Path-free domination

Teresa W. Haynes
Department of Mathematics
East Tennessee State University
Johnson City, TN 37614-0002 USA

Michael A. Henning *
Department of Mathematics
University of Natal
Private Bag X01
Pietermaritzburg, 3209 South Africa

Abstract

For $k \geq 2$, the P_k-free domination number $\gamma(G; -P_k)$ is the minimum cardinality of a dominating set S in G such that the subgraph $\langle S \rangle$ induced by S contains no path P_k on k vertices. The path-free domination number is at least the domination number and at most the independent domination number of the graph. We show that if G is a connected graph of order $n \geq 2$, then $\gamma(G; -P_k) \leq n + 2(k-1) - 2\sqrt{n(k-1)}$, and this bound is sharp. We also give another bound on $\gamma(G; -P_k)$ that yields the corollary: if G is a graph with $\gamma(G) \geq 2$ that is $K_{1,t+1}$-free and $(K_{1,t+1} + e)$-free ($t \geq 3$), then $\gamma(G; -P_3) \leq (t-2)\gamma(G) - 2(t-3)$ and we characterize the extremal graphs for the corollary's bound. Every graph G with maximum degree at most 3 is shown to have equal domination number and P_3-free domination number. We define a graph G to be P_k-domination perfect if $\gamma(H) = \gamma(H; -P_k)$ for every induced subgraph H of G. We show that a graph G is P_3-domination perfect if and only if $\gamma(H) = \gamma(H; -P_3)$ for every induced subgraph H of G with $\gamma(H) = 3$.

*Research supported in part by the National Research Foundation and the University of Natal

Dedicated to Prof. Ernie Cockayne on the occasion of his 60th birthday

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let v be a vertex in V. The *open neighbourhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighbourhood* of v is $N[v] = \{v\} \cup N(v)$. Let $N[v_1, v_2, \ldots, v_k]$ denote the closed neighborhood $N[\{v_1, v_2, \ldots, v_k\}]$ of the set $\{v_1, v_2, \ldots, v_k\}$. A path on k vertices is denoted by P_k. For other graph theory terminology, we follow [4].

A set $S \subseteq V$ is a *dominating set* if every vertex in V is either in S or is adjacent to a vertex in S; that is, $N[S] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of a dominating set in G that is independent. We refer to a minimum dominating set (respectively, minimum independent dominating set) as a γ-set (respectively, i-set) of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [4] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi and Slater [9, 10].

In this paper, we initiate the study of path-free domination in graphs. (This topic has been proposed for directed graphs and the only existing result is the complexity of this problem for directed graphs [1].) Let $k \geq 2$ be an integer. A *P_k-free dominating set* S of a graph G is a dominating set of G where the induced subgraph $\langle S \rangle$ contains no path on k vertices as a (not necessarily induced) subgraph. That is, if S is a P_k-free dominating set of G, then a longest path in $\langle S \rangle$ has order $k-1$. The *P_k-free domination number* $\gamma(G; -P_k)$ is the minimum cardinality of a P_k-free dominating set of G. We refer to a minimum P_k-free dominating set as a $(\gamma; -P_k)$-set.

The P_k-free domination number is a generalization of independent domination number $i(G)$; that is, if $k = 2$, then $\gamma(G; -P_k) = i(G)$.

This differs from the k-dependent number defined by Fink and Jacobson [6, 7] which is a generalization of the independence number $\beta(G)$. For $k \geq 0$, a *k-dependent set* is defined in [6, 7] as a set whose induced subgraph has maximum degree at most k while the *k-dependence number* $\beta_k(G)$ is defined as the maximum cardinality of a k-dependent set. In particular, we note that a P_3-free dominating set is precisely a 1-dependent set and that $\gamma(G; -P_3) \leq \beta_1(G)$.

If $k \geq (n+1)/2$, then, since every γ-set of a graph G on n vertices has cardinality at most $n/2$, $\gamma(G; -P_k) = \gamma(G)$. For $2 \leq k \leq n/2$,

$$\gamma(G) \leq \gamma(G; -P_k) \leq i(G).$$

Both sharpness and strict inequality can be achieved for the parameters in these inequalities. For example, consider a caterpillar. A *caterpillar* is a tree with the property that the removal of its endvertices results in a path. This path is referred to as the *spine* of the caterpillar. If the spine is P_t and each vertex on the spine is adjacent to r endvertices, then the caterpillar is denoted by $C(t, r)$. For $G = C(t, r)$, $t \geq k \geq 2$ and $x = t \pmod{k}$,

$$\gamma(G) = t \leq \gamma(G; -P_k) = (k-1)\lfloor t/k \rfloor + r\lfloor t/k \rfloor + x \leq i(G) = \gamma(G; -P_2).$$

If $t < k$, then $\gamma(G) = \gamma(G; -P_k) = t \leq i(G)$. Hence for each integer $m \geq 1$, there exists a tree T satisfying $\gamma(T; -P_3) - \gamma(T) > m$ and $i(T) - \gamma(T; -P_3) > m$.

Cockayne, Hedetniemi and Miller [3] introduced the following inequality chain involving domination parameters (parameter definitions not given here may be found in [9]):

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \qquad (1)$$

Since this chain first appeared in the literature in 1978, it has been the focus of more than 100 research papers and prompted many open questions such as:

(i) What are upper and lower bounds for each of the parameters in (1)?

(*ii*) Under what conditions are any of the parameters in (1) equal?

(*iii*) Are there other graph parameters whose values are related to those in (1)?

Our observation that for $2 \le k \le n/2$, $\gamma(G) \le \gamma(G;-P_k) \le i(G)$ gives an affirmative answer to Question (iii) by demonstrating where $\gamma(G;-P_k)$ fits in (1). In fact, for $2 \le j \le k \le n/2$, we have $\gamma(G) \le \gamma(G;-P_k) \le \gamma(G;-P_j) \le i(G)$. (Although we are limiting our investigation in this paper to the minimum cardinality of any P_k-free dominating set, we make the following observation. If we define the *upper P_k-free domination number* $\Gamma(G;-P_k)$ to be the maximum cardinality of any minimal P_k-free dominating set, then $ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G;-P_k) \le IR(G)$.)

For a partial answer to Question (i), we determine an upper bound on $\gamma(G;-P_k)$, which yields as a corollary a known bound on $i(G)$. That is, we show that if G is a connected graph of order $n \ge 2$, then $\gamma(G;-P_k) \le n + 2(k-1) - 2\sqrt{n(k-1)}$, and this bound is sharp. Another upper bound on $\gamma(G;-P_k)$ is given and its corollary shows that if G is a graph with $\gamma(G) \ge 2$ that is $K_{1,k+1}$-free and $(K_{1,k+1}+e)$-free ($k \ge 3$), then $\gamma(G;-P_3) \le (k-2)\gamma(G) - 2(k-3)$ and the extremal graphs are characterized.

Question (ii) has plagued researchers for years for parameters such as $\gamma(G)$ and $i(G)$. Since $i(G) = \gamma(G;-P_2)$, it is not surprising that finding conditions for which $\gamma(G) = \gamma(G;-P_k)$ is a difficult problem as well. We address this question for $k = 3$. In particular, we obtain a sufficient condition in terms of forbidden induced subgraphs for the domination number of a graph to equal its P_3-free domination number. As a consequence, every graph G with maximum degree at most 3 has equal domination number and P_3-free domination number. We define a graph G to be P_k-domination perfect if $\gamma(H) = \gamma(H;-P_k)$ for every induced subgraph H of G. We show that a graph G is P_3-domination perfect if and only if $\gamma(H) = \gamma(H;-P_3)$ for every induced subgraph H of G with $\gamma(H) = 3$.

2 Bounds on the P_k-free domination number

First we give some more terminology and a result that will be useful. For each vertex v in a minimal dominating set S of a graph G, the *private neighborhood* $pn(v, S)$ of v is given by $N[v] - N[S - \{v\}]$. If $u \in pn(v, S)$, then either u is isolated in $\langle S \rangle$, in which case $u = v$, or $u \in V - S$ and is adjacent to precisely one vertex of S, namely v. We let $epn(v, S)$, or simply $epn(v)$ if the set S is clear from the context, denote the set of all vertices in $V - S$ that are adjacent with v but with no other vertex of S; that is, $epn(v, S) = pn(v, S) - \{v\}$. The set $epn(v, S)$ is known as the *external private neighborhood* of v. Bollobás and Cockayne [2] established the following property of minimum dominating sets in graphs.

Proposition 1 (Bollobás, Cockayne) *If G is a graph with no isolated vertex, then there exists a γ-set S of G in which $epn(v) \neq \emptyset$ for every vertex v in G.*

Here we address Question (i) and our first result establishes a sharp upper bound on the P_k-free domination number of a connected graph.

Theorem 2 *For $k \geq 2$, if $G = (V, E)$ is a connected graph of order $n \geq 2$, then*

$$\gamma(G; -P_k) \leq n + 2(k-1) - 2\sqrt{n(k-1)},$$

and this bound is sharp.

Proof. Let $D = \{v_1, \ldots, v_b\}$, where $b \geq k$, be a γ-set of G that satisfies the statement of Proposition 1. We introduce the following notation. For $i = 1, \ldots, b$, let $W_i = epn(v_i)$. By our choice of D, we know that $W_i \neq \emptyset$ for all i, and $W_i \cap W_j = \emptyset$ for $1 \leq i < j \leq b$. Since D dominates V, we can partition V into sets $V_1, \ldots, V_b$, where each V_i induces a *connected* graph of radius one, and where $W_i \subseteq V_i$ and v_i dominates V_i. Let S be the set produced by the following algorithm.

Algorithm 1 :
Begin

1. $S \leftarrow \emptyset$, $I \leftarrow \{1, \ldots, b\}$ *and* $i \leftarrow 1$.
2. $S \leftarrow S \cup \{v_i\}$ *and* $I \leftarrow I - \{i\}$.
3. **If** $I = \emptyset$, **then** *stop; otherwise, let* $j \in I$ *and continue.*
4. **If** $\langle S \cup \{v_j\} \rangle$ *contains a path* P_k **then**

    4.1. **For** $w \in V_j$ **do**

    If $d(w, S) > 1$ **then** $S \leftarrow S \cup \{w\}$.

    End for

    4.2. $I \leftarrow I - \{j\}$, *and return to Step* 3;

   otherwise *set* $i \leftarrow j$, *and return to Step* 2.

End

We prove that the set S produced by Algorithm 1 is a P_k-free dominating set of G of cardinality at most $n + 2(k-1) - 2\sqrt{n(k-1)}$. In Step 4 of Algorithm 1, if $\langle S \cup \{v_j\} \rangle$ contains a path P_k as a subgraph, then we proceed systematically through the vertices of V_j, placing a vertex in S only if it not dominated by a vertex already in S. Hence if $\langle S \cup \{v_j\} \rangle$ contains a path P_k in Step 4 of Algorithm 1, then when j is removed from I in Step 4.2, the set S dominates V_j. Furthermore, whenever a vertex v is added to S at any stage of the algorithm, either v belongs to D, in which case a longest path in $\langle S \cup \{v\} \rangle$ has order $k-1$, or v belongs to $V - D$, in which case v is adjacent to no other vertex of S. Hence S is a P_k-free dominating set of G.

It remains for us to show that $|S| \leq n + 2(k-1) - 2\sqrt{n(k-1)}$. For $i = 1, \ldots, b$, let $|V_i| = n_i$. Since $|W_i| \geq 1$, we know that $n_i \geq 2$ for all i. Relabeling the sets if necessary, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_b$. By the Pigeonhole Principle, $n_1 \geq n/b$. Hence, applying mathematical induction, it is straightforward to verify that

$$\sum_{i=1}^{k-1} n_i \geq \frac{n(k-1)}{b}. \tag{2}$$

Since $\langle\{v_1,\ldots,v_{k-1}\}\rangle$ contains no path P_k as a subgraph, $\{v_1,\ldots, v_{k-1}\} \subseteq S$. For $i = 1,\ldots,b$, let $S_i = S \cap V_i$. For each $v_i \in S_i$, since v_i dominates V_i, it is evident from the way in which the set S is constructed that v_i is the only vertex of V_i in S_i. If $v_i \notin S$, then clearly S_i contains at most $n_i - 1$ vertices. Hence, if $b \geq k$, then for $i = k,\ldots,b$,

$$|S_i| \leq n_i - 1. \tag{3}$$

Since S is a P_k-free dominating set of G, we now have

$$\begin{aligned}
\gamma(G;-P_k) \leq |S| &= \textstyle\sum_{i=1}^{k-1} |S_i| + \sum_{i=k}^{b} |S_i| \\
&\leq |\{v_1,\ldots,v_{k-1}\}| + \textstyle\sum_{i=k}^{b} (n_i - 1) \quad \text{(by (3))} \\
&= k - 1 + (n - \textstyle\sum_{i=1}^{k-1} n_i) - (b - k + 1) \\
&\leq n + 2(k-1) - \tfrac{n(k-1)}{b} - b. \qquad \text{(by (2))}
\end{aligned}$$

The last expression is maximized with $b = \sqrt{n(k-1)}$. Thus

$$\gamma(G;-P_k) \leq |S| \leq n + 2(k-1) - 2\sqrt{n(k-1)}.$$

That this upper bound on $\gamma(G;-P_k)$ is sharp may be seen by considering the graph G obtained from a complete graph on $(\ell+1)(k-1)$ vertices, where $k \geq 2$ and $\ell \geq 2$, by attaching to each of its vertices ℓ pendant vertices. Then $n = (\ell+1)^2(k-1)$ and $\gamma(G;-P_k) = (k-1)(\ell^2+1) = n + 2(k-1) - 2\sqrt{n(k-1)}$. This completes the proof of the theorem. □

Since $i(G) = \gamma(G;-P_2)$, the special case of Theorem 2 when $k = 2$ yields the following result due to Gimbel and Vestergaard [8].

Corollary 3 [8] *If G is a connected graph of order $n \geq 2$, then $i(G) \leq n + 2 - 2\sqrt{n}$, and this bound is sharp.*

We say that a graph G is H-free if G does not contain H as an induced subgraph. Furthermore, if $\mathcal{H}$ is a collection of graphs, then we say G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. For $t \geq k \geq 3$, let $\mathcal{H}_t^k = \{H \mid H$ is obtained by joining a central vertex of P_3 or K_3 to every vertex of a graph of order $t-1$ that contains no path $P_k\}$. In particular, $\mathcal{H}_3^3 = \{K_{1,4}, K_{1,4}+e, K_{1,4}+\{e,f\}\}$ where here e and f are nonadjacent edges in the complement of $K_{1,4}$. The next result establishes an upper bound for the P_k-free domination number in terms of the domination number.

Theorem 4 *If G is an $\mathcal{H}_t^k$-free graph, $3 \leq k \leq t$, with $\gamma(G) \geq 2$, then*

$$\gamma(G;-P_k) \leq (t-2)\gamma(G) - 2(t-3).$$

Proof. Let D be a γ-set of G. By assumption, $|D| \geq 2$. Among all subsets of vertices in D that induce a subgraph containing no P_k, let S be chosen to be one, first, of maximum cardinality and, secondly, such that $\langle S\rangle$ has the smallest number of paths P_{k-1} of length $k-2$. Then each vertex v in $D-S$ is adjacent to at least one vertex of S. Suppose some vertex $v \in D-S$ is adjacent to only one vertex w in S. Then w must be the endvertex of a P_{k-1} subgraph in $\langle S\rangle$. Furthermore, $S^* = (S-\{w\}) \cup \{v\}$ induces a subgraph of cardinality equal to that of S and having fewer paths of length $k-2$ than $\langle S\rangle$. However $\langle S^*\rangle$ contains no P_k, contradicting our choice of S. Hence each vertex of $D-S$ is adjacent to at least two vertices of S.

Let Y denote the set of vertices in $V-D$ that are adjacent to no vertex of S in G. Among the subsets of Y whose induced subgraphs have no P_k, let X be one of maximum cardinality. Then every vertex of $Y-X$ has a neighbor in X. Thus $S \cup X$ is a P_k-free dominating set of G, and so $\gamma(G;-P_k) \leq |X| + |S|$. Now, each vertex $v \in D-S$ is adjacent to at most $t-2$ vertices of X, for otherwise v is adjacent to at least $t-1$ vertices of X and (as shown above) to at least two vertices of S, and thus belongs to a graph in $\mathcal{H}_t^k$, a contradiction. This, together with the observation that every vertex of Y (and hence of X) is adjacent to some vertex of $D-S$ in G, implies that $|X| \leq (t-2)(|D| - |S|)$. Since $\gamma(G) = |D|$ and $|S| \geq 2$, it follows that

$$\begin{aligned} \gamma(G; -P_k) &\leq |S| + |X| \\ &\leq |S| + (t-2)(\gamma(G) - |S|) \\ &\leq (t-2)\gamma(G) - (t-3)|S| \\ &\leq (t-2)\gamma(G) - 2(t-3). \end{aligned}$$

□

3 Equality of $\gamma(G)$ and $\gamma(G; -P_3)$

In this section we investigate Question (ii) for the domination and P_3-free domination numbers of a graph. First we observe that if $\gamma(G) = \gamma(G; -P_3)$, then $\gamma(G) = \gamma(G; -P_k)$ for $k \geq 3$. Setting $t = k = 3$ in Theorem 4, we have the following sufficient condition for the domination number of a graph to equal its P_3-free domination number.

Theorem 5 *If G is $\mathcal{H}_3^3$-free, then $\gamma(G) = \gamma(G; -P_3)$.*

As an immediate consequence of Theorem 5, we have the following result due to Favaron [5].

Theorem 6 (Favaron [5]) *If G is a graph with maximum degree at most 3, then $\gamma(G) = \gamma(G; -P_3)$.*

Since $\gamma(H) = 1 = \gamma(H; -P_3)$ for every $H \in \mathcal{H}_3^3$, the hypothesis of Theorem 5 is not a necessary condition. The sufficient condition presented in Theorem 5 can be strengthened slightly. For this purpose, we introduce a family $\mathcal{F}$ of graphs as follows. Let F_1 be the graph obtained from K_3 by attaching two paths of length 1 to each vertex of the K_3. Equivalently, F_1 is obtained from three disjoint P_3s by adding three edges joining the central vertices of the paths. Let $\mathcal{F} = \{F \mid F \cong F_1$ or F is obtainable from F_1 by adding edges joining vertices at distance 3 apart in $F_1\}$.

Theorem 7 *If G is $K_{1,4}$-free and $\mathcal{F}$-free, then $\gamma(G) = \gamma(G; -P_3)$.*

Proof. Let D be a γ-set of G such that $\langle D \rangle$ has minimum size. We show that $\langle D \rangle$ contains no P_3. If this is not the case, then suppose that v_1, v_2, v_3 is a P_3 in $\langle D \rangle$. Then $epn(v_i) \neq \emptyset$ for $i = 1, 2, 3$. If $epn(v_i)$ for $i = 1, 2, 3$ contains a vertex x that dominates $epn(v_i)$, i.e., if x is adjacent to every other vertex of $epn(v_i)$, then $(D - \{v_i\}) \cup \{x\}$ is a γ-set of G of size less than that of D, a contradiction. Hence $|epn(v_i)| \geq 2$ and $epn(v_i)$ contains no vertex that is adjacent to every other vertex of $epn(v_i)$ for $i = 1, 2, 3$. If v_1v_3 is not an edge of G, then v_1, v_2, v_3, and any two non-adjacent vertices in $epn(v_2)$ induce a $K_{1,4}$, contrary to our assumption that G is $K_{1,4}$-free. Hence v_1v_3 must be an edge of G. But then v_1, v_2, v_3, together with two non-adjacent vertices in each of the sets $epn(v_i)$ $(i = 1, 2, 3)$ induce a graph that belongs to $\mathcal{F}$, a contradiction. □

An immediate consequence of Theorem 7 now follows.

Corollary 8 *If a chordal graph G is $K_{1,4}$-free and F_1-free, then $\gamma(G) = \gamma(G; -P_3)$.*

4 P_3-domination perfect graphs

A necessary and sufficient forbidden subgraph list characterizing graphs G having $\gamma(G) = \gamma(G; -P_k)$ is impossible to obtain. This is easy to see since the addition of a new vertex adjacent to all vertices of a graph G produces a graph G' containing G as an induced subgraph with $\gamma(G') = \gamma(G'; -P_k) = 1$. Sumner and Moore [11] defined a graph G to be *domination perfect* if $\gamma(H) = i(H)$ for every induced subgraph H of G. We define a graph G to be *P_k-domination perfect* for $k \geq 3$ if $\gamma(H) = \gamma(H; -P_k)$ for every induced subgraph H of G. As a consequence of Theorem 5 and Corollary 8, we have the following results.

Corollary 9 *Every $\mathcal{H}_3^3$-free graph is P_3-domination perfect.*

Corollary 10 *Every $K_{1,4}$-free and F_1-free chordal graph G is P_3-domination perfect.*

Sumner and Moore [11] established that a graph G is domination perfect if and only if $\gamma(H) = i(H)$ for every induced subgraph H of G with $\gamma(H) = 2$. Since $i(H) = \gamma(H; -P_2)$, this result may be generalized as follows.

Theorem 11 *For $k \in \{2,3\}$, a graph G is P_k-domination perfect if and only if $\gamma(H) = \gamma(H; -P_k)$ for every induced subgraph H of G with $\gamma(H) = k$.*

Proof. The necessity is immediate. To prove the sufficiency, let $k \in \{2,3\}$ and let $G = (V, E)$ be a graph and suppose that $\gamma(H) = \gamma(H; -P_k)$ for every induced subgraph H of G with $\gamma(H) = k$. We show that G is P_k-domination perfect. If this is not the case, then G contains an induced subgraph F with $\gamma(F) < \gamma(F; -P_k)$. Then $\gamma(F) > k$. Among the γ-sets of F, let D be one such that $\langle D \rangle$ has a minimum number of paths P_k. Since $\gamma(F) < \gamma(F; -P_k)$, $\langle D \rangle$ must contain a P_k, say $v_1, v_2, \ldots, v_k$. Then $epn(v_i) \neq \emptyset$ for $1 \leq i \leq k$. Let $X = \cup_{i=1}^{k} epn(v_i)$. Further, let $D' = \{v_1, v_2, \ldots, v_k\}$ and let Y denote those vertices in $V(F) - (D \cup X)$ that are adjacent to no vertex in $D - D'$. Then each vertex in Y is adjacent to at least two of the vertices in D' but to no other vertex of D. Let H be the subgraph induced by $D' \cup X \cup Y$. Then D' is a dominating set of H. If $\gamma(H) \leq k - 1$, then a γ-set of H together with the vertices in $D - D'$ form a dominating set of F of cardinality less than that of $\gamma(F)$, which is impossible. Hence $\gamma(H) = k$, and so, by assumption, $\gamma(H; -P_k) = k$. Thus there exists a γ-set D^* of H that contains no P_k. Since D' contains a P_k, D^* must contain at least one vertex of $X \cup Y$. However, no vertex in $X \cup Y$ is adjacent to any vertex of $D - D'$. Hence, $(D - D') \cup D^*$ is a γ-set of F and, since $k \in \{2,3\}$, it is evident that $\langle (D - D') \cup D^* \rangle$ has fewer P_k subgraphs than $\langle D \rangle$. This contradicts our choice of D. □

Theorem 11 shows that is not necessary to check every induced subgraph of a graph in order to determine if it is P_k-domination perfect. Zverovich and Zverovich [12] provided a forbidden induced subgraph characterization of domination perfect graphs in terms of seventeen forbidden induced subgraphs. We have yet to determine a finite forbidden induced subgraph characterization of P_k-domination perfect graphs. Such a characterization for P_3-domination perfect graphs would, however, involve at least 45 (nonisomorphic) forbidden induced subgraphs each with domination number 3 and P_3-domination number 4.

References

[1] R. Bar-Yehuda and U. Vishkin, Complexity of finding k-path-free dominating sets in graphs. *Inform. Process. Lett.* **14** (1982) 228–232.

[2] B. Bollobás and E.J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance. *J. Graph Theory* **3** (1979) 241–249.

[3] E.J. Cockayne, S.T. Hedetniemi, and D.J. Miller, Properties of hereditary hypergraphs and middle graphs. *Canad. Math. Bull.* **21** (1978) 461–468.

[4] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[5] O. Favaron, k-domination and k-dependence in graphs. *Ars. Combin.* **25C** (1988), 159–167.

[6] J.F. Fink and M.S. Jacobson, n-domination in graphs. *Graph Theory with Applications to Algorithms and Computer Science*, eds. Y. Alavi and A.J. Schwenk, Wiley, New York, 1985, 283–300.

[7] J.F. Fink and M.S. Jacobson, On n-domination, n-dependence and forbidden subgraphs. *Graph Theory with Applications to Algorithms and Computer Science*, eds. Y. Alavi and A.J. Schwenk, Wiley, New York, 1985, 301–311.

[8] J. Gimbel, P.D. Vestergaard, and D. Preben, Inequalities for total matchings of graphs. *Ars Combin.* **39** (1995), 109–119.

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[10] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[11] D.P. Sumner and J.L. Moore, Domination perfect graphs. *Notice Amer. Math. Soc.* **26** (1979) A-569.

[12] I.E. Zverovich and V.E. Zverovich, An induced subgraph characterization of domination perfect graphs. *J. Graph Theory* **20** (1995) 375–395.