The Scope of Three Colouring Conjectures

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ABSTRACT. It is shown that the Overfull Conjecture, which would provide a chromatic index charaterization for a large class of graphs, and the Conformability Conjecture, which would provide a total chromatic number characterization for a large class of graphs , both in fact apply to almost all graphs, whether labelled or unlabelled. The arguments are based on Polya's theorem, and are elementary in the sense that practically no knowledge of random graph theory is presupposed. It is similarly shown that the Biconformability Conjecture, which would provide a total chromatic number characterization for a large class of equibipartite graphs, in fact applies to almost all equibipartite graphs.

1 Introduction

The graphs in this paper will all be simple and it is well known that the edge chromatic number of a graph G, $\chi'(G)$, satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. As is also well known [13] , the problem of deciding whether a graph is Class 1 or Class 2, that is whether $\chi'(G)$ satisfies the left hand or right hand equality respectively, is NP-hard, thus a reasonable characterization of those graphs which are Class 1 is unlikely to exist. However, the Overfull Conjecture, if true, would provide such a characterization for all graphs which satisfy $\Delta(G) > \frac{1}{3}|V(G)|$. A graph G is *overfull* if $|E(G)| \geq \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor + 1$. The Overfull Conjecture states:

Conjecture 1. *Let $\Delta(G) > \frac{1}{3}|V(G)|$. Then G is Class 2 if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

For information on this conjecture see [2,3,12]. Since there is unlikely to be a characterization for all graphs, it is interesting to observe that the Overfull Conjecture actually applies to almost all graphs (see Theorem 1 and 2), and thus, if true, would provide a characterization for almost all graphs. Here the phrase "almost all graphs have property P" has the limited meaning that $\lim_{n\to\infty} \frac{P(n)}{A(n)} = 1$, where $P(n)$ is the cardinality of the set of all graphs of order n having property P, and $A(n)$ is the cardinality of the set of all graphs of order n. The graphs in question may all be labelled or all be unlabelled.

A result in this vein already exists: Erdös and Wilson [5] showed that almost all graphs are class 1, because almost all graphs have just one vertex of maximum degree (and such graphs are class 1). Their result, though, does not classify graphs into class 1 or class 2.

A similar situation is obtained with the total chromatic number of a graph G, $\chi_T(G)$, and another well known conjecture states that $\Delta(G)+1 \leq \chi_T(G) \leq \Delta(G) + 2$. Graphs which do or do not satisfy the left hand inequality are called Type 1 or Type 2 respectively. The next conjecture involves conformable graphs. A graph G is said to be *conformable* if G can be properly vertex coloured with $\Delta(G) + 1$ colours such that the number of colour classes of parity different from that of $|V(G)|$ is at most $\sum_{v\in V(G)}(\Delta(G) - d(v))$, which is called the deficiency of G and is denoted def(G). Sanchez-Arroyo [16] showed that it is NP-hard to determine the total chromatic number of a graph, even if it is bipartite. But the same considerations apply for total colouring as for the edge colouring, and the Conformability and Biconformability Conjectures (below) both concern possible classifications into Type 1 and Type 2 graphs of high maximum degree.

Conjecture 2. (The Conformability Conjecture) *Let $\Delta(G) > \frac{1}{2}(|V(G)| + 1)$. Then G is Type 2 if and only if G contains a subgraph H with $\Delta(H) = \Delta(G)$ that is either non-conformable or, in the case when $\Delta(G)$ is even, $K_{\Delta(G)+1}$ with one edge subdivided.*

For further information about this conjecture, see [4,7,8,9,10]. Note that the Conformability Conjecture as stated in [4] has had to be modified in the light of the Chen and Fu result [1]; in [7] a substantial argument is given to explain why no further modification is likely to be needed.

The final conjecture is the Biconformability Conjecture which just concerns bipartite graphs. Let $G = (A, B)$ be a bipartite graph and c: $V(G) \to \{1, 2, \ldots, \Delta(G) + 1\}$. Let $A_i = A \cap c^{-1}(i)$ and $B_i = B \cap c^{-1}(i)$ and

$a_i = |A_i|$ and $b_i = |B_i|$. Also, if $W \subseteq V(G)$ let $V_{<\Delta}(W)$ denote the set of all vertices of W which have degree in G less than $\Delta(G)$. A bipartite graph G is said to be *biconformable* if there exists a proper vertex colouring $c\colon V(G) \to \{1, 2, \ldots, \Delta(G)+1\}$ such that

$$|V_{<\Delta}(A \backslash A_i)| \geq b_i - a_i, \ |V_{<\Delta}(B \backslash B_i)| \geq a_i - b_i \text{ and } \operatorname{def}(G) \geq \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|.$$

Conjecture 3. *Let G be a bipartite graph and let $\Delta(G) > \frac{3}{14}|V(G)|$. Then G is Type 2 if and only if G contains a non-biconformable subgraph H with $\Delta(H) = \Delta(G)$.*

For further information about this conjecture, see [4,8]. Note in particular that the definition of biconformability in [4] has been modified in [8], and that a further modification will be needed to cover an awkward case discovered recently by Bor, Chen and Fu; the details of this have not yet been published.

2 Graphs with $\Delta(G) \geq \frac{1}{2}|V(G)|$

We show first that almost all labelled graphs satisfy $\Delta(G) \geq \frac{1}{2}|V(G)|$, and then, using Polya's famous asymptotic result (see below), that almost all unlabelled graphs satisfy $\Delta(G) \geq \frac{1}{2}|V(G)|$. Thus the class of graphs, whether labelled or unlabelled, which are the subject of Conjecture 1, or of Conjecture 2, includes almost all graphs.

The distribution of degrees in a graph for almost all graphs is a question that has been studied considerably, and in fact the results of Theorems 1 and 2 seem to be quite well-known. All the same, we have not been able to find these results stated explicitly. Our short proofs should at least make these results accessible.

Theorem 1. *Almost all labelled graphs G satisfy $\Delta(G) \geq \frac{1}{2}|V(G)|$.*

Proof: For each $n \geq 1$, let n^* be the larger root of $x^2 + x - n$. Then $n^* = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4n}$ and so n^* is less than $\sqrt{n}$ by a small amount. Moreover, $n^* = \sqrt{n - \sqrt{n - \sqrt{n - \ldots}}}$ and $n^* = \sqrt{n - n^*}$. Let $n_0 = \lfloor n^* \rfloor$, then $n_0 \leq \sqrt{n - n^*}$.

Consider a bipartite graph H of order n with vertex sets $\{v_1, v_2, \ldots, v_{n_0}\}$ and $\{v_{n_0+1}, \ldots, v_n\}$, and with each of the $n_0(n - n_0)$ edges present with probability $\frac{1}{2}$. What is the probability that v_1 has degree at most $\frac{n}{2}$? Observe that

$$\frac{n}{2} = \frac{1}{2}(n - n_0) + \frac{1}{2}n_0 \leq \frac{1}{2}(n - n_0) + \frac{1}{2}\sqrt{n - n_0} = \frac{1}{2}(n - n_0) + \sqrt{\frac{1}{4}(n - n_0)}.$$

Consequently, by the DeMoivre-Laplace Limit Theorem [6 viii.3], $P(d(v_1) \leq \frac{n}{2}) \rightarrow N(1) < 1$, where $N(x)$ is the normal distribution function. Therefore, there exists an $\varepsilon < 0$ and an integer N_0 such that $N(1) + \varepsilon < 1$ and, for all $n \geq N_0$, $P(d(v_1) \leq \frac{n}{2}) \leq N(1) + \varepsilon$.

Thus for each $\delta > 0$ there exists a positive integer N such that the probability that all of $v_1, v_2, \ldots, v_{n_0}$ have degree at most $\frac{n}{2}$ is less than δ for each $n \geq N$.

If G is a labelled graph with vertex set $\{v_1, \ldots, v_n\}$, we let G_B be the bipartite graph obtained from G by deleting all internal edges in the sets $\{v_1, \ldots, v_{n_0}\}$ and $\{v_{n_0+1}, \ldots, v_n\}$. Since the degree of a vertex in G is at least as big as the degree of the vertex in G_B, by the above considerations the probability that each vertex of G has degree less than $\frac{n}{2}$ goes to zero as n goes to infinity. Therefore, almost all labelled graphs G satisfy $\Delta(G) \geq \frac{n}{2}$. □

We remark that we consider G_B instead of G to make $\{P(d(v_i) \leq \frac{n}{2}) \mid 1 \leq i \leq n_0\}$ a set of independent probabilities.

Let R_n and r_n be the number of all labelled and unlabelled graphs respectively of order n, that satisfy $\Delta(G) \geq \frac{n}{2}$. Let G_n and g_n be the number of all labelled and unlabelled graphs respectively of order n. Then Theorem 1 says that $\frac{R_n}{G_n} \sim 1$. Since the number of isomorphically distinct labellings of a graph G is given by $\frac{n!}{|\Gamma(G)|}$, where $\Gamma(G)$ denotes the automorphism group of the graph G, it follows that $r_n \geq \frac{R_n}{n!}$. Therefore,

$$1 \geq \frac{r_n}{g_n} \geq \frac{R_n}{n!}\frac{1}{g_n} \sim \frac{R_n}{n!}\frac{n!}{G_n} = \frac{R_n}{G_n} \sim 1$$

where we have used the aforementioned well known result of Polya [14] which says that $g_n \sim \frac{G_n}{n!}$. Thus we have proved the following:

Theorem 2. *Almost all unlabelled graphs G have maximum degree at least $\frac{1}{2}|V(G)|$.*

3 Bipartite graphs with $\Delta(G) \geq \frac{1}{4}|V(G)|$

A bipartite graph G is called *equibipartite* if it has a bipartition (A, B) with $|A| = |B|$ such that each edge joins a vertex in A to a vertex in B. In this section we show first that almost all labelled equibipartite graphs satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$. For equibipartite graphs the term unlabelled is not at first sight clearly defined. We explain how it can be interpreted in three different ways (bicoloured, bipartitioned, and unassigned), and we show that whichever meaning is taken, it is nonetheless true that almost all unlabelled equibipartite graphs satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$. Thus Conjecture 3 applies to almost all equibipartite graphs, whether labelled or unlabelled.

A *bicoloured equibipartite graph* is an equibipartite graph in which each vertex is assigned one of two colours in such a way that each vertex is adjacent only to vertices of the opposite colour and the number of vertices of each colour is equal. For example the pairs of graphs C1a and C2a and also C1b and C2b of Figure 1 are considered to be coloured differently.

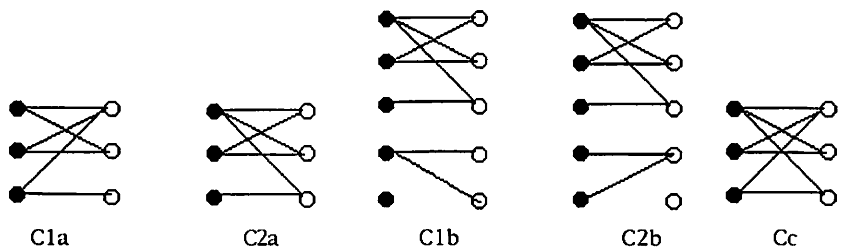

Figure 1

However, we note that there are bicoloured graphs G for which there is no natural mate G' with $G \neq G'$: for example, graph Cc of Figure 1.

We define an equivalence relation $\sim_{\text{bic}}$ on the set of (0, 1) matrices of order n by $C \sim_{\text{bic}} D$ if and only if there exist permutation matrices P and Q such that $C = PDQ$. The number of distinct bicoloured equibipartite graphs is the number of equivalence classes under $\sim_{\text{bic}}$, and each matrix in the equivalence class corresponding to a bicoloured equibipartite graph G is a bipartite adjacency matrix for G (an $n \times n$ matrix where $|A| = |B| = n$). We think of a labelled bicoloured equibipartite graph as having labels $1, 2, \ldots, n$ of one colour affixed to the vertices of A and the labels $1, 2, \ldots, n$ of the other colour affixed to the vertices of B. The number of labelled bicoloured equibipartite graphs is just the number of (0, 1) matrices of order n, as labelling G just selects one particular matrix from the equivalence class for G under $\sim_{\text{bic}}$.

Theorem 3. *Almost all labelled bicoloured equibipartite graphs G of order $2n$ satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$.*

Proof: Consider an $n \times n$ bipartite adjacency matrix $A(G)$ for the graph G. In order for $\Delta(G) < \frac{1}{4}|V(G)|$, each of its row sums must be less than $\frac{n}{2}$. For each row this has probability at most $\frac{1}{2}$. Thus the probability of the graph G satisfying $\Delta(G) < \frac{1}{4}|V(G)|$ is at most $\left(\frac{1}{2}\right)^n$, which clearly tends to zero. Hence the probability that $\Delta(G) \geq \frac{1}{4}|V(G)|$ tends to one. □

A second type of bipartite graph to be discussed will be called a *bipartitioned equibipartite graph*, which is a bipartite graph in which the vertex set has been partitioned into two parts of equal size such that no formal identification is given to either part. Considering the above examples, C1a and C2a are considered the same, but C1b and C2b are considered to be different. Again we note that there are bipartitioned graphs that correspond to just one bicoloured graph, for instance an uncoloured Cc in Figure 1.

The number of bipartitioned equibipartite graphs is the number of equivalence classes under the equivalence relation $\sim_{\text{bip}}$ on the set of $(0,1)$-matrices of order n, $C \sim_{\text{bip}} D$ if and only if there exist permutation matrices P and Q such that $C = PDQ$ or $C^t = PDQ$ where C^t is the transpose of C. If an equivalence class under $\sim_{\text{bip}}$ contains a symmetric matrix then it is also an equivalence class under $\sim_{\text{bic}}$; otherwise it is the union of two equivalence classes under $\sim_{\text{bic}}$. A labelled bipartitioned equibipartite graph has the labels $1, 2, \ldots, n$ affixed to the vertices of each part of the bipartition. The labelled bipartitioned equibipartite graphs G_1 and G_2 with respective bipartite adjacency matrices $A(G_1)$ and $A(G_2)$ (row and column numbers corresponding to the labels) are the same if and only if $A(G_1) = A(G_2)$ or $A(G_1) = A(G_2)^t$.

The final type of bipartite graph we consider is when the uncoloured C1b and C2b are considered to be equal. These will be called *unassigned equibipartite graphs* and are defined to be bipartite graphs for which it is possible to partition the vertex sets into two parts of equal size, but for which no formal partition is given.

We note that two unassigned equibipartite graphs are considered to be the same if and only if they are isomorphic (as graphs). A labelling of an unassigned equibipartite graph with vertex partition (A, B) consists of affixing the labels $1, 2, \ldots, n$ to the vertices of A and to the vertices of B. The labelled unassigned equibipartite graphs G_1 and G_2 are the same if there is a label-preserving isomorphism between them.

Lemma 4. *Almost all labelled equibipartite graphs are connected.*

Proof: Let $G = (A, B)$ be a random equibipartite graph with $|A| = |B| = n$. Consider a pair of vertices in A or B to be "bad" if there is no vertex adjacent to both. Let $X(G)$ be the number of "bad" pairs of vertices in G and p be the probability of an edge (normally $p = \frac{1}{2}$ in this paper). Then the expectation $E(X)$ of X satisfies $E(X) \leq n^2(1 - p^2)^n$, which tends to zero as n tends to infinity. Therefore the probability that $X(G) = 0$ tends to one. □

Let C_n and c_n be the number of labelled and unlabelled bicoloured equibipartite graphs of order $2n$ respectively that are connected. Similarly define B_n and b_n to be the number of labelled and unlabelled bicoloured equibipartite graphs of order $2n$ respectively. Then Lemma 4 can be restated.

Lemma 4'.

$$\frac{C_n}{B_n} \to 1 \text{ as } n \to \infty.$$

The following lemma will be used to prove analogous results for the unlabelled case.

Lemma 5.

$$b_n \to \frac{B_n}{(n!)^2} \text{ as } n \to \infty.$$

Proving this lemma requires further terminology to be defined.

Let $b_{n,q}$ be the number of unlabelled bicoloured equibipartite graphs of order $2n$ which have exactly q edges and let $b(x) = \sum_{q=0}^{n^2} b_{n,q}x^q$. From Harary and Palmer [11] $b(x) = Z(S_n \times S_n, 1+x)$, where S_n denotes the symmetric group on n elements and $Z(A)$ is the cycle index for a given group A. It is also given in [10] that

$$Z(S_n \times S_n) = \frac{1}{n!n!} \sum_{(\alpha,\beta)} \prod_{r,t=1}^{n,n} s_{[r,t]}^{(r,t)j_r(\alpha)j_t(\beta)},$$

where $j_i(\gamma)$ is the number of cycles of length i in the disjoint cycle decompostion of γ, $[r,t]$ and (r,t) denote the l.c.m. and g.c.d. of r and t respectively. This however is clearly the same as $\frac{1}{n!n!}\sum_{(j,\iota)} h(j,\iota)\prod_{r,t=1}^{n,n} s_{[r,t]}^{(r,t)j_r\iota_t}$, where (j,ι) is an ordered pair of partitions of n with $j = (j_1, j_2, \ldots, j_n)$ and $\iota = (\iota_1, \iota_2, \ldots, \iota_n)$ where j_t and ι_t are the respective number of parts of size t, and $h(j,\iota)$ is the number of ordered permutation pairs whose cycle structures are j and ι respectively. Also let $q(j,\iota)$ denote the number of edge cycles determined by the partition pair (j,ι).

Proof of Lemma 5: The number $q(j,\iota)$ of edge cycles determined by the permutation pairs corresponding to the partition pair (j,ι) is

$$q(j,\iota) = \sum_{(r,t=1}^{n,n} (r,t)j_r\iota_t. \tag{1}$$

For each nonnegative integer k $(0 \le k \le 2n)$ we let $b_n^{(k)}$ be the contribution by all partition pairs (j,ι) such that $j_1+\iota_1 = 2n-k$. So then $\sum_{k=0}^{2n} b_n^{(k)} = b_n$ and

$$b_n^{(k)} = \frac{1}{n!n!}\sum_{(j,\iota)} h(j,\iota)2^{q(j,\iota)},$$

where the sum is over all (j,ι) such that $j_1+\iota_1 = 2n-k$. Notice that $b_n^{(0)} = \frac{2^{n^2}}{n!n!}$, $b_n^{(2)} = b_n^{(0)}\frac{n(n-1)}{2^n}$, and $b_n^{(3)} = b_n^{(0)}\frac{n(n-1)(n-2)}{3\cdot 2^{2n-1}}$, which leads one to guess the following:

$$\text{For each positive integer } m, b_n \sim \sum_{k=0}^{m-1} b_n^{(k)}. \tag{2}$$

To see this, first establish an upper bound for $b_n^{(k)}$. For each k, consider those partition pairs (j,ι) with $j_1+\iota_1 = 2n-k$. On substituting $j_1 = \iota_1 =$

$2n - \frac{k}{2}$ and $j_2 = \iota_2 = \frac{k}{4}$ in the right hand side of (1) an upper bound for $q(j, \iota)$ is obtained to be

$$q(j, \iota) \leq n^2 - \frac{3nk}{8}.$$

Also, the number of permutations of n objects with exactly $(n - \frac{k}{2})$ objects fixed is less than or equal to $n!/(n - \frac{k}{2})!$ [14 p. 59]. So then we can say

$$\begin{aligned} b_n^{(k)} &\leq \frac{1}{n!n!} \frac{n!}{(n-\frac{k}{2})!} \frac{n!}{(n-\frac{k}{2})!} 2^{n^2 - \frac{3nk}{8}} \\ &\leq b_n^{(0)} n^k 2^{\frac{-3nk}{8}} \\ &= b_n^{(0)} (\frac{n}{2^{3n\backslash 8}})^k. \end{aligned} \tag{3}$$

Summing (3) over k from m to $2n$ we get

$$\sum_{k=m}^{2n} b_n^{(k)} \leq \sum_{k=m}^{2n} b_n^{(0)} (n/2^{\frac{3n}{8}})^k = b_n^{(0)} \sum_{k=m}^{2n} (n/2^{\frac{3n}{8}})^k.$$

This sum is geometric with common ratio approaching zero as n approaches infinity, so we can write

$$\sum_{k=m}^{2n} b_n^{(k)} \leq C b_n^{(0)} (n/2^{\frac{3n}{8}}) \tag{4}$$

where $C > 1$ and close to one for large n.

Next summing $b_n^{(k)}$ from zero to $(m-1)$ we get

$$\sum_{k=0}^{m-1} b_n^{(k)} \leq b_n \leq \sum_{k=0}^{m-1} b_n^{(k)} + b_n^{(0)} O((n/2^{\frac{3n}{8}})^m).$$

Dividing gives: $1 \leq b_n \sum_{k=0}^{m-1} b_n^{(k)} \leq 1 + O(n^m/2^{\frac{3nm}{8}})$. This verifies (2).

Continuing with the proof of the lemma we return to (3) with $k = m$ and see that $b_n^{(m)} \leq b_n^{(0)} (n/2^{\frac{3n}{8}})^m$. Therefore,

$$\sum_{k=m}^{2m} b_n^{(k)} \leq b_n^{(0)} O(n^m/2^{\frac{3nm}{8}}) \tag{5}$$

and from (4) it follows that

$$\sum_{k=2m+1}^{2n} b_n^{(k)} \leq b_n^{(0)} O(n^m/2^{\frac{3nm}{8}}). \tag{6}$$

Combining (5) and (6) it follows that $\sum_{k=m}^{2n} b_n^{(k)} \leq b_n^{(0)} O(n^m/2^{\frac{3nm}{8}})$. Setting $m = 3$ and adding $b_n^{(0)} + b_n^{(2)}$ to both sides yields $b_n = b_n^{(0)}(1 + \frac{n(n-1)}{2^n} + O(n^3 2 - \frac{9n}{2}))$ and since $b_n^{(0)} = \frac{2^{n^2}}{n!n!}$, Lemma 5 now follows. □

Lemma 6. $c_n \geq \frac{C_n}{(n!)^2}$

Proof: The number of labellings of any equibipartite bicoloured graph (preserving bicolouring) equals $\frac{(n!)^2}{|\Gamma_1(G)|}$, where $\Gamma_1(G)$ is the automorphism group of G that preserves the bicolouring. □

Lemma 7. *$b_n \sim c_n$ for sufficiently large n.*

Proof: By Lemmas 5 and 6,

$$1 \geq \frac{c_n}{b_n} \geq \frac{C_n}{(n!)^2} \cdot \frac{1}{b_n} \sim \frac{C_n}{(n!)^2} \cdot \frac{(n!)^2}{B_n} = \frac{C_n}{B_n} \to 1.$$

Hence the result follows. □

We remark that Lemma 7 has a short direct proof (without Lemma 5) using the expected number of bad vertices as in the proof of Lemma 4. However, we need Lemma 5 to prove other results.

Let p_n denote the number of unlabelled bipartitioned equibipartite graphs of order $2n$. The following lemma says that almost no bicoloured equibipartite graphs have a symmetric adjacency matrix.

Lemma 8. $b_n \sim 2p_n$

Proof: As in the proof of Lemma 5, we define $p_{n,q}$ to be the number of unlabelled bipartitioned equibipartite graphs of order $2n$ which have exactly q edges and let $p(x) = \sum_{q=0}^{n^2} p_{n,q} x^q$. Again we cite [11] and obtain $p(x) = Z([S_n]^{S_2}, 1 + x)$, where $Z([S_n]^{S_2}) = \frac{1}{2}(Z(S_n \times S_n) + Z'_n)$ and

$$Z'_n = \frac{1}{n!} \sum_{(j)} h(j) \prod_{k \text{ odd}} s_k^{j_k} \prod_{k \text{ even}} s^{k\binom{j_k}{2} + [\frac{k}{2}]j_k} \prod_{r<t} s_{2[r,t]}^{(r,t)j_r j_t}$$

From Lemma 5 it is enough to show that the contribution of Z'_n is almost zero. Let p'_n be the contribution of Z'_n to $Z([S_n]^{S_2})$.

Proceding as in the proof of Lemma 5 we see that

$$q(j) = \sum_{k \text{ odd}} j_k + \sum_{k \text{ even}} k\binom{j_k}{2} + [\frac{k}{2}]j_k + \sum_{r<t} (r,t) j_r j_t. \tag{7}$$

Now for $k = 0$ through n let $p_n'^{(k)}$ denote the contribution to p'_n determined by all partitions (j) having exactly $n - k$ parts of size one. As before,

$p'_n = \sum_{k=0}^{n} p_n'^{(k)}$ and $p_n'^{(k)} = \frac{1}{n!}\sum_{(j)} h(j)2^{q(j)}$, where the sum is over all partitions (j) with $j_1 = n - k$.

Next we show that: for each positive integer m,

$$p'_n \sim \sum_{k=0}^{m-1} p_n'^{(k)}. \tag{8}$$

For each k consider those partitions (j) with $j_1 = n - k$. substituting with $j_1 = n - k$ and $j_2 = \frac{k}{2}$ into the right hand side of (7) maximizes $q(j)$ and so $q(j) \leq \binom{n+1}{2} - \frac{1}{2}(nk + k - \frac{k^2}{2})$

Therefore as in Lemma 5,

$$p_n'^{(k)} \leq p_n'^{(0)}(n^2/2^{n+1-\frac{k}{2}})^{k/2} \tag{9}$$

However, $k \leq n$ so it follows that

$$p_n'^{(k)} \leq p_n'^{(0)}(n^2/2^{(n/2)+1})^{k/2}. \tag{10}$$

Summing (10) from $k = m$ to n yields

$$\sum_{k=m}^{n} p_n'^{(k)} \leq p_n'^{(0)} \sum_{k=0}^{n}(n^2/2^{(n/2)+1})^{k/2}. \tag{11}$$

The right hand side of (11) is a geometric series whose common ratio approaches zero as n increases, so

$$\sum_{k=m}^{n} p_n'^{(k)} \leq Cp_n'^{(0)}(n^2/2^{n/2})^{m/2}, \tag{12}$$

where $C > 1$ and close to one for large n.

It now follows that $\sum_{k=m}^{n-1} p_n'^{(k)} \leq p_n \leq \sum_{k=m}^{n-1} p_n'^{(k)} + O(n^m/2^{nm/4})$. Dividing yields $1 \leq p'_n / \sum_{k=m}^{n-1} p_n'^{(k)} \leq 1 + O(n^m/2^{nm/4})$ which verifies (8).

Returning to (9) with $k = m$ it is seen that $p_n'^{(m)} \leq p_n'^{(0)}(n^2/2^{n+1-\frac{m}{2}})^{m/2} = 2^{m(m-2)/4}p_n'^{(0)}(n^2/2^n)^{m/2}$ and hence

$$\sum_{k=m}^{2m} p_n'^{(k)} \leq p_n'^{(0)}O(n^m/2^{nm/2}). \tag{13}$$

From (12) it follows that $\sum_{k=2m+1}^{n} p_n'^{(k)} \leq p_n'^{(0)}O((n^2/2^{n/2})^{(2m+1)/2}) = p_n'^{(0)}O((n^m/2^{nm/2})$. Combining this with (13) then gives $\sum_{k=m}^{n} p_n'^{(k)} \leq p_n'^{(0)}O(n^m/2^{nm/2})$. Setting $m = 3$ and noting that $p_n'^{(0)} = \frac{2^n}{n!}$ it now follows that $p'_n = \frac{2^n}{n!}(1 + O(\frac{n^2}{2^n}))$ which tends to zero as n tends to infinity. Thus Lemma 8 follows. □

Lemma 9. $c_n \sim 2p_n$

Proof: Lemmas 7 and 8. □

Let u_n be the number of unlabelled unassigned equibipartite graphs of order $2n$. Let $u_{1,n}$ be the number of such graphs that correspond to exactly one bipartitioned equibipartite graph of order $2n$; let $u_{\geq 2,n}$ be the number of such graphs that correspond to two or more bipartitioned equibipartite graphs of order $2n$. Also let c_n^* be the number of unlabelled bipartitioned equibipartite graphs of order $2n$ that are connected.

Lemma 10. $u_{1,n} \geq c_n^*$

Proof: Any bipartitioned equibipartite graph consisting of one connected component clearly has only one equibipartition. □

Lemma 11. *Almost all bicoloured equibipartite graphs have a trivial automorphism group.*

Proof: Let G be a bicoloured graph with automorphism group $\Gamma(G)$. Then G can be labelled in $n!n!/|\Gamma(G)|$ ways. Therefore, $B_n = \sum_{G \in b_n} \frac{n!n!}{|\Gamma(G)|}$ and so $\frac{B_n}{n!n!} = \sum_{G \in b_n} \frac{1}{|\Gamma(G)|}$. Since the left hand side is asymptotic to b_n, almost every term on the right must contribute a one. □

Lemma 12. *Almost all bipartitioned equibipartite graphs have a trivial automorphism group.*

Proof: Adapt the proof of Lemma 11. □

Lemma 13. $c_n^* \sim p_n$

Proof: Any bipartitioned equibipartite graph yields at most $2(n!)^2$ labelled bicoloured equibipartite graphs, so $c_n^* \geq \frac{C_n}{2(n!)^2}$. Therefore, by Lemmas 5 and 8, $1 \geq \frac{c_n^*}{p_n} \geq \frac{C_n}{2(n!)^2} \cdot \frac{1}{p_n} \sim \frac{C_n}{2(n!)^2} \cdot \frac{2(n!)^2}{B_n} = \frac{C_n}{B_n} \sim 1$. and it is clear that the result follows. □

Let u_n^Δ be the number of these unlabelled unassigned equibipartite graphs of order $2n$ with $\Delta(G) \geq \frac{1}{4}|V(G)|$.

Lemma 14. $u_n \sim p_n$

Proof: Any connected unassigned equibipartite graph has exactly one equibipartition. Therefore, $u_n \geq c_n^*$. But also every unassigned equibipartite graph can be bipartitioned in at least one way. Therefore, $p_n \geq u_n$ and so by Lemma 13, $c_n^* \sim p_n \geq u_n \geq c_n^*$. □

Let b_n^Δ and B_n^Δ be the number of unlabelled and labelled bicoloured equibipartite graphs of order $2n$ respectively that satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$.

Theorem 15. *$b_n^\Delta \sim b_n$, so almost all unlabelled bicoloured equibipartite graphs G satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$.*

Proof: By Theorem 3 and Lemma 5, with the obvious notation $1 \geq \frac{b_n^\Delta}{b_n} \geq \frac{B_n^\Delta}{(n!)^2} \cdot \frac{1}{b_n} \sim \frac{B_n^\Delta}{(n!)^2} \cdot \frac{(n!)^2}{B_n} = \frac{B_n^\Delta}{B_n} \sim 1$ and so the result follows. □

Similarly define p_n^Δ to be the number of unlabelled bipartitioned equibipartite graphs of order $2n$.

Theorem 16. *$p_n^\Delta \sim p_n$, so almost all unlabelled bipartitioned equibipartite graphs G satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$.*

Proof: Any bipartitioned graph of order $2n$ yields at most $2(n!)^2$ labelled bicoloured equibipartite graphs, thus $p_n^\Delta \geq \frac{B_n^\Delta}{(n!)^2}$. Using Lemma 8, it follows that $1 \geq \frac{p_n^\Delta}{p_n} \geq \frac{B_n^\Delta}{2(n!)^2} \cdot \frac{1}{p_n} \sim \frac{B_n^\Delta}{2(n!)^2} \cdot \frac{2}{b_n} \sim \frac{B_n^\Delta}{2(n!)^2} \cdot \frac{2(n!)^2}{B_n} = \frac{B_n^\Delta}{B_n} \sim 1$. Therefore, $p_n^\Delta \sim p_n$ as required. □

Lemma 17. *$u_n^\Delta \sim p_n^\Delta$*

Proof: As in Lemma 14, $p_n^\Delta \geq u_n^\Delta \geq c_n^{*\Delta}$. From Theorem 16 and Lemma 13 almost all bipartitioned equibipartite graphs are connected, and almost all satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$, so almost all are connected and satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$. Therefore, $c_n^{*\Delta} \sim p_n$ and thus by Theorem 16, $c_n^{*\Delta} \sim p_n^\Delta \geq u_n^\Delta \geq c_n^{*\Delta}$ and the result follows. □

Theorem 18. *$u_n^\Delta \sim u_n$ so almost all unlabelled unassigned equibipartite graphs G satisfy $\Delta(G) \geq \frac{1}{4}|V(G)|$.*

Proof: This follows from Lemmas 17 and 14, and Theorem 16. □

Therefore, Conjecture 3 applies to almost all labelled or unlabelled equibipartite graphs regardless of how equivalency is defined.

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