A Note on MAD Spanning Trees

Peter Dankelmann
Department of Mathematics
University of Natal
Durban 4041, South Africa

ABSTRACT. The question whether every connected graph G has a spanning tree T of minimum average distance such that T is distance preserving from some vertex is answered in the negative. Moreover it is shown that, if such a tree exists, it is not necessarily distance preserving from a median vertex.

Over the past two decades the average distance of graphs has sparkled considerable interest (see e.g. [2, 3, 7, 8]). One particular problem that received some attention is that of finding a minimum average distance (MAD) spanning tree of a given connected graph G, i.e. a spanning tree T such that T has minimum average distance among all spanning trees of G. It was proved by Johnson, Lenstra, and Rinnooy-Kan [6] that this problem is NP-complete. Entringer, Kleitman, and Szekely [4] showed that every connected graph G contains a spanning tree T that is distance preserving from some vertex in G such that the average distance of T is less than twice the average distance of G. This prompted the following questions by Entringer [5]:

1. Does every graph have a MAD spanning tree that is also distance preserving from some vertex?

2. If the graph G has a MAD spanning tree T that is distance preserving, is it distance preserving from a median vertex?

In this note it is shown that both questions have a negative answer.

We employ the notation and terminology of [1]. In particular, if G is a connected graph with vertex set V, then the *distance* of G, $d(G)$, is defined as

$$d(G) = \sum_{\{v,w\} \subset V} d_G(v, w),$$

JCMCC 32 (2000), pp. 93-95

where $d_G(v,w)$ denotes the distance between v and w. The *average distance* of G, $\mu(G)$, is the average of all distances in G, i.e. $\mu(G) = \binom{|V|}{2}^{-1} \sum_{|\{v,w\}\subset V} d_G(v,w)$.

Definition 1. *For positive integers k, r let H be the graph obtained from a cycle $C = (v_1, v_2, \dots, v_{2k-1}, v_1)$ and $2r$ vertices $w_1, w_2, \dots, w_r$, $x_1, x_2, \dots, x_r$ by joining the w_i with v_k and x_i with v_{k+1} for $i = 1, \dots, r$. Let H' be a disjoint copy of H with vertex set $\{v_1', \dots, v_{2k-1}', w_1', \dots, w_r', x_1', \dots, x_r'\}$. Define $G(k,r)$ as the graph obtained from the union of H and H' by adding the edge v_1v_1'.*

Theorem 1. *Let v be a vertex of $G(k,r)$ and let T be a spanning tree distance preserving from v. Then for $k \geq 3$ and $r \geq 6$, T is not a MAD spanning tree.*

Proof: Without loss of generality we can assume that v is in H'. Then T does not contain the edge $e = v_kv_{k+1}$ but the edge $f = v_{k+1}v_{k+2}$. Let $T_1 = T - f + e$. It is easy to verify that T_1 is also a spanning tree of $G(k,r)$.

We now prove that replacing the edge f by the edge e decreases the distance of T (and thus T is not a MAD spanning tree). The only distances that are changed are distances between the w_i and the x_j (each goes down by $2k-3$), between w_i and v_{k+1} (down by $2k-3$), between each vertex in $\{x_i\} \cup \{v_{k+1}\}$ and every vertex in H' (up by 1). It is easy to check that all the other changes in the distances cancel out. Hence we have, for $k \geq 3$ and $r \geq 6$

$$\begin{aligned} d(T_1) &= d(T) - r^2(2k-3) - r(2k-3) + (r+1)(2k-1+2r) \\ &= d(T) - (r+1)((2k-5)(r-1)-4) \\ &< d(T). \end{aligned}$$

Thus T is not a MAD spanning tree. □

Definition 2. *Let $H = H(k,r)$ be the graph obtained from a cycle $C = (v_1, v_2, \dots, v_{2k-1}, v_1)$ of order $2k-1$ and $5r$ isolated vertices $x_1, x_2, \dots, x_r$, $y_1, y_2, \dots, y_r$, $z_1, z_2, \dots, z_{3r}$, by joining x_i to v_k and y_i to v_{k+1} for $i = 1, \dots, r$, and z_j to v_1 for $j = 1, \dots, 3r$.*

Theorem 2. (i) *The graph H contains a MAD spanning tree which is distance preserving from some vertex.*

(ii) *Let T be a spanning tree of H distance preserving from a vertex of minimum distance in H. Then T is not a MAD spanning tree.*

Proof: (i) Every spanning tree of H is obtained by deleting a cycle edge e. It is easy to see that it is distance preserving from the vertex on the cycle opposite e. Hence H has a MAD spanning tree which is distance preserving from some vertex.

(ii) It is easy to check that the median of H consists only of v_1. Let T be the spanning tree of H distance preserving from v_1. Then $T = H - e$ where $e = v_k v_{k+1}$. We now show that the spanning tree $T_1 = H - f$ with $f = v_{k+1}v_{k+2}$, which is distance preserving from v_2, has a lower distance than T, hence T is not a MAD spanning tree.

As above we show that replacing the edge f in T by the edge e decreases the distance of T (and thus T is not a MAD spanning tree). The only distances that are changed are distances between the x_i and the y_j (each goes down by $2k-3$), between x_i and v_{k+1} (down by $2k-3$), between the vertices in $\{y_i\} \cup \{v_{k+1}\}$ and z_j (up by 1). It is easy to check that all the other changes in the distances cancel out. Hence we have, for $k \geq 4$

$$\begin{aligned} d(T_1) &= d(T) - r^2(2k-3) - r(2k-3) + (r+1)3r \\ &= d(T) - r(r+1)(2k-6) \\ &< d(T). \end{aligned}$$

Thus T is not a MAD spanning tree. □

References

[1] G. Chartrand and O.R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill, Inc., New York (1993).

[2] P. Dankelmann, O.R. Oellermann and H.C. Swart, The average Steiner distance of a graph, *J. Graph Theory* **22** (1996), 15–22.

[3] P. Dankelmann, O.R. Oellermann, and H.C. Swart, On three conjectures of GRAFFITI, *Journal of Combinatorial Mathematics and Combinatorial Computing* (to appear).

[4] R.C. Entringer, D. Kleitman, and L. Szekely, A note on spanning trees with minimum average distance, *Bull. Inst. Combin. Appl.* **17** (1996), 71–78.

[5] R.C. Entringer, Distance in graphs: Trees, *J. Combin. Math. Combin. Comput.* **24** (1997), 65–84.

[6] D.S. Johnson, J.K. Lenstra, and A.H.G. Rinnooy-Kan, The complexity of the network design problem, *Networks* **8** (1978), 279–285.

[7] M. Kouider and P. Winkler, Mean distance and minimum degree, *J. Graph Theory* **25** (1997), 95–99.

[8] J. Plesnik, On the sum of all distances in a graph or digraph, *J. Graph Theory* **8** (1984), 1–21.