rangle$ $\langle 0\ 8\ 30\ 58\ 79\rangle$ $\langle 0\ 9\ 27\ 46\ 82\rangle$ $\langle 0\ 1\ 3\ 7\ 38\rangle$ $\langle 0\ 9\ 27\ 39\ 50\rangle$ $\langle 0\ 10\ 32\ 57\ 76\rangle$ $\langle 0\ 13\ 28\ 42\ 82\rangle\} \times 2$ $\langle 0\ 5\ 25\ 49\ 71\rangle$ $\langle 0\ 6\ 38\ 49\ 56\rangle$ $\langle 0\ 8\ 27\ 43\ 69\rangle$ $\langle 0\ 1\ 5\ 7\ 11\rangle$ $\langle 0\ 3\ 11\ 43\ 90\rangle$ $\langle 0\ 4\ 12\ 25\ 90\rangle$ $\langle 0\ 5\ 21\ 30\ 78\rangle$ $\langle 0\ 7\ 35\ 56\ 70\rangle$.

In the case $g \equiv 0 \pmod 4$, the following theorem is most useful to us.

Lemma 4.1. *Let $n \geq 5$ be an integer. Then $n \in PBD(\{5,6\},5)$.*

Proof: If $n \equiv 0$ or $1 \pmod 3$ then there exists a $B[n,6,5]$, [10] and hence $n \in PBD(\{5,6\},5)$.

For $n \equiv 2 \pmod 3$ take a $B[n+1,6,5]$ and delete one point from this design to obtain the result.

Theorem 4.4. *There exists a $(5,5)$-GDD of type g^n for all integers $n \geq 5$ and $g \equiv 0 \pmod 4$.*

Proof: By the previous lemma $n \in PBD(\{5,6\},5)$ for all $n \geq 5$, and by Theorem 1.1 a $(5,1)$-GDD of type g^5 and type g^6 exists for all $g \equiv 0 \pmod 4$. Apply Theorem 2.1 and the result follows.

Theorem 4.5. *Let α be a positive odd integer. Then there exists a $(5,5\alpha)$-GDD of type g^n for all g and $n \geq 5$ satisfying the necessary conditions with the possible exception of $(g,n) = (6,19)(6,23)(6,27)$ and $\lambda = 5$.*

Proof: We first prove the theorem for $\lambda = 5$. If g is odd then the result follows from Theorem 4.1. If $g \equiv 0 \pmod 4$ then the result follows from Theorem 4.4. If $g = 2$ or 6 the result is given in Theorem 4.3. For $g \equiv 2 \pmod 4$, $g \neq 2,6$ write $g = 2m$ then take a $(5,5)$-GDD of type 2^n and then apply Theorem 2.7 with $\mu = 1$ and m is odd. We now construct a $(5,15)$-GDD of type 6^n, $n = 19, 23, 27$. For this purpose take a $(5,5)$-GDD of type 2^n, $n = 19, 23, 27$ and then apply Theorem 2.3 with $\mu = 3$ and $m = 3$.

For $\lambda \geq 15$ write $\lambda = 10s + 15t$ and then the result is obtained by taking s copies of a $(5,10)$-GDD of type g^n together with t copies of a $(5,15)$-GDD of type g^n.

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