The intersection problem for maximum pentagon packings

Barbara M. Maenhaut[1]
Centre for Combinatorics
University of Queensland
QLD, 4072, Australia

Abstract

Let K_n be the complete graph on n vertices. Let $I(X)$ denote the set of integers k for which a pair of maximum pentagon packings of graph X exist having k common 5-cycles. Let $J(n)$ denote the set $\{0, 1, 2, \ldots, P-2, P\}$, where P is the number of 5-cycles in a maximum pentagon packing of K_n. This paper shows that $I(K_n) = J(n)$, for all $n \geq 1$.

1 Introduction

The intersection problem for a combinatorial structure is the problem of determining the possible numbers of common objects (such as cycles) in two combinatorial structures (such as cycle systems) based on a common underlying set. This problem has been considered for many types of combinatorial structures. In 1975, Lindner and Rosa considered the intersection problem for Steiner triple systems [7]. More recently the problem has been considered by Fu for pentagon systems [5] and by Billington for m-cycle systems [2]. Here the intersection problem for maximum pentagon packings will be considered.

First we need a few definitions and some convenient notation.

Throughout this paper, let K_n represent the complete graph on n vertices, and let K_n^* represent the complete graph on n vertices with a one-factor removed. If A and B are edge-disjoint graphs, let $A + B$ denote the graph formed by their union. If B is a subgraph of A, let $A \setminus B$ denote the graph A with the edges of B removed. The complete multipartite graph with m parts of size n will be denoted by $K_{n(m)}$, while the complete multipartite graph with m_1 parts of size n_1 and m_2 parts of size n_2 will be denoted by $K_{n_1(m_1),n_2(m_2)}$. Let $e(X)$ represent the number of edges in the graph X, let C_n represent a cycle of length n, and let F represent a one-factor of

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the graph in question. Finally, the notation $n \cdot G$ denotes a graph which consists of n edge-disjoint copies of the graph G.

A **pentagon packing** of the graph X, $PP(X)$, is a family of edge-disjoint pentagons in X. A **maximum pentagon packing** of X, an $MP(X)$, is a $PP(X)$ such that no other $PP(X)$ contains more pentagons. We will refer to an $MP(K_n)$ simply as an $MP(n)$. The **leave** of a $PP(X)$ is the graph which is the complement of the union of the pentagons in the $PP(X)$. Since a pentagon packing must have each vertex of even degree, the number of pentagons in an $MP(n)$ cannot exceed $n(n-1)/10$, for n odd, and $n(n-2)/10$, for n even. Let $mp(n)$ represent the number of pentagons in an $MP(n)$.

Rosa and Znám [8] have shown that the number of pentagons in an $MP(n)$ is

$$mp(n) = \begin{cases} \lfloor e_n/5 \rfloor & \text{if } n \not\equiv 7,9 \bmod 10 \\ \lfloor e_n/5 \rfloor - 1 & \text{if } n \equiv 7,9 \bmod 10 \end{cases}$$

where $e_n = n(n-1)/2$ if n is odd and $e_n = n(n-2)/2$ if n is even.

Table 1 may provide the reader with a clearer idea of what an $MP(n)$ looks like.

No. of vertices	No. of pentagons	Leave
$n \equiv 1,5 \bmod 10$	$\frac{n^2-n}{10}$	none
$n \equiv 3 \bmod 10$	$\frac{n^2-n-6}{10}$	C_3
$n \equiv 7,9 \bmod 10$	$\frac{n^2-n-12}{10}$	 or 
$n \equiv 0,2 \bmod 10$	$\frac{n^2-2n}{10}$	F
$n \equiv 4,8 \bmod 10$	$\frac{n^2-2n-8}{10}$	C_4 and F (see Remark 1.1)
$n \equiv 6 \bmod 10$	$\frac{n^2-2n-4}{10}$	$\frac{n}{2}+2$ edges (see Remark 1.2)

Table 1: The structure of maximum pentagon packings of K_n

Remark 1.1 *The C_4 and F will occur in one of three configurations. Suppose the C_4 is (v_1, v_2, v_3, v_4). The one-factor may contain the following edges: **i)** both $\{v_1, v_3\}$ and $\{v_2, v_4\}$ **ii)** either $\{v_1, v_3\}$ or $\{v_2, v_4\}$ **iii)** neither $\{v_1, v_3\}$ nor $\{v_2, v_4\}$. Any of the three configurations may occur as can be illustrated by an $MP(8)$.*

Remark 1.2 *An $MP(n)$ is in fact an $MP(K_n^*)$ for $n \equiv 0,2,4,8 \bmod 10$, but $n \equiv 6 \bmod 10$ is a special case since a maximum pentagon packing of K_6 leaves 5 edges which do not contain a one-factor. Consequently if we consider the graph $K_{10x+6} = K_{10x+6} \backslash K_6 + K_6$, a maximum pentagon packing of $K_{10x+6} \backslash K_6$ has a leave of a one-factor ($5x$ edges), and a maximum* *pentagon packing of K_6 has a leave of 5 edges, either a double ended arrow or a star. Therefore the leave of an $MP(K_n)$, where $n = 10x + 6$, is $(n-6)/2 + 5$ edges which almost contain a one-factor.*

Adams, Bryant and Khodkar [1] have shown that for all odd (even) integers n and all non-negative integers r and s satisfying $3r + 5s = \frac{n(n-1)}{2}$ $\left(3r + 5s = \frac{n(n-2)}{2}\right)$, the edge-set of K_n (K_n^*) can be partitioned into r 3-cycles and s 5-cycles. When s is as large as possible, and $n \not\equiv 4, 6, 8 \bmod 10$, the constructions in [1] provide an $MP(n)$. The small $MP(n)$s, for $n \equiv 4, 6, 8 \bmod 10$, given in the appendices rely on work by Heinrich and Rosa [6] and by El-Zanati [4]. Combining those small cases with the constructions from [1] we can obtain $MP(n)$s for all n.

Let $I(X)$ denote the set of integers k for which a pair of $MP(X)$s exist having k common cycles. Let $J(n)$ denote the set $\{0, 1, 2, \ldots, mp(n) - 2, mp(n)\}$. It is clear that $I(K_n) \subseteq J(n)$. The aim of this paper is to show that $I(K_n) = J(n)$.

2 The small cases

The graph H [1], shown in Figure 1, can be decomposed into 5-cycles in various ways. Hence, many of the small $MP(n)$s used in this paper have been constructed using copies of H, as was done in [1]. The follow-

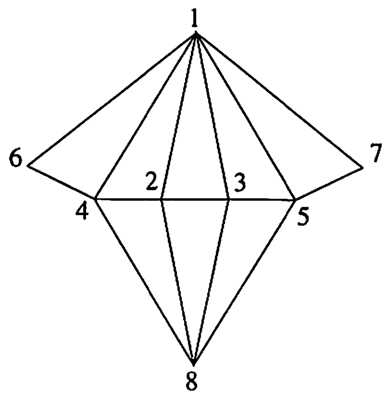

Figure 1: The graph H

ing three decompositions of H into 5-cycles show that $I(H) = \{0, 1, 3\}$.
$D_1 = \{16423, 14835, 12857\}$ $\quad D_2 = \{16423, 14857, 12835\}$
$D_3 = \{16482, 17583, 14235\}$

Another necessary tool in describing the possible intersections of the $MP(n)$s is a specific type of 5-cycle trade of volume 2. For a general discussion of trades in graphs, see Billington and Hoffman [3]. Here only the specific trade we need will be introduced. A **5-cycle trade of volume 2** is a graph X whose ten edges can be partitioned into a pair of 5-cycles in at least two different ways. For example, the graph shown in Figure 2 can be decomposed into the 5-cycles $\{12345, 16378\}$ or into the 5-cycles $\{16345, 12378\}$.

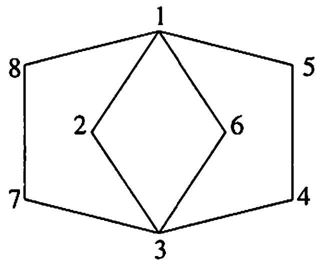

Figure 2: A 5-cycle trade of volume 2

Notice that if we are given two 5-cycles in which vertices a and b occur at distance two in each 5-cycle, then we have a 5-cycle trade of volume 2 as shown above. We will refer to such a trade as an **exchange** of two cycles. For instance $\{axbyz, aubvw\}$ is an exchange of two cycles as it can be replaced by $\{aubyz, axbvw\}$.

The idea of an exchange of two cycles can be extended to any number of cycles. For example, the graph created by the union of the three cycles $\{arbst, aubvw, axbyz\}$ could also be partitioned into the cycles $\{aubst, arbvw, axbyz\}$ or $\{aubst, axbvw, arbyz\}$, using an exchange of two cycles or an exchange of three cycles. Hence if X were the graph created by the union of n such cycles, X would contain exchanges of two to n cycles, and $I(X) = \{0, 1, 2, \ldots, n-2, n\}$.

The proofs of the following three lemmas are found in the appendices.

Lemma 2.1 *For* $n = 1, \ldots, 32$, $I(K_n) = J(n)$.

Lemma 2.2 *For* $t = 0, 1, \ldots, 9$,
$I(K_{10+t} \setminus K_t) = \left\{0, 1, 2, \ldots, \frac{e(K_{10+t} \setminus K_t)}{5} - 2, \frac{e(K_{10+t} \setminus K_t)}{5}\right\}$.

Lemma 2.3 $I(K_{5(3)}) = \{0, 1, \ldots, 13, 15\}$.

Notice that maximum pentagon packings of $K_{10+t} \backslash K_t$ leave either no edges or a one-factor, and that maximum pentagon packings of $K_{5(3)}$ leave no edges.

3 The larger cases

Lemma 3.1 *Let S be the set of graphs $\{K_{10(3)}, K_{10(4)}, K_{10(3),20(1)}, K_{10(6)}, K_{10(7)}, K_{10(6),20(1)}\}$. For each $s \in S$, $I(s) = \{0, 1, \ldots, \frac{e(s)}{5} - 2, \frac{e(s)}{5}\}$.*

Proof By Lemma 3.3 from [1] each of these graphs can be decomposed into edge-disjoint copies of $K_{5(3)}$. Hence the result follows from Lemma 2.3. □

Lemma 3.2 *For $t = 3, \ldots, 9$,*
$I(K_{30+t} \setminus K_t) = \left\{0, 1, \ldots, \frac{e(K_{30+t} \setminus K_t)}{5} - 2, \frac{e(K_{30+t} \setminus K_t)}{5}\right\}$.

Proof From [1] $K_{30+t} \setminus K_t = K_{10(3)} + 3 \cdot (K_{10+t} \setminus K_t)$. The result follows from Lemmas 2.2 and 3.1. □

Lemma 3.3 *For $n = 33, \ldots, 89$, $I(K_n) = J(n)$.*

Proof As shown in [1] the following constructions give K_n for $n = 33, \ldots, 89$.

$$\begin{aligned}
K_{30+t} &= K_{30+t} \setminus K_t + K_t \\
K_{40+t} &= K_{10(4)} + 4 \cdot (K_{10+t} \setminus K_t) + K_t \\
K_{50+t} &= K_{10(3),20(1)} + 3 \cdot (K_{10+t} \setminus K_t) + K_{20+t} \\
K_{60+t} &= K_{10(6)} + 6 \cdot (K_{10+t} \setminus K_t) + K_t \\
K_{70+t} &= K_{10(7)} + 7 \cdot (K_{10+t} \setminus K_t) + K_t \\
K_{80+t} &= K_{10(6),20(1)} + 6 \cdot (K_{10+t} \setminus K_t) + K_{20+t}
\end{aligned}$$

By Lemmas 2.1, 2.2, 3.1 and 3.2, the result follows. □

Lemma 3.4 *For all $x \geq 3$, let S be the set of graphs $\{K_{30(x)}, K_{30(x),10(1)}, K_{30(x),20(1)}\}$. For each $s \in S$, $I(s) = \left\{0, 1, \ldots, \frac{e(s)}{5} - 2, \frac{e(s)}{5}\right\}$.*

Proof By Lemma 3.6 in [1], each of the graphs in S can be decomposed into edge-disjoint copies of $K_{5(3)}$, and the result follows from Lemma 2.3. □

Lemma 3.5 *For $n \geq 90$, $I(K_n) = J(n)$.*

Proof From Lemma 3.7 in [1] the following construction gives K_n for $n \geq 90$. For $t = 0, 1, \ldots, 9$, $u = 0, 10, 20$, and $x \geq 3$:

$$K_{30x+u+t} = K_{30(x),u(1)} + x \cdot (K_{30+t} \setminus K_t) + K_{u+t}$$

The result follows from Lemmas 2.1, 3.2 and 3.4. □

Combining Lemmas 2.1, 3.3 and 3.5, we obtain the following main result.

Theorem 3.6 *For all $n \geq 1$, $I(K_n) = J(n)$.*

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A Proof of Lemma 2.1

In the following tables, the first column contains an $MP(n)$ and subsequent columns contain $MP(n)$s which intersect the first one in the cycles which are left blank. For $n \geq 12$, a pair of subscripts after a set of cycles indicate the two vertices which occur at distance 2 in each cycle of the set. In sets of cycles without a subscript, semicolons are used to separate exchanges of two or three cycles and within those exchanges the cycles intersect in the first and third vertices. Two sets of cycles separated by an $\mathcal{R}$ indicate that the first set of cycles can be replaced by the second set. The two sets of cycles use the same edges but have no cycles in common.

$I(K_5) = J(5)$	
01234	01243
02413	02314

$I(K_6) = J(6)$	
01345	01543
02153	02135

$I(K_7) = J(7)$		
03152		02543
01234	01245	04215
06245	06234	06231

$I(K_8) = J(8)$			
17524			15723
05163		05134	05241
04326	04372	02736	04362
01372	01326	01623	03716

$I(K_9) = J(9)$					
03184					03154
05176				05146	05236
14286			14236	16237	14276
36475		23465	25346	24365	17348
38562	34562	26357	27458	27458	26538
34527	38527	37458	37568	35768	46857

$I(K_{10}) = J(10)$							
05416							05419
13958						43958	43958
37468					37968	37968	37968
27849				27843	27843	27813	27813
51963			57963	57963	57463	57463	57463
02579		02679	02519	02519	02519	02519	02516
01234	08234	08234	08234	08294	08294	08294	08294
08267	01267	01257	01267	01267	01267	01267	01267

$I(K_{11}) = J(11)$
Intersection problem has been solved for pentagon systems [5].

$I(K_{12}) = J(12)$
From [6] $MP(12) = A + B + C$ where

$A = \{a5b20, a7b42, a9b14, a6b31, a8b03\}_{ab}$,
$B = \{18360, 19346\}_{13}$ and
$C = \{12947, 23508, 45270, 56789, 57968\}\ \mathcal{R}$
$\quad C' = \{12547, 23507, 49280, 56879, 58967\}$.
From the exchanges in A and B, $\{5, 6, \ldots, 10, 12\} \subseteq I(K_{12})$. Replacing C with C', $\{0, 1, \ldots, 5, 7\} \subseteq I(K_{12})$. Combining these we obtain the required intersection values.

$I(K_{13}) = J(13)$
From [1] $K_{13} = 5 \cdot H + C_3$ so an $MP(13)$ simply consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, the result follows.

$I(K_{14}) = J(14)$
From [6] $MP(14) = A + B + C$ where
$A = \{c02d5, c24d6, c41d7, c13d8, c30d9\}_{cd}$,
$B = \{a57b0, a79b1, a96b2, a68b3, a85b4\}_{ab}$ and
$C = \{06381, 05328; 17492, 16439; 52704, 56789\}$.
From the exchanges in A and B, $\{6, 7, \ldots, 14, 16\} \subseteq I(K_{14})$. C has been grouped into three exchanges of two cycles, so $\{0, 2, 4, 6\} \subseteq I(C)$. Combining these we obtain the required intersection values.

$I(K_{15}) = J(15)$
Intersection problem has been solved for pentagon systems [5].

$I(K_{16}) = J(16)$
From [4] $MP(16) = A + B + C + D$ where
$A = \{1a0b7, 3a2b9, 5a4b1, 7a6b3, 9a8b5\}_{ab}$,
$B = \{0c1d8, 2c3d0, 4c5d2, 6c7d4, 8c9d6\}_{cd}$,
$C = \{1e0f9, 3e2f1, 5e4f3, 7e6f5, 9e8f7\}_{ef}$ and
$D = \{83052, 49618, 65870, 21436, 09274, abfce, acdef\}\ \mathcal{R}$
$\quad D' = \{83092, 43658, 61870, 21496, 05274, abfec, aedcf\}$.
From the exchanges in A, B and C, $\{7, 8, \ldots, 20, 22\} \subseteq I(K_{16})$. Replacing D with D', $\{0, 1, \ldots, 13, 15\} \subseteq I(K_{16})$. Combining these we obtain the required intersection values.

$I(K_{17}) = J(17)$
From [1] $K_{17} = 8 \cdot H + 2 \cdot C_5 + 2 \cdot C_3$, giving the $MP(17)$ consisting of the 5-cycles from the decomposition of the copies of H given in [1] and the two extra 5-cycles $(e8dc4)$ and $(e5da7)$. Now $I(H) = \{0, 1, 3\}$ and the two 5-cycles form an exchange of two cycles. Combining these we obtain the required intersection values.

$I(K_{18}) = J(18)$
From [6] $MP(18) = A + B + C + D + E$ where
$A = \{1a0b7, 3a2b9, 5a4b1, 7a6b3, 9a8b5\}_{ab}$,

$B = \{0c1d8, 2c3d0, 4c5d2, 6c7d4, 8c9d6\}_{cd}$,
$C = \{1e0f9, 3e2f1, 5e4f3, 7e6f5, 9e8f7\}_{ef}$,
$D = \{8g1h3, 3g4h0, 0g6h5, 5g7h2, 2g9h8\}_{gh}$ and
$E = \{61849, 65870; 21436, 27409; fahbg, fcheb; ecgdf, eaghd\}$.
From the exchanges in A, B, C and D, $\{8, 9, \ldots, 26, 28\} \subseteq I(K_{18})$. E has been grouped into four exchanges of two cycles, so $\{0, 2, 4, 6, 8\} \subseteq I(E)$. Combining these we obtain the required intersection values.

$I(K_{19}) = J(19)$
From [1] $K_{19} = 11 \cdot H + 2 \cdot C_3$, so an $MP(19)$ consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, we obtain the required intersection values.

$I(K_{20}) = J(20)$
From [6] $MP(20) = A + B + C + D + E + F + G$ where
$A = \{1a0b7, 3a2b9, 5a4b1, 7a6b3, 9a8b5\}_{ab}$,
$B = \{0c1d8, 2c3d0, 4c5d2, 6c7d4, 8c9d6\}_{cd}$,
$C = \{1e0f9, 3e2f1, 5e4f3, 7e6f5, 9e8f7\}_{ef}$,
$D = \{8g1h3, 3g4h0, 0g6h5, 5g7h2, 2g9h8\}_{gh}$,
$E = \{6i1j5, 5i2j8, 8i3j7, 7i4j0, 0i9j6\}_{ij}$,
$F = \{ajbef, agbfc; ifjde, igjhd; hcdbi, hfdge; aicje, abcgh\}$ and
$G = \{49618, 21436, 27409\}\ \mathcal{R}\ G' = \{43618, 27496, 21409\}$.
From the exchanges in A, B, C, D and E, as well as the four exchanges of two cycles in F, $\{3, 4, \ldots, 34, 36\} \subseteq I(K_{20})$. Replacing G with G', $\{0, 1, \ldots, 31, 33\} \subseteq I(K_{20})$. Combining these we obtain the required intersection values.

$I(K_{21}) = J(21)$
Intersection problem has been solved for pentagon systems [5].

$I(K_{22}) = J(22)$
From [1] $K_{22} = 13 \cdot H + 5 \cdot C_5$, so an $MP(22)$ consists of the 5-cycles from the decomposition of the copies of H given in [1] and the five extra 5-cycles $\{0g7al, 12eh7, 1b4hk, 25a8l, 5c8gi\}$. Since $I(H) = \{0, 1, 3\}$, $\{5, 6, \ldots, 42, 44\} \subseteq I(K_{22})$. Replacing the other cycles by $\{0g8al, 12ehk, 17h4b, 25c8l, 5a7gi\}$, $\{0, 1, \ldots, 37, 39\} \subseteq I(K_{22})$. Combining these we obtain the required intersection values.

$I(K_{23}) = J(23)$
From [1] $MP(23) = A + B + C$, where
$A = \{01756, 04735, 0b7da; 14203, 1l2b5, 1j2i8; 07j98, 0djb9, 0hj6c;$
$hbm0k, hem48, h5mi4; 7hcd2, 7gc46, 7mca9; 1elga, 1hl6b, 1mlck\}$
$B = \{0f6ej, 0g6kl; 3ba82, 3ja68; 12kgd, 1ik3f; 5261c, 596d8; akm9l, ahmji;$
$2c3mf, 2h3dm; 36mg9, 3am8g; 4dbf9, 4lbkf; j4kdl, j5k9c; h9i5d, h6idf;$
$43e2a, 4bea5; f8k7a, fjke7; 8bgi7, 8jgfe\}$ and

$C = \{0ecfi, l8cbi, l7ci3, 92ged, 91g4e, ihg5e\}\ \mathcal{R}$

$C' = \{0ecbi, l7cfi, l8ci3, 91ged, 92g5e, ihg4e\}$.

A has been grouped into six exchanges of three cycles and B into 13 exchanges of two cycles. From these exchanges $\{6, 7, \ldots, 48, 50\} \subseteq I(K_{23})$. Replacing C with C', $\{0, 1, \ldots, 42, 44\} \subseteq I(K_{23})$. Combining these we obtain the required intersection values.

$I(K_{24}) = J(24)$

From [6] $MP(24) = A + B + C + D + E + F$ where

$A = \{k12l6, k23l7, k34l8, k40l9, kbclg, kdeli, kealj\}_{kl}$,

$B = \{m02n5, m24n6, m41n7, m13n8, m30n9, macnf, mebnh, mbdni, mdanj\}_{mn}$,

$C = \{081a9, 0c15d, 0e17f, 0g19h, 0i1bj\}_{01}$,

$D = \{56789, 57968, fghij, fhjgi\}$,

$E = \{fkabl, hkcdl, 8aieh\}\ \mathcal{R}\ E' = \{h8abl, fkcdl, kaieh\}$ and

$F = \{0b53a, 0k5l1; 1d25f, 1h28j; 2936b, 2c37e, 2f3dg; d8g6j, d9gef; i2j54, i3ja5, i9j7c; 607a2, 6d7ga; 3b46e, 3g47h; 4ahb8, 4dh5c; 4fc8e, 4jc6h; engmc, e5gc9; f6ib9, f8i7b\}$.

From the exchanges in A, B and C, the two $MP(5)$s in D and the exchanges of two and three cycles in F, $\{3, 4, \ldots, 50, 52\} \subseteq I(K_{24})$. Replacing E with E', $\{0, 1, \ldots, 47, 49\} \subseteq I(K_{24})$. Combining these we obtain the required intersection values.

$I(K_{25}) = J(25)$

Intersection problem has been solved for pentagon systems [5].

$I(K_{26}) = J(26)$

From [6] $MP(26) = A + B + C + D + E + F$ where

$A = \{k12l6, k23l7, k34l8, k40l9, kablf, kbclg, kcdlh, kdeli, kealj\}_{kl}$,

$B = \{m56n0, m67n2, m78n4, m89n1, m95n3, mfgna, mghnc, mhine, mijnb, mjfnd\}_{mn}$,

$C = \{o02p5, o24p7, o41p9, o13p6, o30p8, oacpf, oceph, oebpj, obdpg, odapi\}_{op}$,

$D = \{081a9, 0c15d, 0e17f, 0g19h, 0i1bj\}_{01}$,

$E = \{klmno, kmpln\}$ and

$F = \{0b53a, 0k5l1; 1d25f, 1h28j; 2936b, 2c37e, 2f3dg; d8g6j, d9gef; i2j54, i3ja5, i9j7c; f6ib9, f8i7b; ieh8a, ifhjg; 607a2, 6d7ga; 3b46e, 3g47h; 4ahb8, 4dh5c; 4fc8e, 4jc6h; 57968, 5e9cg\}$.

From the exchanges in A, B, C and D, as well as the $MP(6)$ in E, $\{26, 27, \ldots, 60, 62\} \subseteq I(K_{26})$. F has been grouped into exchanges of two and three cycles, so $\{0, 1, \ldots, 24, 26\} \subseteq I(F)$. Combining these we obtain the required intersection values.

$I(K_{27}) = J(27)$

From [1] $MP(27) = A + B + C$, where

$A = \{blc20, bpcma; 0562d, 0p6mj; 06cdl, 0ocaq; 70hep, 7mh8e; 90g2e, 9mgpi;$

$0e48m, 0k45n; 2ln81, 2bnkq; 1m2k3, 172ni; 4oi7b, 4cikh; jal1c, jelgh;$
$i2jl8, iqjke; 4anh3, 49npj; 37ahc, 3fagi; 3apqd, 3mpdb; g6lh9, gklij;$
$k1qmb, kcq46; 9bq52, 9lqg7; 76q3l, 7hqn4; 5c8q7, 5g8p9; fdh5b, fohb8;$
$2ai0f, 24i6h; d1p28, d4p3n; eno7d, egodm; 193e6, 1g30a; 1nc04, 1bcfh;$
$ebgf1, ecg4f; 789fj, 7c9qf\},$
$B = \{j1o36, j9o23, jboa8; 58oqe, 5joea, 5lokf; mopl4, m5phi, mkpfl\}$ and
$C = \{8k510, 835o6, bi5d6, 6adif, 69djn, 7kdgn\}\ \mathcal{R}$
$C' = \{83510, 8k5d6, bi5o6, 69dif, 6adgn, 7kdjn\}.$
A has been grouped into 27 exchanges of two cycles and B into three exchanges of three cycles. From these exchanges $\{6, 7, \ldots, 67, 69\} \subseteq I(K_{27})$. Replacing C with C', $\{0, 1, \ldots, 61, 63\} \subseteq I(K_{27})$. Combining these we obtain the required intersection values.

$I(K_{28}) = J(28)$
From [6] $MP(28) = A + B + C + D + E + F + G$ where
$A = \{k12l6, k23l7, k34l8, k40l9, kablf, kbclg, kcdlh, kdeli, kealj\}_{kl},$
$B = \{m56n0, m67n2, m78n4, m89n1, m95n3, mfgna, mghnc, mhine,$
$mijnb, mjfnd\}_{mn},$
$C = \{o02p5, o24p7, o41p9, o13p6, o30p8, oacpf, oceph, oebpj, obdpg, odapi\}_{op},$
$D = \{q57r0, q79r1, q96r2, q68r3, q85r4, qhjrb, qgird\}_{qr},$
$E = \{081a9, 0c15d, 0e17f, 0g19h, 0i1bj\}_{01},$
$F = \{0b53a, 0k5l1; 1d25f, 1h28j; 2936b, 2c37e, 2f3dg; d8g6j, d9gef; i2j54,$
$i3ja5, i9j7c; 607a2, 6d7ga; 3b46e, 3g47h; 4ahb8, 4dh5c; 4fc8e, 4jc6h;$
$a8hei, arhfq; f6ib9, f8i7b; qkrmp, qlrom; okprn, olpnq\}$ and
$G = \{9e5gc, qcrgj, qerfi\}\ \mathcal{R}\ G' = \{9ergc, qe5gj, qcrfi\}.$
From the exchanges in A, B, C, D and E, as well as the exchanges of two and three cycles in F, $\{3, 4, \ldots, 70, 72\} \subseteq I(K_{28})$. Replacing G with G', $\{0, 1, \ldots, 67, 69\} \subseteq I(K_{28})$. Combining these we obtain the required intersection values.

$I(K_{29}) = J(29)$
From [1] $MP(29) = A + B$, where
$A = \{1jc02, 17c4k; 09irl, 0niar; bki0f, bci3l; 1q3k9, 1a3js; 3rkh1, 3ckm7;$
$48pji, 4lpnf; g7qs6, grqis; e2imp, elifo; 3gc1m, 3nc8e; ach35, amh7l;$
$45l9p, 4hlnd; 81brd, 8ibdh; kdqh2, kgq4o; 7sh6d, 7ehgb; g1rm4, gfr4j;$
$lorn1, ljr9s; 67asm, 68ank; 25qk8, 29qf6; 7ksf2, 7ps3f; m2rpc, merco;$
$8fkp3, 8jkab; b3dc9, bsdgm; 7r8gi, 7n89o; ecldi, eklmj; m8opd, mnod9;$
$o6pg5, obp5i; 6rh93, 6bhpi; e4nqa, ebn6q\}$ and
$B = \{01o34, 0jo23, 0soqp; 06578, 0b5mq, 0k5fm; 059ed, 0796e, 0a9ng;$
$58se1, 5rs2n, 5csne; 2461p, 2a6jb, 2l6cq; dahi1, d5hj2, dfhnj;$
$47jf1, 49j5s, 4ajqb; f9g2c, flgoa, fegap\}.$
A has been grouped into 28 exchanges of two cycles and B into eight exchanges of three cycles. Combining these we obtain the required intersection values.

$I(K_{30}) = J(30)$
From [6] $MP(30) = A+B+C+D+E+F+G+H$ where
$A = \{k12l6, k23l7, k34l8, k40l9, kablf, kbclg, kcdlh, kdeli, kealj\}_{kl}$,
$B = \{m56n0, m67n2, m78n4, m89n1, m95n3, mfgna, mghnc, mhine, mijnb, mjfnd\}_{mn}$,
$C = \{o02p5, o24p7, o41p9, o13p6, o30p8, oacpf, oceph, oebpj, obdpg, odapi\}_{op}$,
$D = \{q57r0, q79r1, q96r2, q68r3, q85r4, qhjrb, qjgrc, qgird, qifre\}_{qr}$,
$E = \{s06t1, s62t3, s2at4, sa7t8, s70tb, s5etd, se9tf, s9cth, scgti, sg5tj\}_{st}$,
$F = \{081a9, 0c15d, 0e17f, 0g19h, 0i1bj\}_{01}$,
$G = \{0b53a, 0k5l1; 1d25f, 1h28j; 2936b, 2c37e, 2f3dg; d8g6j, d9gef, d7ga6; a8hei, arhfq; i2j54, i3ja5, i9j7c; 3b46e, 3g47h; 4ahb8, 4dh5c; 4fc8e, 4jc6h; f6ib9, f8i7b\}$ and
$H = \{kpqtm, psqlm, rslom, otknq, snork, kosmq, nprtl, nrlpt\}\ \mathcal{R}$
$H' = \{kpqlm, rsqtm, pslom, otkqn, nsork, koqms, npltr, ntprl\}$.
From the exchanges in A, B, C, D, E and F, as well as the exchanges of two and three cycles in G, $\{8, 9, \ldots, 82, 84\} \subseteq I(K_{30})$. Replacing H with H', $\{0, 1, \ldots, 74, 76\} \subseteq I(K_{30})$. Combining these we obtain the required intersection values.

$I(K_{31}) = J(31)$
Intersection problem has been solved for pentagon systems [5].

$I(K_{32}) = J(32)$
From [1] $K_{32} = 32 \cdot H$, so an $MP(32)$ consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, we obtain the required intersection values.

B Proof of Lemma 2.2

For $t = 0, 1, 2$, an $MP(K_{10+t} \setminus K_t)$ is an $MP(K_{10+t})$, and thus $I(K_{10+t} \setminus K_t) = J(10+t)$.

$I(K_{13} \setminus K_3) = \{0, 1, \ldots, 13, 15\}$
From [1] $K_{13} \setminus K_3 = 5 \cdot H$, so an $MP(K_{13} \setminus K_3)$ consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, the result follows.

$I(K_{14} \setminus K_4) = \{0, 1, \ldots, 14, 16\}$
From [1] $MP(K_{14} \setminus K_4) = A+B+C$, where
$A = \{07165, 04159; 1b4a9, 1849d; 2d3c5, 24358; ca396, c7368; 27d4c, 26db9\}$,
$B = \{b38d0, b2ad5, b6a57\}$ and
$C = \{08746, c0a79, 1cb8a\}\ \mathcal{R}\ C' = \{0a746, b879c, 08a1c\}$.
A has been grouped into five exchanges of two cycles, while B can be

replaced by $B' = \{b38d5, b2ad0, b6a57\}$ or by $B'' = \{b38d5, b6ad0, b2a57\}$. From $I(A)$ and $I(B)$, $\{3, 4, \ldots, 14, 16\} \subseteq I(K_{14} \setminus K_4)$. Replacing C with C', $\{0, 1, \ldots, 11, 13\} \subseteq I(K_{14} \setminus K_4)$. Combining these we obtain the required intersection values.

$I(K_{15} \setminus K_5) = \{0, 1, \ldots, 17, 19\}$
From [1] $MP(K_{15} \setminus K_5) = A + B$, where
$A = \{07168, 0518a; 1b2ac, 192ce; 2d387, 2537e; 3c5ea, 395ab; 4b7ad, 467d9;$
$60975, 6c9e3; 9bd1a, 98de6; 5d628, 5b6a4\}$ and
$B = \{c748b, c4eb0, c8e0d\}$.
A has been grouped into eight exchanges of two cycles, so $\{0, 2, 4, \ldots, 14, 16\} \subseteq I(A)$. B can be replaced by $B' = \{c748b, c8eb0, c4e0d\}$ or by $B'' = \{c74e8, c48b0, cd0eb\}$, so $I(B) = \{0, 1, 3\}$. Combining these we obtain the required intersection values.

$I(K_{16} \setminus K_6) = \{0, 1, \ldots, 18, 20\}$
From [4] $MP(K_{16} \setminus K_6) = A + B + C + D$ where
$A = \{a17b0, a39b2, a51b4, a73b6, a95b8\}_{ab}$,
$B = \{c08d1, c20d3, c42d5, c64d7, c86d9\}_{cd}$,
$C = \{e19f0, e31f2, e53f4, e75f6, e97f8\}_{ef}$ and
$D = \{83052, 49618, 65870, 21436, 09274\}\ \mathcal{R}$
$D' = \{83092, 43658, 61870, 21496, 05274\}$.
From the exchanges in A, B and C, $\{5, 6, \ldots, 18, 20\} \subseteq I(K_{16} \setminus K_6)$. Replacing D with D', $\{0, 1, \ldots, 13, 15\} \subseteq I(K_{16} \setminus K_6)$. Combining these we obtain the required intersection values.

$I(K_{17} \setminus K_7) = \{0, 1, \ldots, 21, 23\}$
From [1] $MP(K_{17} \setminus K_7) = A + B + C$, where
$A = \{0918a, 071ac; cbdf1, c2d1e; b1ge2, b3gc8; 2f3a9, 2739g; 3c4be, 384ed;$
$0b978, 0e9cd; 698ef, 6d8gb\}$,
$B = \{4ag7d, 4g5d9, 4759f\}$ and
$C = \{685fc, 6e5c7, ea5b7, g6af0, gdabf, 82a7f\}\ \mathcal{R}$
$C' = \{6e5fc, 685b7, ea5c7, gdaf0, g6a7f, 82abf\}$.
A has been grouped into seven exchanges of two cycles, while B can be replaced by $B' = \{4ag7d, 475d9, 4g59f\}$ or by $B'' = \{4ag59, 4g75d, 47d9f\}$. From $I(A)$ and $I(B)$, $\{6, 7, \ldots, 21, 23\} \subseteq I(K_{17} \setminus K_7)$. Replacing C with C', $\{0, 1, \ldots, 15, 17\} \subseteq I(K_{17} \setminus K_7)$. Combining these we obtain the required intersection values.

$I(K_{18} \setminus K_8) = \{0, 1, \ldots, 22, 24\}$
From [6] $MP(K_{18} \setminus K_8) = A + B + C + D + E$ where
$A = \{c24f6, c41f7, c13f8, c30f9\}_{cf}$,
$B = \{g28hc, g80h4, g05h1, g53hf, g32h7\}_{gh}$,
$C = \{a45bc, a52b1, a27b3, a70bf, a04b8\}_{ab}$,
$D = \{dc0e1, d02e3, d2fe4, df5e7, d5ce8\}_{de}$ and

$E = \{17492, 16439, 06381, 56789, 57968\}\ \mathcal{R}$

$E' = \{16492, 17439, 06781, 58369, 56897\}$.

From the exchanges in A, B, C and D, $\{5, 6, \ldots, 22, 24\} \subseteq I(K_{18} \setminus K_8)$. Replacing E with E', $\{0, 1, \ldots, 17, 19\} \subseteq I(K_{18} \setminus K_8)$. Combining these we obtain the required intersection values.

$I(K_{19} \setminus K_9) = \{0, 1, \ldots, 25, 27\}$

From [1] $K_{19} \setminus K_9 = 9 \cdot H$, so an $MP(K_{19} \setminus K_9)$ consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, the result follows.

C Proof of Lemma 2.3

$I(K_{5(3)}) = \{0, 1, \ldots, 13, 15\}$

From [1] $K_{5(3)} = 5 \cdot H$, so an $MP(K_{5(3)})$ consists of the 5-cycles from the decomposition of the copies of H given in [1]. Since $I(H) = \{0, 1, 3\}$, the result follows.