Some New Direct Constructions for $(v, \{5, w^*\}, 1)$ PBDs

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ABSTRACT. For $w \leq 33$, the known necessary conditions for existence of a $(v, \{5, w^*\}, 1)$ PBD, namely $v, w \equiv 1 \bmod 4$, $v \geq 4w + 1$ and $v \equiv w$ or $4w + 1 \bmod 20$ are known to be sufficient in all but 26 cases. This paper provides several direct constructions which reduce the number of exceptions to 8.

1 Introduction

A design is a pair $(X, \mathcal{A})$ where X denotes a set of points of finite cardinality, v, and $\mathcal{A}$ is a family of subsets of X, called *blocks*. If v is the total number of points in the design, K is a set of positive integers such that such that each block has a size from K, and if every pair of points appears in one block, then such a design is called a *pairwise balanced design*, or a $(v, K, 1)$ PBD. In addition, a PBD on v points with one block of size w and all other blocks sizes from K is called a $(v, K \cup \{w^*\}, 1)$ PBD. Further blocks of size w are possible if $w \in K$.

Another useful type of design that we use is a *group divisible design* or a K-GDD of type $(h_1)^{u_1}, (h_2)^{u_2} \ldots (h_m)^{u_m}$. Here the block sizes come from K; and the point set X is partionable into u_1 subsets of size h_1, u_2 subsets of size $h_2 \ldots u_m$ subsets of size h_m. In addition, 2 points appear in exactly 1 block if they are in different groups and no blocks if they are in the same group.

JCMCC 32 (2000), pp. 97-102

A *parallel class* of blocks in a design is a set of blocks in which each point appears exactly once. A design whose blocks can be partitioned into parallel classes is called *resolvable*.

This paper is mainly concerned with the existence of $(v, \{5\} \cup \{w^*\}, 1)$ PBDs (also denoted as $(v, \{5, w^*\}, 1)$ PBDs). These designs have a close relation to several other combinatorial structures; for instance, in [6], they are used to construct several group divisible designs with block size 5. The existence of $(v, \{5, w^*\}, 1)$ PBDs was investigated in [1] for $w \in \{21, 25\}$ and in [4] for $w \in \{9, 13\}$; later in [3], a more thorough investigation was done for $w \leq 33$. Known necessary conditions for existence of a $(v, \{5, w^*\}, 1)$ PBD are $w \equiv 1 \bmod 4$ and $v \equiv w$ or $4w+1 \bmod 20$. The following theorem summarises the known results quoted in [3] for $w \leq 33$.

Theorem 1.1 *A $(v, \{5, w^*\}, 1)$ PBD exists if either:*

1. $w = 9$, $v \equiv 9, 17 \bmod 20$, *and* $v \geq 37$ *except possibly for* $v = 49$.
2. $w = 13$, $v \equiv 13 \bmod 20$, *and* $v \geq 53$.
3. $w = 17$, $v \equiv 9, 17 \bmod 20$, *and* $v \geq 69$, *except possibly for* $v \in \{77, 89, 109, 129, 137, 149, 169, 189, 209, 229, 249, 269, 289\}$.
4. $w = 21$, $v \equiv 1, 5 \bmod 20$, *and* $v \geq 85$, *except possibly for* $v = 125$.
5. $w = 25$, $v \equiv 1, 5 \bmod 20$, *and* $v \geq 101$, *except possibly for* $v = 141$.
6. $w = 29$, $v \equiv 9, 17 \bmod 20$, *and* $v \geq 117$, *except possibly for* $v \in \{137, 157, 177, 217, 237, 277, 337, 397, 417\}$.
7. $w = 33$, $v \equiv 13 \bmod 20$, *and* $v \geq 133$, *except possibly for* $v = 153$.

In addition it is proved in [3] that if $v, w \equiv 1$ or $5 \bmod 20$, $w \geq 21$, and $v \geq 5w-4$, then there exists a $(v, 5, 1)$ BIBD containing a sub-$(w, 5, 1)$ BIBD except possibly for $(v, w) \in \{(125, 21), (141, 25), (425, 65)\}$. However, a $(425, \{5, 65^*\}, 1)$ PBD in obtainable using the 5-GDD of type 60^7 given in [6]: construct a $(65, 5, 1)$ BIBD on each group of this GDD (except the last) plus 5 infinite points (in each case with 1 block on infinite points), delete the blocks containing the infinite points, and form a block of size 65 on the last group plus the infinite points. A $(125, \{5, 21^*\}, 1)$ PBD is given later in lemma 2.1. Thus $(141, 25)$ remain the only unknown case.

In this paper, we reduce the above lists of unknown $(v, \{5, w^*\}, 1)$ PBDs, especially for $w = 17$ and 29. Specifically, we obtain the following improvement on Theorem 1.1:

Theorem 1.2 *Suppose $v, w \equiv 1 \bmod 4$, $v \geq 4w+1$, $w \leq 33$ and $v \equiv w$ or $4w+1 \bmod 20$. Then a $(v, \{5, w^*\}, 1)$ PBD exists, except possibly for $(v,w) \in \{(49,9), (77,17), (89,17), (137,17), (141,25), (137,29), (397,29), (153,33)\}$.*

2 The Constructions

All the constructions in this paper are direct difference type constructions. We start with those which make use of the cyclic group Z_{v-w}.

Lemma 2.1 *A $(v, \{5, w^*\}, 1)$ PBD exists for the following values of v and w:*

1. *$w = 17$ and $v \in \{129, 149, 169, 189, 209, 229, 249, 289\}$.*
2. *$w = 21$ and $v = 125$.*
3. *$w = 29$ and $v \in \{157, 217, 237, 277\}$.*

Proof: In each case we take $X = Z_{v-w}$ plus w infinite points. We give a number of blocks of size 4 or 5 which should be cycled mod g for $g = v - w$. With just one exception (the block $\{0, 4, 44, 72\}$ for $(v, w) = (157, 29)$) all values in any given block of size 4 are distinct mod 4; this ensures that the translates of each such block can be partitioned into 4 parallel classes. For $(v, w) = (157, 29)$, the translates of $\{0, 4, 44, 72\}$ mod 128 can also be partitioned into 4 parallel classes; more generally, if u is a power of 2 such that $4u$ divides g and the elements of a block B of size 4 equal $0, u, 2u$ and $3u \bmod 4u$, then one parallel class can be obtained by adding the values $4ux + y$ $(0 \leq x \leq (g/u) - 1,\ 0 \leq y \leq u - 1)$ to B. Three further block-disjoint parallel classes are then obtained by adding $u, 2u$ and $3u$ to the blocks in this parallel class. In all cases there is one base block of size 4 of the form$\{0, h, 2h, 3h\}$ for $h = g/4$; the translates of this block form one parallel class on the non-infinite points. All base blocks including this short one, are given below. Finally, each infinite point should be added to a parallel class generated by one of the base blocks of size 4. □

v	w	Base Blocks
125	21	$\{0,4,28,40,60\}, \{0,8,53,69,90\}, \{0,1,3,10\}, \{0,5,11,38\}, \{0,13,30,55\}, \{0,15,46,65\}, \{0,18,41,75\}, \{0,26,52,78\}$
129	17	$\{0,4,24,36,66\}, \{0,8,48,73,99\}, \{0,16,43,60,97\}, \{0,1,3,10\}, \{0,5,11,34\}, \{0,14,49,67\}, \{0,19,41,74\}, \{0,28,56,84\}$

149 17 {0,1,4,9,22}, {0,12,29,61,102}, {0,16,50,52,76}, {0,20,48,88,125},
{0,6,25,87}, {0,10,53,67}, {0,11,46,85}, {0,15,38,69}, {0,33,66,99}

169 17 {0,1,4,9,22}, {0,2,12,29,45}, {0,20,64,70,100}, {0,24,56,84,103},
{0,40,54,93,141}, {0,7,62,85}, {0,15,61,86}, {0,26,57,115},
{0,34,69,111}, {0,38,76,114}

189 17 {0,4,16,36,64}, {0,8,47,84,130}, {0,2,40,58,103}, {0,24,51,57,68},
{0,21,73,80,102}, {0,5,72,95,98}, {0,1,31,118}, {0,9,19,34},
{0,13,66,107}, {0,35,49,110}, {0,43,86,129}

209 17 {0,4,16,44,76}, {0,8,43,88,130}, {0,36,92,115,117}, {0,20,21,73,166},
{0,50,68,83,134}, {0,5,24,34,91}, {0,9,64,78,127}, {0,7,13,110},
{0,17,71,102}, {0,37,59,98}, {0,3,30,41}, {0,48,96,144}

229 17 {0,4,36,64,104}, {0,16,63,92,190}, {0,12,42,96,103}, {0,20,72,87,113},
{0,24,105,154,193}, {0,6,8,56,185}, {0,13,44,115,166}, {0,1,10,75,80},
{0,17,86,111}, {0,21,66,155}, {0,3,14,37}, {0,55,73,150}, {0,53,106,159}

249 17 {0,4,32,92,100}, {0,16,80,91,178}, {0,40,84,135,147}, {0,72,115,124,149},
{0,24,38,65,166}, {0,17,18,48,171}, {0,5,76,118,187}, {0,7,13,112,186},
{0,20,35,56,82}, {0,29,102,151}, {0,23,33,126}, {0,37,39,94}, {0,3,22,89},
{0,58,116,174}

289 17 {0,4,48,104,124}, {0,8,40,51,82}, {0,12,92,129,147}, {0,28,142,167,270},
{0,24,45,108,239}, {0,36,85,132,231}, {0,72,88,111,138}, {0,1,7,60,122},
{0,17,52,90,171}, {0,9,116,183,186}, {0,54,64,69,166}, {0,13,71,162},
{0,29,75,94}, {0,61,83,246}, {0,79,93,238}, {0,68,136,204}

157 29 {0,12,20,36,103}, {0,7,34,48,86}, {0,4,44,72}, {0,1,6,111}, {0,9,35,74},
{0,3,13,98}, {0,19,21,78}, {0,29,51,82}, {0,11,58,73}, {0,32,64,96}

217 29 {0,4,56,76,84}, {0,12,48,51,134}, {0,16,35,60,130}, {0,11,32,82,100},
{0,24,64,79,171}, {0,1,6,151}, {0,9,22,87}, {0,29,63,90}, {0,7,33,146},
{0,31,45,142}, {0,23,53,126}, {0,57,59,178}, {0,47,94,141}

237 29 {0,4,68,88,200}, {0,16,60,91,174}, {0,24,39,56,97}, {0,28,35,82,100},
{0,36,95,128,141}, {0,40,130,155,178}, {0,1,10,147}, {0,81,107,170},
{0,21,87,98}, {0,27,29,186}, {0,37,43,122}, {0,3,45,102}, {0,5,19,74},
{0,52,104,156}

277 29 {0,4,16,40,108}, {0,8,80,83,174}, {0,76,85,99,128}, {0,56,100,121,237},
{0,28,35,86,88}, {0,32,54,102,247}, {0,20,39,84,89}, {0,47,96,126,238},
{0,13,31,118}, {0,17,63,170}, {0,25,98,135}, {0,15,41,134},
{0,27,61,158}, {0,71,77,210}, {0,81,123,182}, {0,62,124,186}

It should also be noted that if 10 infinite points are removed from the $(125, \{5, 21^*\}, 1)$ PBD, then the result is a $(115, \{4, 5, 11^*\}, 1)$ PBD. This PBD was given as unknown in [2] and [5] Also in [5], it was mistakenly stated that a $(v, \{4, 5, 11^*\}, 1)$ PBD was unknown for $v = 139$; the correct unknown value is $v = 130$. In fact, a $(139, \{4, 5, 11^*\}, 1)$ PBD is obtainable by deleting 18 infinite points from the $(157, \{5, 29^*\}, 1)$ PBD above.

Lemma 2.2 *1. If* $(v, w) \in \{(109, 17), (177, 29), (269, 17), (417, 29)\}$, *then there exists a* $(v, \{5, w^*\}, 1)$ *PBD.*

2. There exists a 5-GDD of type $44^7 28^1$ *and a* $(v, \{5, w^*\}, 1)$ *PBD for* $(v, w) = (337, 29)$.

Proof: The constructions for these designs (including the GDD in (2)) are like those in the previous lemma, except here we use a non-cyclic group of order $g = v - w$, namely $GF(4, x^2 = x+1) \times Z_{g/4}$. Here too (for the designs in (1)) there is one base block $\{(0,0), (1,0), (x,0), (x^2,0)\}$ whose translates form one parallel class on the non-infinite points. The base blocks are given below. No base block of size 4 contains 2 points with identical components from GF(4), so again, each such block generates 4 parallel classes on the non-infinite points. For the GDD in (2), one group consists of the 28 infinite points and the others are of the form $GF(4) \times \{u, u+7, u+14 \ldots u+70\}$ for $0 \le u \le 6$. A $(337, \{5, 29^*\}, 1)$ PBD is then obtainable from this design by adding a 29th infinite point and forming a $(45, 5, 1)$ BIBD on this point plus each group of size 44. □

v	w	Base Blocks
109	17	$\{(0,0),(0,4),(0,13),(0,16),(0,21)\}, \{(0,0),(1,3),(x,5),(x^2,16)\}$,
		$\{(0,0),(0,1),(1,2),(x,3),(x^2,9)\}, \{(0,0),(1,5),(x,9),(x^2,17)\}$,
		$\{(0,0),(1,4),(x,15),(x^2,5)\}, \{(0,0),(1,7),(x,4),(x^2,13)\}$,
		$\{(0,0),(1,0),(x,0),(x^2,0)\}$
177	29	$\{(0,0),(0,1),(0,3),(0,24),(1,22)\}$, (Multiply by $(1,1)$, $(x,10)$ and $(x^2,26)$)
		$\{(0,0),(1,y),(x,10y),(x^2,26y)\}$ for $y = 3, 7, 9, 24, 27, 29, 32$,
		$\{(0,0),(1,0),(x,0),(x^2,0)\}$
269	17	$\{(0,0),(0,10),(0,21),(0,34),(0,40)\}, \{(0,0),(0,1),(0,9),(1,39),(1,53)\}$,
		$\{(0,0),(0,12),(1,20),(1,40),(x,39)\}, \{(0,0),(1,21),(x,58),(x^2,31)\}$,
		$\{(0,0),(0,22),(1,37),(x,2),(x^2,49)\}, \{(0,0),(1,1),(x,4),(x^2,16)\}$
		(Multiply all these blocks except the first and last by $(x,4)$ and
		$(x^2,16)$ to give 8 further base blocks),
		$\{(0,0),(1,0),(x,0),(x^2,0)\}$
417	29	$\{(0,0),(0,16),(1,10),(x,36),(x,41)\}, \{(0,0),(0,15),(1,61),(x,84),(x^2,19)\}$,

$\{(0,0),(0,23),(0,41),(x,26),(x^2,79)\}, \{(0,0),(1,3),(1,15),(1,73),(1,74)\}$,
$\{(0,0),(0,9),(0,17),(1,25),(1,75)\}$ (Multiply all these blocks by
$(x,35)$ and $(x^2,61)$ to give 10 further base blocks)
$\{(0,0),((1,y),(x,35y),(x^2,61y)\}$ for $y = 1,2,5,7,13,19,30$,
$\{(0,0),(1,0),(x,0),(x^2,0)\}$

337 29 $\{(0,0),(0,37),(0,64),(x,5),(x^2,73)\}, \{(0,0),(1,4),(x,1),(x^2,24)\}$,
$\{(0,0),(0,54),(1,2),(1,38),(1,71)\}, \{(0,0),(1,30),(x,4),(x^2,52)\}$,
$\{(0,0),(0,12),(0,29),(1,32),(1,34)\}$ (Multiply these blocks by
$(x,23)$ and $(x^2,67)$ to give 10 further base blocks)
$\{(0,0),((1,1),(x,23),(x^2,67)\}$

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