Minimum coverage by dominating sets in graphs

C.B. Smart and P.J. Slater
Mathematical Sciences Department
The University of Alabama in Huntsville
Huntsville, AL 35899

ABSTRACT. The $[0,\infty)$-valued dominating function minimization problem has the $[0,\infty)$-valued packing function as its linear programming dual. The standard $\{0,1\}$-valued minimum dominating set problem has the $\{0,1\}$-valued maximum packing set problem as its binary dual. The recently introduced complementary problem to a minimization problem is also a maximization problem, and the complementary problem to domination is the maximum enclaveless problem. This paper investigates the dual of the enclaveless problem, namely, the domination-coverage number of a graph. Specifically, let $\eta(G)$ denote the minimum total coverage of a dominating set. The number of edges covered by a vertex v equals its degree, $\deg v$, so $\eta(G) = \text{MIN}\{\sum_{s\in S} \deg s \colon S \text{ is a dominating set}\}$. Bounds on $\eta(G)$ and computational complexity results are presented.

1 Introduction

Many graph-theoretic subset problems such as those involving domination, packing, covering, and independence can be described in matrix form. Such graph theoretic minimization and maximization problems expressed as linear programming problems have dual maximization and minimization problems, respectively. Extending results in [23,26], Slater [27] formally defines "complementary" problems and presents a linear algebraic framework involving duality and complementarity showing the relationships among many graph-theoretic subset problems such as those involving domination, independence, enclavelessness, packing, and covering. As for duality, complementarity involves one minimization and one maximization problem. (See (4) below.) The complementation theorem [23,26,27] applied to specific pairs of complementary problems produces results including Gallai's covering/independence theorem [10] with $\alpha(G) + \beta(G) = |V(G)|$ and the

JCMCC 32 (2000), pp. 33-50

domination/enclaveless theorem [23] that $\gamma(G) + \Psi(G) = |V(G)|$. In this paper we begin the study of the parameter $\eta(G)$, the domination-coverage number of graph G, which was defined in [27] as the dual of the enclaveless number $\Psi(G)$. Specifically, $\eta(G)$ is the minimum total coverage of a dominating set.

For graph $G = (V, E)$ with vertex set V of order $|V(G)| = n$ and edge set E of size $|E(G)| = m$, we let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. The open neighborhood of $v_i \in V(G)$ is $N(v_i) = \{v_j : v_i v_j \in E(G)\}$, and the closed neighborhood is $N[v_i] = N(v_i) \cup \{v_i\}$. The closed neighborhood matrix $N = [n_{i,j}]$ is the n-by-n binary matrix with $n_{i,j} = 1$ if $v_i \in N[v_j]$ and $n_{i,j} = 0$ otherwise. Vertex v_i dominates each $v \in N[v_i]$, including v_i itself, and edges e_h and e_k dominate each other if they have a vertex in common. Vertex v_i and edge $e_h = v_i v_j$ are said to cover each other.

Vertex set $S \subseteq V(G)$ is a dominating set if each v_i is in S or is adjacent to a vertex in S. That is, we must have $\bigcup_{s \in S} N[s] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. See, for example, Hedetniemi and Laskar [17] for a collection of papers on graph domination. Letting $X_S = [s_1, s_2, \ldots, s_n]^t$ be the characteristic column vector for S with $s_i = 1$ if $v_i \in S$ and $s_i = 0$ if $v_i \notin S$, and letting $\vec{1}_n = [1, 1, \ldots, 1]^t$ be the all ones column n-tuple, S dominates if and only if $N \cdot X_S \geq \vec{1}_n$.

This paper is only one of many recent papers to study combinatorial problems which are posed as linear programming problems. For example, Chung, et al. [6] have studied the fractional covering problem for hypergraphs as well as its linear programming dual, and Tipnis and Trotter [30] have studied vertex-packing problems in graphs. Fisher [9] continues the investigation of fractional domination, which was previously studied by Farber [8]. Also, Karchmer, et al. [21] have recently applied linear programming to the problem of communication complexity, and Boyd and Hao [5] have made use of linear programming techniques in the design of communication networks.

$$\text{(1)} \qquad \begin{aligned} &\gamma_Y(G) = \text{MIN} \sum_{i=1}^{n} x_i \\ &\text{Subject to } N \cdot X_S \geq \vec{1}_n \\ &\text{With } x_i \in Y \end{aligned}$$

Y-domination for arbitrary subsets Y of the reals, $Y \subseteq \mathbb{R}$, is defined in Bange, et al [4] and by Goddard and Henning [13], and Y-valued parameters are defined in [26,27] for other graph parameters. For normal domination we take $Y = \{0, 1\}$ in (1), so $\gamma(G) = \gamma_{\{0,1\}}(G)$. If $Y = \mathbb{R}^+ = [0, \infty)$ then

(1) defines a linear programming problem with a dual packing problem.

$$(2) \qquad \begin{aligned} &\rho_Y(G) = \text{MAX} \sum_{i=1}^{n} x_i \\ &\text{Subject to } N \cdot X \leq \vec{1}_n \\ &\text{With } x_i \in Y \end{aligned}$$

We say that γ_Y and ρ_Y are Y-duals. In particular, the packing number $\rho(G) = \rho_{\{0,1\}}(G)$, and γ and ρ are binary duals.

For $S \subseteq V(G)$ vertex v is an enclave of S if $N[v] \subseteq S$, and v is an isolate of S if $v \notin S$ and $N(v) \subseteq S$. Alan Goldman suggested the study of enclaveless sets (see [23]) and isolate-free sets (see Maurer [22]). Specifically, the enclaveless number of G, denoted $\Psi(G)$, is the maximum cardinality of a vertex set $S \subseteq V(G)$ such that S has no enclaves. Note that S is enclaveless if $N[v_i]$ is not a subset of S for $1 \leq i \leq n$. Equivalently, we must have $|N[v_i] \cap S| \leq \deg v_i$. Let $D = [\deg v_1, \deg v_2, \ldots, \deg v_n]^t$.

$$(3) \qquad \begin{aligned} &\Psi_Y(G) = \text{MAX} \sum_{i=1}^{n} x_i \\ &\text{Subject to } N \cdot X \leq D \\ &\text{With } x_i \in Y \end{aligned}$$

As noted in [23], domination and enclavelessness are complementary properties, that is, S is a dominating set if and only if $V(G) - S$ is enclaveless. It follows that $\gamma(G) + \Psi(G) = n$. More generally, $Y \subseteq \mathbb{R}$ is called complementable if $x \in Y$ implies $1 - x \in Y$ (in particular, $\{0, 1\}$ being complementable), and we have the following theorem.

Theorem 1. [27] *If Y is complementable and $\Psi_Y(G) < \infty$, then $\gamma_Y(G) + \Psi_Y(G) = n$.*

As in [27], we define the general Y-complementarity problem for a complementable set $Y \subseteq \mathbb{R}$. Let M denote an arbitrary k-by-h real matrix; let $C = [c_1, c_2, \ldots, c_h]^t$ be the vector of coefficients for an objective function; and let $B = [b_1, b_2, \ldots, b_k]^t$ be a vector of constraint values. Denote the jth row sum of M by $r_j = \sum_{i=1}^{h} m_{j,i}$, and let the row sum vector of M be $L_M = [r_1, r_2, \ldots, r_k]^t$. Note that a binary h-vector $X = [x_1, x_2, \ldots, x_h]^t$ can be considered to be the characteristic function of a set S of columns of M, those columns with $x_j = 1$. Clearly, $\vec{1}_h - X = [1 - x_1, 1 - x_2, \ldots, 1 - x_h]^t$ is the characteristic function of the complement of S. In general, for a complementable set Y we have $X \in Y^h$ if and only if $\vec{1}_h - X \in Y^h$.

PRIMAL

$$Z = \text{MIN} \sum_{i=1}^{h} c_i x_i$$

(4) Subject to $M \cdot X \geq B$

With $x_i \in Y$

DUAL

$$Z^* = \text{MAX} \sum_{j=1}^{k} b_j x_j$$

Subject to $M^t \cdot X \leq C$

With $x_j \in Y$

COMPLEMENT

$$Z^{\#} = \text{MAX} \sum_{i=1}^{h} c_i x_i$$

Subject to $M \cdot X \leq L_M - B$

With $x_i \in Y$

Theorem 2. (Matrix Complementation [27]) *For any k-by-h matrix M, h-tuple C, k-tuple B, and complementable set $Y \subseteq \mathbb{R}$, let $L = [r_1, r_2, \ldots, r_k]^t$ be the row sum vector of M. Then if the primal problem in (4) has a feasible solution, then either $Z = -\infty$ and $Z^{\#} = \infty$, or else $Z + Z^{\#} = \sum_{i=1}^{h} c_i$.*

Proof: If $Z = K > -\infty$, then given $\epsilon > 0$ there exists an n-tuple X in Y^h with $M \cdot X \geq B$ and $\sum_{i=1}^{h} c_i x_i \leq K + \epsilon$.

Now $M \cdot X \geq B$
if and only if for $1 \leq j \leq k$ we have $m_{j,1}x_1 + m_{j,2}x_2 + \cdots + m_{j,h}x_h \geq b_j$
if and only if for $1 \leq j \leq k$ we have $r_j - (m_{j,1}x_1 + m_{j,2}x_2 + \cdots + m_{j,h}x_h) \leq r_j - b_j$
if and only if for $1 \leq j \leq k$ we have $m_{j,1}(1 - x_1) + m_{j,2}(1 - x_2) + \cdots + m_{j,h}(1 - x_h) \leq r_j - b_j$
if and only if $M(\vec{1}_h - X) \leq L - B$. By assumption, $\vec{1}_h - X \in Y^h$. Thus $Z^{\#} \geq \sum_{i=1}^{h} c_i(1 - x_i) = \sum_{i=1}^{h} c_i - \sum_{i=1}^{h} c_i x_i \geq \sum_{i=1}^{h} c_i - (K + \epsilon)$. Hence $Z + Z^{\#} \geq \sum_{i=1}^{h} c_i - \epsilon$. And if $Z^{\#} = J < \infty$, then given $\epsilon > 0$ there exists an h-tuple X with $M \cdot X \leq L - B$ and $\sum_{i=1}^{h} c_i x_i \geq J - \epsilon$. It follows that $M(\vec{1}_h - X) \geq B$ and $Z \leq \sum_{i=1}^{h} c_i(1 - x_i) = \sum_{i=1}^{h} c_i - \sum_{i=1}^{h} c_i x_i \leq \sum_{i=1}^{h} c_i - J + \epsilon$. Hence $Z + Z^{\#} \leq \sum_{i=1}^{h} c_i + \epsilon$. Consequently, $Z + Z^{\#} = \sum_{i=1}^{h} c_i$.

If $Z = -\infty$ and $-\infty < K$, then there exists an n-tuple X in Y^h with $M \cdot X \geq B$ and $\sum_{i=1}^{h} c_i x_i < K$. As above, $M(\vec{1}_h - X) \leq L - B$ with $\vec{1}_h - X \in Y^h$ by complementarity. Thus $Z^{\#} \geq \sum_{i=1}^{h} c_i - \sum_{i=1}^{h} c_i x_i \geq \sum_{i=1}^{h} c_i - K$. So $Z^{\#} = \infty$. Similarly, $Z^{\#} = \infty$ implies $Z = -\infty$, completing the proof.

Two examples will illustrate the many corollaries of the theorem: a weighted Gallai Theorem (covering/independence) and a weighted domination/enclaveless theorem. Assume each vertex $v_i \in V(G)$ has a weight c_i. For $S \subseteq V(G)$ define the weight of S to be $w(S) = \sum_{v_i \in S} c_i$. Let $w\alpha_Y$ and $w\gamma_Y$ denote the minimum weights of Y-covering and Y-dominating sets, respectively, using just $w\alpha$ and $w\gamma$ when $Y = \{0,1\}$. Let $w\beta_Y$ and $w\Psi_Y$ denote the maximum weights of Y-independent and Y-enclaveless sets, respectively, using $w\beta$ and $w\Psi$ for $Y = \{0,1\}$.

Corollary 1. *If Y is complementable, then*

a. $w\alpha_Y(G) + w\beta_Y(G) = w(V(G))$.

b. $w\gamma_Y(G) + w\Psi_Y(G) = w(V(G))$.

When $Y = \{0,1\}$ we get the following.

Corollary 2.

a. $w\alpha(G) + w\beta(G) = w(V(G))$, *and*

b. $w\gamma(G) + w\Psi(G) = w(V(G))$.

With $Y = \{0,1\}$ and $C = \vec{1}_n$, we get the following.

Corollary 3.

a. $\alpha(G) + \beta(G) = n$, *and*

b. $\gamma(G) + \Psi(G) = n$.

Corollary 3a. is Gallai's Theorem [12]; using different notation 3b. appeared in [25].

The Y-domination-coverage number defined in (5) is the Y-dual of the enclaveless problem (3).

$$\text{(5)} \qquad \begin{aligned} &\eta_Y(G) = \text{MIN} \sum_{i=1}^{n} (\deg v_i) x_i \\ &\text{Subject to } N \cdot X \geq \vec{1}_n \\ &\text{With } x_i \in Y \end{aligned}$$

Of particular interest, of course, is $\eta(G) = \eta_{\{0,1\}}(G)$ for which $\eta(G) = \text{MIN}\{\sum_{s\in S} \deg s \mid S \text{ is a dominating set}\}$. More generally, the complementable set $Y_k = \{1-k, 2-k, \ldots, k-2, k-1, k\}$ will be of interest. Note that $\deg v_i$ is the number of edges covered by v_i, so $\sum_{s\in S} \deg s$ is the total amount of coverage done by S. For the constraint $N \cdot X \geq \vec{1}_n$ in (5) the solution vector X must produce a Y-dominating function on $V(G)$. Thus the domination-coverage number $\eta(G)$ is, in fact, equal to the minimum coverage done by a dominating set. Using LP-duality we get the next result, and Theorem 4 is obvious.

Theorem 3. ([27]) $\Psi(G) \leq \Psi_{\mathbb{R}} + (G) = \eta_{\mathbb{R}} + (G) \leq \eta(G)$.

Theorem 4. ([27]) *If $Y1 \subseteq Y2$, then $\eta_{Y2}(G) \leq \eta_{Y1}(G)$.*

2 Examples and Bounds

Let G be the graph obtained from a complete graph K_t on a set $\{v_1, v_2, \ldots, v_t\}$ by adding two endpoints v_i^1 and v_i^2 adjacent to each v_i. For G we have $n = 3t$ and $m = t(t-1)/2 + 2t$. Any dominating set S contains v_i or both v_i^1 and v_i^2, and it follows that $\eta(G) = 2t$. More generally, given graphs F on t vertices and H on r vertices let $G = F \circ H$ be the graph on $t(1+r)$ vertices obtained from one copy of F and t copies of H by adding edges connecting each $v \in V(F)$ to every vertex w in a distinct copy of H. Graph G above is $K_t \circ \overline{K}_2$. In general, $\eta(F \circ \overline{K}_r) = tr$. Note that for $F \circ \overline{K}_r = G$ the unique $\eta(G)$-set consists of the tr endpoints while $\gamma(G) = t$.

Also, for example, consider the complete multipartite graph $G = K_{n1,n2,\ldots nt}$ with $n = n_1 + n_2 + \ldots n_t$, and $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$. Note that there are t sets of maximal mutually nonadjacent vertices. We can classify the minimal dominating sets of G into two groups. One is to use all the vertices in one of the maximal sets of mutually nonadjacent vertices. The number of such vertices is n_j for some $1 \leq j \leq t$. The amount of coverage done by such a dominating set is $n_j \sum_{i\neq j} n_i = n_j(n - n_j)$. The other type of dominating set consists of two vertices, one from each of two distinct maximal sets of mutually nonadjacent vertices of sizes, say, n_j and n_k. The amount of coverage done by these two vertices is $n_j + n_k + 2\sum_{i\neq j,k} n_i$. If $n_t = 1$, then the first of these two sums is smaller, and $\eta(G) = n - 1$. Otherwise, suppose $j < k$, so that $n_j + n_k + 2\sum_{i\neq j,k} n_i \leq 2nj + 2\sum_{i\neq j,k} n_i = 2\sum_{i\neq k} n_i \leq n_k \sum_{i\neq k} n_i$. Therefore, since the smallest of the sums $n_j + n_k + 2\sum_{i\neq j,k} n_i = 2n - n_j - n_k$ is the one for which $j = 1$, $k = 2$, we have $\eta(G) = n1 + n2 + 2\sum_{i=3}^{n} n_i = 2n - n_1 - n_2$.

Theorem 5. *If the minimum degree $\delta(G) \geq 1$, then $n/2 \leq \eta(G) \leq m$.*

Proof: To see that $n/2 \leq \eta(G)$, let S be any minimal dominating set of G. If $|S| \geq n/2$ then clearly $\sum_{s\in S} \deg s \geq n/2$. If $|S| < n/2$ then

$|V - S| \geq n/2$, and thus S dominates at least $n/2$ vertices outside of S which implies S covers at least $n/2$ edges. To see that $\eta(G) \leq m$, let S be any maximal independent set of G. Since S is independent, each edge of G is covered at most once, and since S is dominating we have that $\eta(G) \leq m$.

Theorem 6. *$\eta(G) = m$ if and only if each component of G is a C_4 or a $K_{1,n-1}$.*

Proof: First note that it it easy to see that $\eta(C_4) = 4$ and $\eta(K_{1,n-1}) = n - 1$. Conversely, suppose that $\eta(G) = m$. Assume that some component H of G contains two vertices u, v such that $d(u, v) = 3$. Let u, w, x, v be a path of length three between u and v. Let S be any maximal independent set containing u and v. Then S is a dominating set of G and $w, x \notin S$. Therefore, S is a dominating set of G that covers at most $m - 1$ edges of G. Therefore, $\eta(G) = m$ implies that the diameter of each component is at most two. Suppose that $\eta(G) = m$ and that the diameter of each component is at most two. Letting H be any component of G with $|V(H)| = k$, note that if H contains a vertex of degree $k - 1$, then $\eta(G) = m$ only if $H = K_{1,n-1}$. Otherwise, let $u, v \in V(H)$ such that $uv \notin E(H)$. Then since $\text{diam}(H) = 2$, there exists $w \in V(H)$ such that $uw, vw \in E(H)$. Since $\deg w < k - 1$, there exists $x \in V(H)$ that is not adjacent to w. However, since $\text{diam}(H) = 2$, x is adjacent to some vertex that is adjacent to w. Without loss of generality, suppose that $ux \in E(H)$. If $vx \notin E(H)$ then let S be a maximal independent set of H that contains x and v. S is a dominating set of H that does not cover edge uw and which covers all other edges at most once. Thus $vx \in E(H)$. If these are the only vertices of H, then $H = C_4$. Otherwise, there is some vertex $y \in V(H)$ distinct from the other four vertices. Suppose that $uy \in E(H)$. Then let $S1 = \{u, y\}$. $S1$ dominates (at least) vertices u, v, w, x, and y. Edges ux and uw are not covered by $S1$ and at most one edge incident with these vertices is covered twice. (If $vy \in E(H)$ it is covered twice by $S1$.) Let $S2$ be a maximal independent set of $H - N(S1)$. Then $S1 \cup S2$ is a dominating set of H that covers at most $m - 1$ edges. Therefore, $\eta(G) = m$ if and only if each component of G is a C_4 or a $K_{1,n-1}$.

Theorem 7. *If the minimum degree $\delta(G) \geq 1$, $\eta(G) = n/2$ if and only if $G \cong H \circ K_1$, where H is any graph.*

Proof: If $G \cong H \circ K_1$, it is easy to see that that $\eta(G) = n/2$. Conversely, suppose that $\eta(G) = n/2$. Let S be an $\eta(G)$ set. Since each vertex of S covers at least one edge, $|S| \leq n/2$. Also, $\eta(G) = D \cdot X_s = (D + \vec{1}_n - \vec{1}_n) \cdot X_s = (D + \vec{1}_n) \cdot X_s - |S|$. Because S dominates $(D + \vec{1}_n) \cdot X_s \geq n$ vertices, if $|S| < n/2$, then $\eta(G) > n/2$. Thus, $|S| = n/2$. This implies that each element of S must dominate itself and exactly one other vertex, so we have that $v \in S$ implies $\deg v = 1$. This gives us the result that $G \cong H \circ K_1$ for some graph H.

By noting that any $\eta(G)$ set is a dominating set of G and each vertex in an $\eta(G)$ set must cover at least as many edges as the minimum degree of G and no more than the maximum degree of G, $\Delta(G)$, we have the following result.

Theorem 8. $\gamma(G) \cdot \delta(G) \leq \eta(G) \leq \gamma(G) \cdot \Delta(G)$.

Corollary. $\delta(G) = \Delta(G) = r$ *implies* $\eta(G) = \gamma(G) \cdot r$.

For example, both K_n and C_n are regular, which implies $\eta(K_n) = n - 1$and $\eta(C_n) = 2\lceil n/3 \rceil$.

For domination each vertex is dominated at least once; for efficient domination each vertex is dominated at most once and we seek to dominate as many vertices as possible under this constraint. For "redundance" we seek to minimize the total amount of domination done, given that every vertex gets dominated at least once. Studies of efficiency and redundance include [1, 2, 3, 14, 15, 19, 20]. Because v_i dominates $|N[v_i]| = 1 + \deg v_i$ vertices, we have $F(G) = F_{\{0,1\}}(G)$ defined as follows.

Efficient Y-Domination

$$\begin{aligned} & F_Y(G) = \text{MAX} \sum_{i=1}^{n} (1 + \deg v_i) x_i \\ (6) \qquad & \text{Subject to } N \cdot X \leq \vec{1}_n \\ & \text{With } x_i \in Y \end{aligned}$$

Y-Redundance

$$\begin{aligned} & R_Y(G) = \text{MIN} \sum_{i=1}^{n} (1 + \deg v_i) x_i \\ (7) \qquad & \text{Subject to } N \cdot X \leq \vec{1}_n \\ & \text{With } x_i \in Y \end{aligned}$$

The closed neighborhood order domination (CLOD) and closed neighborhood order packing (CLOP) parameters arose as the positive integer duals of F and R, respectively. See [24, 25, 18]. Their formulations follow with D^* denoting the column vector $[1 + \deg v_1, 1 + \deg v_2, \ldots, 1 + \deg v_n]^t$.

CLOD

$$\begin{aligned} & W_Y(G) = \text{MIN} \sum_{i=1}^{n} x_i \\ (8) \qquad & \text{Subject to } N \cdot X \geq D^* \\ & \text{With } x_i \in Y \end{aligned}$$

$$P_Y(G) = \text{MAX} \sum_{i=1}^{n} x_i \tag{7}$$
$$\text{Subject to } N \cdot X \leq D^*$$
$$\text{With } x_i \in Y$$

We next compare $\eta(G)$ to the closed neighborhood parameters $F(G)$, $W(G)$, $P(G)$, and $R(G)$.

Theorem 9. *$n \leq \gamma(G) + \eta(G) \leq R(G)$, and $F(G) = n$ implies that $n = \gamma(G) + \eta(G) = R(G)$.*

Proof: From Theorem 1, $\gamma(G) + \Psi(G) = n$, and from Theorem 3, $\Psi(G) \leq \eta(G)$, so together we have that $\gamma(G) + \eta(G) \geq n$. Also, if S is an $R(G)$ set, then S dominates G and S covers $R(G) - |S|$ edges. Since $|S| \geq \gamma(G)$, we have that $\eta(G) \leq R(G) - \gamma(G)$. Noting that $F(G) = n$ if and only if $R(G) = n$ (see [14]) completes the proof.

For example, since $F(P_n) = n$, and since $\gamma(P_n) = \lceil n/3 \rceil$, we have that $\eta(P_n) = n - \lceil n/3 \rceil$. However, $\gamma(G) + \eta(G) = n$ does not imply $F(G) = n$. See Figure 1.

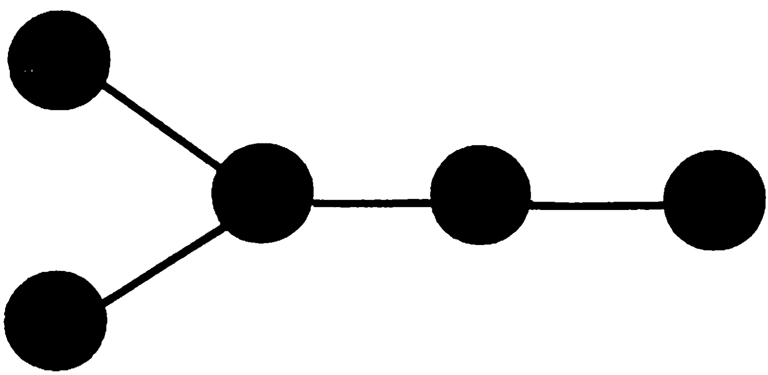

Figure 1.
Tree T for which $F(T) = 4 < \gamma(T) + \eta(T) = 5 < 6 = R(T)$.

Note that $\eta(K \circ K_1) = n/2 < n = F(K \circ K_1) = W(K \circ K_1) = P(K \circ K_1)$.

On the other hand, consider the graph H in Figure 2. It is easy to see that $F(H) = 3p$, $R(H) = 6p$, and since H is regular we have that $W(H) = P(H) = n = 4p$, and since the degree of each vertex is $3p - 1$, $\eta(H) = 6p - 2$. Thus, $F(H) < W(H) = n = P(H) < \eta(H) < R(H)$. From the graphs in Figures 1 and 2 we have that $\eta(G)$ is pairwise incomparable to $F(G)$, $P(G)$, and $W(G)$. Also the class of graphs in Figure 2 is one in which $\eta(G) > n$.

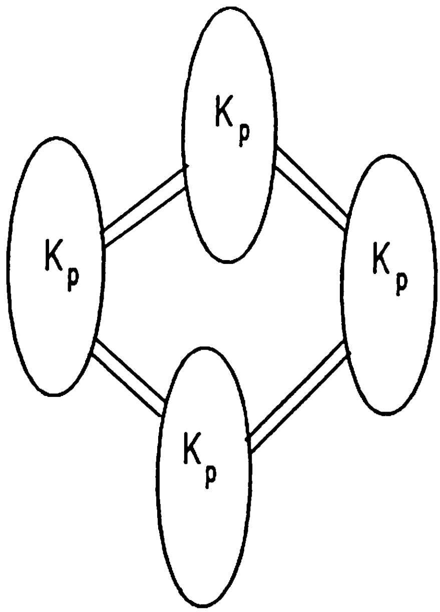

Figure 2.
$F(G) = 3p < W(G) = 4p = n = P(G) < \eta(G) = 6p - 2 < R(G) = 6p$.

Also, we have the following Nordhaus-Gaddum-type bounds for η.

Theorem 10. $n - 1 \leq \eta(G) + \eta(\overline{G}) \leq \binom{n}{2}$.

Proof: To prove the lower bound, first suppose that $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$. Then by Theorem 5, $\eta(G) \geq n/2$ and $\eta(\overline{G}) \geq n/2$. Thus, we need only consider the case in which one of G or $\overline{G}$ has isolates. Suppose that G has $k \geq 1$ isolated vertices with the set of isolated vertices being $X = \{x_1, x_2, \ldots, x_k\}$. Then by Theorem 5, $\eta(G) \geq (n-k)/2$. Suppose also that there are $j \geq 0$ vertices of degree $n-1-k$ in G with the set of such vertices being called S. Then let $H = \overline{G} - X - S$. Note that $|V(H)| = n - j - k$. Now if any vertex in X in used as a dominating set of $\overline{G}$ then the amount of coverage done is $n-1$. Otherwise, the j vertices of S and some of the vertices of H are used to dominate $\overline{G}$. Noting that the subgraph induced by the vertices of H has minimum degree of at least 1, we have $\eta(G)+\eta(\overline{G}) \geq (n-k)/2+jk+(n-j-k)/2+k\gamma(H) = n+jk-j/2+k(\gamma(H)-1)$. If $|V(H)| \neq \phi$, then $\eta(G) + \eta(\overline{G}) \geq n$. Otherwise, $G \cong K_j + \overline{K}_k$. If $j = 0$ we then have $\eta(G)+\eta(\overline{G}) = n-1$, and if $j = 1$ we have $\eta(G)+\eta(\overline{G}) = n-1+k-1$. Finally $j \geq 2$ and $k \geq 1$, so we have that $\eta(G)+\eta(\overline{G}) = \min\{j-1+n-1, j-1+jk\} = \min\{n+j-2, j(k+1)-1\} = \min\{k+2j-2, j(k+1)-1\} = k+2j-2 \geq n$, which completes the proof that $n-1 \leq \eta(G)+\eta(\overline{G})$. This proof also shows that this lower bound is attained if and only if G or $\overline{G}$ is a complete graph. Also note that $G = K_2 \cup \overline{K}_k$ is a family of graphs for which $\eta(G)+\eta(\overline{G}) = n$.

The upper bound is easy to see since $\eta(G) + \eta(\overline{G}) \leq |E(G)| + |E(\overline{G})| = \binom{n}{2}$.

Noting that this bound is sharp only for those graphs which have the property that $\eta(G) = |E(G)|$ and $\eta(\overline{G}) = |E(\overline{G})|$, we see that the upper bound is sharp if G is a C_4, P_2, or a P_3. Otherwise from Theorem 5 the only other graphs to consider are $K_{1,n-1}$ for $n \geq 3$. But for $K_{1,n-1}$, the complement is an isolate plus a complete graph on $n-1$ vertices. Thus, the upper bound is sharp if and only if G or $\overline{G}$ is a C_4, P_2, or a P_3.

If G or $\overline{G}$ is a complete graph, then $\eta(G) \cdot \eta(\overline{G}) = 0$. However, if neither G nor $\overline{G}$ has any isolates then we can improve upon this as follows.

Theorem 11. *If $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, then $n^2/4 \leq \eta(G) \cdot \eta(\overline{G}) \leq \binom{n}{2}^2/4$.*

Proof: The lower bound follows from Theorem 5. To get the upper bound, note that $\eta(G) \cdot \eta(\overline{G}) \leq |E(G)|(\binom{n}{2} - |E(G)|)$. Letting $f(x) = x(a-x)$, it follows from calculus that $\max_{x \in \mathbb{R}} f(x) = a^2/4$, which gives us the bound.

Note that $\eta_Y(G) \geq 0$ where $Y \subseteq \mathbb{R}^+$. However, if we extend our scope to let Y include negative numbers, then there exist graphs for which $\eta_Y(G) < 0$. For example, with Y_k defined as in Section 1, and tree T as in Figure 3, $\eta(T) = 4$ and as indicated $\eta_{Yk}(T) = 4 - 2k$. Also, $\eta_Z(T) = \eta_Q(T) = \eta_{\mathbb{R}}(T) = -\infty$. By noticing that if $\eta_Y(G) < 0$, we can simply double the weights of any domination-coverage function and it is still a dominating function, then we have the following.

Theorem 12. *If $\eta_Y(G) < 0$ then $\eta_Y(G) = -\infty$ where $Y = \{\mathbb{R}, Q, Z\}$.*

Theorem 13. *$\eta_{Y1}(G) = -\infty$ if and only if $\eta_{Y2}(G) = -\infty$, where $Y_1, Y_2 \in \{\mathbb{R}, Q, Z\}$.*

In order to calculate $\eta_{\{0,1\}}(G)$ for some graph G we might attempt to calculate the associated parameter $\eta_{[0,1]}(G)$. However, this does not always work, as illustrated by the graph H in Figure 4. Since H is regular of degree $2p$ and $\gamma(H) = 2$, then from the corollary to Theorem 7 we have $\eta_{\{0,1\}}(H) = 4p > n = 3p$. However, note that by placing a weight of $1/(2p+1)$ on each vertex we have a $[0,1]$-dominating function of H whose coverage is $3p(2p/(2p+1))$. Also, by placing a weight of $2p/(2p+1)$ on each vertex, we have a $[0,1]$-enclaveless function whose weight sum is $3p(2p/(2p+1))$. Therefore, from Theorem 2 we have $\eta_{[0,1]}(G) = 3p(2p/(2p+1)) < 3p = n < 4p = \eta_{\{0,1\}}(G)$.

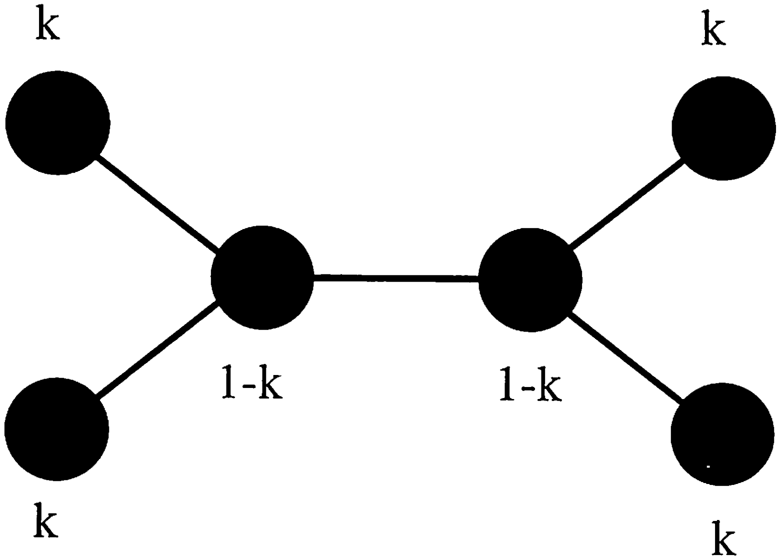

Figure 3.
Tree T with $\eta_{Yk}(T) = 6 - 2k$ and $\eta_Z(T) = \eta_Q(T) = \eta_R(T) = -\infty$.

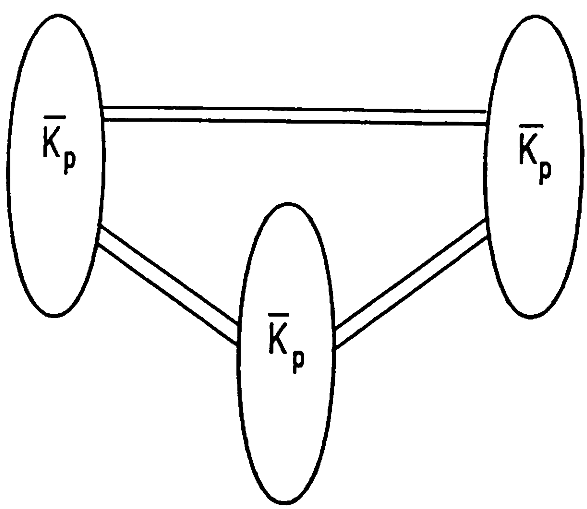

Figure 4.
Graph H for which $\eta_{[0,1]}(H) < \eta_{\{0,1\}}(H)$.

The preceding example leads us to wonder about the complexity of the question "Is $\eta(G) \leq J$?", which we consider in the next section.

3 Complexity Results

In this section we prove that determining if $\eta(G) \leq J$ is NP-complete and that determining if $\eta(G) = \Psi(G)$ is NP-complete even for the class of chordal graphs.

From Garey and Johnson[11], we know that domination is NP-complete, even for the class of chordal graphs.

PROBLEM: DOMINATING SET.
INSTANCE: A chordal graph $H = (V, E)$ and a positive integer $j \leq |V|$.
QUESTION: Is $\gamma(H) \leq j$ (that is, is there a vertex set $S \subseteq V$ such that S is a dominating set with $|S| \leq j$)?

We will show a polynomial time reduction of the above problem to domination-coverage to show that it is also NP-complete.

PROBLEM: DOMINATION-COVERAGE.
INSTANCE: A chordal graph $G = (V, E)$ and a positive integer $k \leq |E|$.
QUESTION: Is $\eta(G) \leq k$ (that is, is there a dominating set of G whose coverage is less than or equal to k)?

Theorem 14. *Problem Domination-Coverage is NP-complete, even for the class of chordal graphs.*

Proof: Given a graph G of order n and size m, form graph G' as in Figure 5. This construction can clearly be done in polynomial time. Note that if in G', the vertices of G are dominated only by vertices in G, then the minimum amount of coverage done by a dominating set is $m\gamma(G) + \sum_{i=1}^{n}(m - \deg v_i) + n(mn) = m\gamma(G) + mn - 2m + mn^2$ (achieved by using every t_i and y_k, together with a minimum dominating set of G). Otherwise if $k \geq 1$ of the vertices in the $K_{m-\deg v_i}$'s are used (we can assume that they are from distinct cliques and that they come from those indexed from 1 to k) then the amount of coverage done is at least $\sum_{i=1}^{k}(m - \deg v_i) + k + kmn + \sum_{i=k+1}^{n}(m - \deg v_i) + mn^2 = k(1+mn) + mn - 2m + mn^2 > m\gamma(G) + mn - 2m + mn^2$. Therefore, $\gamma(G) \leq j$ if and only if $\eta(G') \leq mj + mn - 2m + mn^2$, and since G is chordal implies that G' is chordal, the proof is complete.

In recent years there has been an increasing interest in the complexity of questions involving the comparison of two graph parameters. One such well-known question is "Does $i(G) = \beta(G)$?", that is, is every maximal independent set of maximum size. Graphs which satisfy this property are said to be well-covered. Chvatal and Slater [7] proved that the question "Is $i(G) < \beta(G)$?" is NP-complete. It is also quite natural to ask about the complexity of questions concerning the equality of parameters which are the Y-duals of each other.

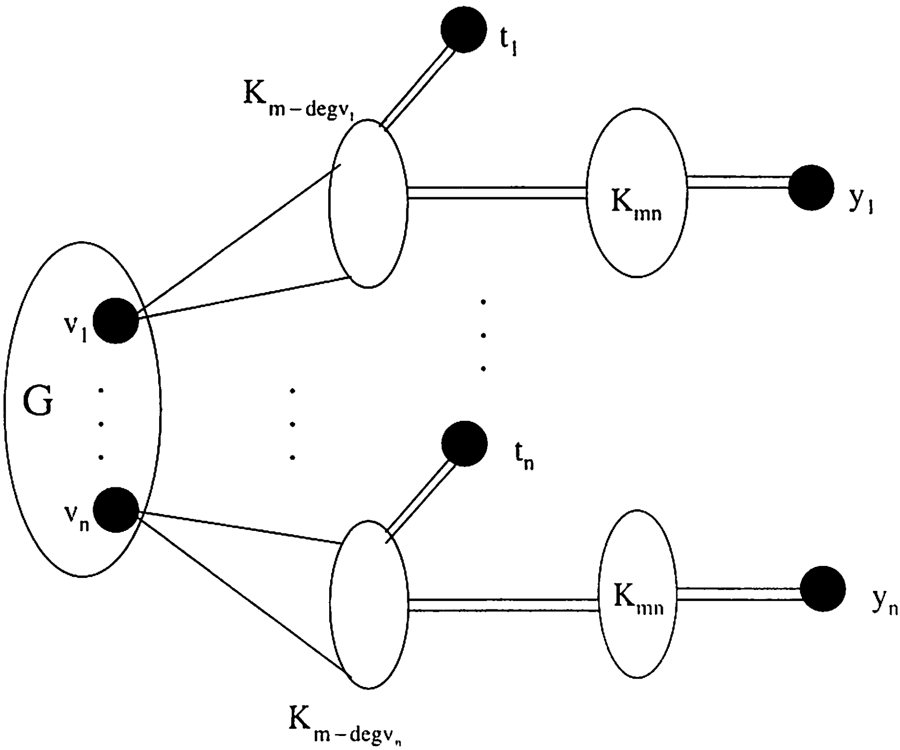

Figure 5. Reduction from Domination

For example, we have recently shown (see [28,29]) that the questions "Is $F(G) = W(G)$?" and "Is $P(G) = R(G)$?" are NP-complete, even for the class of bipartite graphs. This leads us to consider the complexity of the question "Is $\eta(G) = \Psi(G)$?". From Garey and Johnson [11] we know that the following problem is NP-complete.

PROBLEM: ONE-IN-THREE 3-SATISFIABILITY WITH NO NEGATED LITERALS
INSTANCE: Set U of boolean variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ and no $c \in C$ contains a negated literal.
QUESTION: Is there a truth assignment for U such that each clause in C has exactly one true literal?

PROBLEM: $\eta - \Psi$ EQUALITY.
INSTANCE: A graph $G = (V, E)$
QUESTION: Does $\eta(G) = \Psi(G)$?

Theorem 15. *Determining if $\eta(G) = \Psi(G)$ is NP-complete, even for the class of chordal graphs.*

Proof: The reduction is from Exact 1-in-3 3SAT with no negated literals. For $U = \{u_1, u_2, \ldots, u_N\}$, let $C = (u_{11} \vee u_{12} \vee u_{13}) \wedge (u_{21} \vee u_{22} \vee u_{23}) \wedge (u_{31} \vee u_{32} \vee u_{33}) \cdots \wedge (u_{M1} \vee u_{M2} \vee u_{M3})$. Notice that $|U| = N$ and $|C| = M$. Given an instance of Exact 1-in-3 3SAT with no negated literals construct the graph G of order $n = M + 2N$ in Figure 6, where each c_i is made adjacent to the vertices corresponding to u_{i1}, u_{i2}, and u_{i3}. Note that $\gamma(G) = N$. We show that the instance of Exact 1-in-3-3SAT has a satisfying truth assignment if and only if $\eta(G) = \Psi(G)$. To see this, note that since $\Psi(G) + \gamma(G) = n$, $\eta(G) = \Psi(G)$ if and only if $\eta(G) + \gamma(G) = n$. Suppose that the associated instance of Exact 1-in-3-3SAT with no negated literals has a satisfying truth assignment. Define X to be the set of vertices to which vertex u_i belongs if and only if literal u_i is assigned a value of true, to which vertex v_i belongs if and only if literal u_i is assigned a value of false and every $c_j \notin X$. It can easily be seen that since the associated instance of Exact 1-in-3-3SAT with no negated literals has a satisfying truth assignment, then X is a dominating set and no two vertices of G dominate the same vertex. Thus, X is an efficient dominating set of G. Therefore, by Theorem 9, we have that $\eta(G) + \gamma(G) = n$. Conversely, assume the associated instance of Exact 1-in-3-3SAT with no negated literals does not have a satisfying truth assignment. For an $\eta(G)$-set S if we use only some of the vertices u_i and v_j, (that is, no $c_k \in S$) then more than $N + M$ edges are covered, which implies that $\eta(G) + \gamma(G) > n$. If any vertex c_j is in S, $1 \leq j \leq n$, then at least $M + N + 2$ edges are covered, and thus $\eta(G) + \gamma(G) > n$. Noting that G is chordal completes the proof.

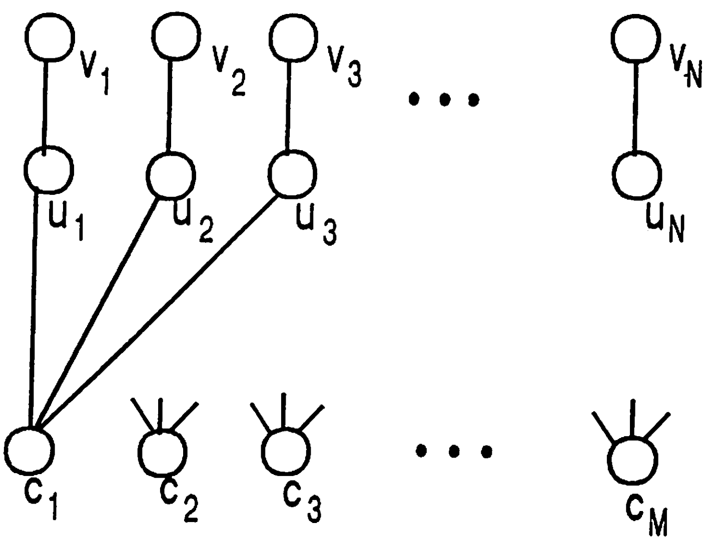

Figure 6.
Reduction from Exact 1-in-3 3SAT with no negated literals

By noticing, as already pointed out, that $\eta(G) = \Psi(G)$ if and only if $\eta(G) + \gamma(G) = n$, we have the following corollary.

Corollary. *Deciding if $\eta(G)+\gamma(G) = n$ is NP-complete, even for the class of chordal graphs.*

4 Conclusion

The parameter studied in this paper is only one of many arising out of the duality-complementarity cycles of the matrix formulations of graph parameters presented in [27]. For example another of these new parameters is the maximum number of edges that can be covered by an open packing, where an open packing is a subset S of the vertex set that satisfies the condition that given any vertex v, at most one vertex from S appears in v's open neighborhood. This is studied in [12].

Also, while the problem of determining $\eta(G)$ is NP-complete for general graphs, it can be solved in linear time for series-parallel graphs. However, this parameter does not fit the template of Grinstead and Slater [15]. It does, however, fit a different template that also includes the efficient open domination parameter mentioned in the previous paragraph, as well as several other new parameters arising from the framework in [27]. This template will be presented in later work.

References

[1] D.W. Bange, A.E. Barkauskas, and P.J. Slater, Disjoint dominating sets in trees, Sandia Laboratories Report, SAND78-1087J, 1978.

[2] D.W. Bange, A.E. Barkauskas, and P.J. Slater, Efficient dominating sets in graphs, *Applications of Discrete Math.* (Proc. Third SIAM Conf. on Discrete Math., 1986), SIAM, Philadelphia (1988), 189–199.

[3] D.W. Bange, A.E. Barkauskas, L.H. Host, and P.J. Slater, Efficient near-domination of grid graphs, *Congressus Numerantium* **58** (1987), 83–92.

[4] D.W. Bange, A.E. Barkauskas, L.H. Host, and P.J. Slater, Generalized Domination and Efficient Domination in Graphs, *Discrete Math.* **159** (1996), 1–11.

[5] S.C. Boyd and T. Hao, An integer polytope related to the design of survivable communication networks, *SIAM J. Discrete Math.* **6** No. 4 (1993), 612–630.

[6] F.R.K. Chung, Z. Furedi, M.R. Garey, and R.L. Graham, On the fractional covering number of hypergraphs, *SIAM J. Discrete Math.* **1** No. 1 (1988), 45–49.

[7] V. Chvatal and P.J. Slater, A note on well-covered graphs, *Annals of Discrete Math.* **55** (1993), 179–182.

[8] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, *Discrete Applied Math.* **7** (1984), 115–130.

[9] D.C. Fisher, Domination, fractional domination, 2-packing, and graph products, *SIAM J. Discrete Math.* **7** No. 3 (1994), 493–498.

[10] T. Gallai, Uber extreme Punkt-und Kantenmengen, Ann. Univ. Sci. Budapest, *Eotvos Sect. Math.* **2** (1959), 133–138.

[11] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, New York (1979).

[12] H. Gavlas, K. Schultz, and P.J. Slater, Efficient open domination in graphs, *Scientia*, to appear.

[13] W. Goddard and M.A. Henning, personal communication.

[14] D.L. Grinstead and P.J. Slater, Fractional domination and fractional packing in graphs, *Congressus Numerantium* **71** (1990), 153–172.

[15] D.L. Grinstead and P.J. Slater, A recurrence template for several parameters in series-parallel graphs, *Discrete Applied Math.* **54** (1994), 151–168.

[16] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., 1998.

[17] S.T. Hedetniemi and R.L. Laskar, guest editors, *Discrete Math.* **86** (1990), 1–280.

[18] M.A. Henning and P.J. Slater, Closed neighborhood order dominating functions, *Australasian J. Comb.* **17** (1998), 77–88.

[19] T.W. Johnson and P.J. Slater, Maximum independent, minimally c-redundant sets in graphs, *Congressus Numerantium* **74** (1990), 193–211.

[20] T.W. Johnson and P.J. Slater, Maximum independent, minimally redundant sets in series-parallel graphs, *Quaestiones Mathematicae* **16** (1993), 351–370.

[21] M. Karchmer, E. Kushilevitz, and N. Nisan, Fractional covers and communication complexity, *SIAM J. Disc. Math.* **8** No. 1 (1995), 76–92.

[22] S.B. Maurer, Vertex colorings without isolates, *J. Combinatorial Theory B* **27** (1979), 294–319.

[23] P.J. Slater, Enclaveless sets and MK-systems, *J. of Research of the National Bureau of Standards* **82** (1977), 197–202.

[24] P.J. Slater, Closed neighborhood order domination and packing, *Congressus Numerantium* **97** (1993), 33–43.

[25] P.J. Slater, Packing into closed neighborhoods, *Bull. Inst. Comb. Appls.* **13** (1995), 23–33.

[26] P.J. Slater, Generalized graph parameters: Gallai Theorems I, *Bull. Inst. Comb. Appls.* **17** (1996), 27–37.

[27] P.J. Slater, LP-duality, complementarity, generality of graphical parameters, *Domination in Graphs Advanced Topics*, Chapter 1, eds. T.W. Haynes, S.T.Hedetniemi, and P.J. Slater, Marcel Dekker Inc., 1998, 1–30.

[28] C.B. Smart and P.J. Slater, Complexity results for closed neighborhood order parameters, *Congressus Numerantium* **112** (1995), 83–96.

[29] C.B. Smart and P.J. Slater, Closed neighborhood order parameters: Complexity, algorithms, and inequalities, submitted for publication.

[30] S.K. Tipnis and L.E. Trotter, Jr., Node-packing problems with integer rounding properties, *SIAM J. Discrete Math.* **2** No 3 (1989), 407-412.