# Wojcicka's Theorem: Complete, Consolidated Proof

Lohini Moodley*

**Abstract**

The conjecture by E. Wojcicka, that every 3-domination-critical graph with minimum degree at least two is hamiltonian, has recently been proved in three different papers by five different authors. We survey the results which lead to the proof of the conjecture and consolidate them to form a unit.

Dedicated to Professor Ernie Cockayne
on the occasion of his 60th birthday

## 1 Introduction

All graphs $G = (V(G), E(G))$ will be finite, undirected and without loops or multiple edges. Any basic graph-theoretic definitions and notation not defined below comply with [3], while definitions and notation pertaining to domination and related concepts can be found in [8]. Specifically, we denote the domination and independence numbers of a graph $G$ by $\gamma(G)$ and $\beta(G)$ respectively.

The graph $G = (V, E)$ is called *k-edge-domination-critical*, abbreviated *k-$\gamma$-critical*, if $\gamma(G) = k$, and for every edge $e \in E(\overline{G})$, $\gamma(G + e) = k - 1$. (It is easy to see that the addition of an edge to any graph cannot decrease the domination number by more than one.) A *dominating cycle* of $G$ is a cycle such that each edge of $G$ is incident with at least one vertex of the cycle. The length of a longest cycle in $G$ is called the *circumference* of $G$ and is denoted by $c(G)$. Let $\omega(G)$ denote the number of components of $G$. Then $G$ is said to be 1-*tough* if for each cut-set $S$ of $G$, $\omega(G - S) \leq |S|$.

If $A$, $B \subseteq V(G)$, where $A$ and $B$ are disjoint, then we use $E(A, B)$ to denote the set of edges between vertices in $A$ and vertices in $B$.

In general, a graph $G$ is critical with respect to a property **P** if $G$ possesses the property, but no proper induced subgraph, no proper spanning

*Department of Mathematics, University of South Africa, Pretoria, SOUTH AFRICA

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subgraph, or no proper spanning supergraph (as the case may be) of $G$ possesses **P**. Studying criticality with respect to major graph-theoretical properties is important as it affords a deeper understanding of the property and the class of graphs that possesses it, and because induction (using properties of subgraphs of a graph $G$ to deduce properties of $G$) is often used in graph-theoretical arguments.

Sumner and Blitch initiated the study of edge-domination-critical graphs in [11]. This work was continued by Wojcicka in [14] and these two papers gave rise to two major conjectures, namely Conjectures 1 and 2, which are stated below.In [11], Sumner and Blitch showed that the 1-$\gamma$-critical graphs are precisely the complete graphs, and that $G$ is 2-$\gamma$-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars. They further showed that a disconnected 3-$\gamma$-critical graph is the disjoint union of a complete graph and a 2-$\gamma$-critical graph, and therefore concentrated their research on the connected 3-$\gamma$-critical graphs. These are some of the results that they obtained: Every 3-$\gamma$-critical graph of order $n$ and size $m$

- contains a triangle,
- satisfies $m \leq {}_{n-2}C_2$,
- has a 1-factor if $n$ is even and
- has its clique number bounded below by 3 and above by $n - 3$.

Sumner and Blitch [11] also pointed out that all the examples of 3-$\gamma$-critical graphs with $n \geq 7$ that they had constructed contained hamiltonian paths, and thought it logical to ask when a $k$-$\gamma$-critical graph would contain a hamiltonian path or cycle. In [14], Wojcicka proved the following result:

**Theorem 1** *If $G$ is a connected, 3-$\gamma$-critical graph on more than six vertices, then $G$ has a hamiltonian path.*

She then formulated the following conjecture:

**Conjecture 1** Every connected 3-$\gamma$-critical graph $G$ with $\delta(G) \geq 2$ is hamiltonian.

This conjecture is henceforth referred to as *Wojcicka's Conjecture*. Obviously, graphs with end-vertices are not hamiltonian, but it was proved by Xue and Chen in [15] that 3-$\gamma$-critical graphs with end-vertices yield hamiltonian graphs when all end-vertices are removed. We state this result here for the sake of completeness.

**Theorem 2** *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta(G) = 1$ and let $V_1 = \{v \in V(G) \mid \deg(v) = 1\}$. Then $G - V_1$ is hamiltonian.*

Sumner and Blitch also formulated the following conjecture in [11]:

**Conjecture 2** If $G$ is $k$-$\gamma$-critical, then $i(G) = k$.

Both these conjectures have generated a great deal of research. Conjecture 1 has been solved. In [5], Favaron, Tian and Zhang showed that (i) 3-$\gamma$-critical graphs with $\delta \geq 2$ satisfy $\beta \leq \delta + 2$, and went on to prove the conjecture for $\beta \leq \delta + 1$. The outstanding case ($\beta = \delta + 2$) was proved by Tian, Wei and Zhang in [13]. The proofs make use of the following results established by Flandrin, Tian, Wei and Zhang in [6]: 3-$\gamma$-critical graphs with $n$ vertices and minimum degree $\delta \geq 2$ are (ii) 1-tough and have the property that (iii) each longest cycle is a dominating cycle. Using (ii) and (iii), they proved that (iv) the circumference of these graphs is at least $n - 1$. In [5], Favaron, Tian and Zhang used (ii) and (iv) to prove Wojcicka's Conjecture for $\beta \leq \delta + 1$. Furthermore, they showed that if $\beta = \delta + 2$, then (v) every maximum independent set contains all vertices of degree $\delta$, and (vi) if $x$ is a vertex of degree $\delta$, then $N(x)$ induces a clique. Tian, Wei and Zhang then showed in [13] that (vii) there is only one vertex of degree $\delta$, and used (ii), (iv), (v), (vi) and (vii) to show that 3-$\gamma$-critical graphs with $\beta = \delta + 2$ are hamiltonian, thus completing the proof of Wojcicka's Conjecture.

The proofs are long and technical and contain many repetitions and some omissions, and are difficult to follow. In this survey we consolidate the work done in [5, 6, 10, 11, 13, 14, 15], of which [5, 6, 13] are the most important, on this conjecture. Several of the proofs contain lemmas and other results that are used in more than one paper or more than one proof in the same paper. We formulate these results separately and ordered them under the headings *Independent sets* (Section 2, containing general results regarding the independent sets of 3-$\gamma$-critical graphs), *Cut-sets, cut-vertices and end-vertices* (Section 3, containing the proof of (ii) and general results on cut-sets, cut-vertices and end-vertices of 3-$\gamma$-critical graphs), *Independence numbers* (Section 4, results on independence number and independent domination number, including the proofs of (i), (v) and (vi)) and *Longest cycles* (Section 5, results pertaining to longest cycles, including the proofs of (iii), (iv) and (vii)). The proof of Wojcicka's Conjecture in the case $\beta \leq \delta + 1$ [5] is given in Section 6, while the proof of the case $\beta = \delta + 2$ is given in Section 7.

We have attempted to make these long and difficult proofs as accessible as possible, thus making the proof of this major conjecture available as one unit.

Further results on $k$-$\gamma$-critical graphs for $k \geq 4$ can be found in [1, 2, 4, 9]. Those readers who are interested in other issues surrounding criticality with respect to domination are referred to a survey of various types of criticality of domination, independence and irredundance by Grobler in

[7] and the recent survey of domination critical graphs by Sumner and Wojcicka in [12].

Finally, we use the notation "■" to indicate the end of the proof of a theorem or lemma, and "□" to indicate the end of the proof of a lemma within the proof of a theorem.

# 2 Independent sets

By definition, if $G$ is 3-$\gamma$-critical and $u$ and $v$ are any two nonadjacent vertices of $G$, then $\gamma(G+uv) = 2$. In [11], Sumner and Blitch observed that there therefore exists a vertex $x \in V(G) \setminus \{u, v\}$ such that in $G$, $\{u, x\}$ dominates $G-v$ but not $v$ (denoted by $[u, x] \rightarrow v$) or $\{v, x\}$ dominates $G-u$ but not $u$ (denoted by $[v, x] \rightarrow u$). This observation will be used often in the proofs that follow.

The following lemmas relate to the independent sets of connected, 3-$\gamma$-critical graphs and will be used often in the proofs that follow. Sumner and Blitch proved the next result in [11], using the condition $k \geq 4$ to ensure that $W \cap Y(W) = \phi$ and that $y_1 y_2 ... y_{k-1}$ is a path in $G$, but as observed in [6], the result can be easily verified for $k \in \{2, 3\}$ by using the fact that $G$ is 3-$\gamma$-critical.

**Lemma 3** *Let $G$ be a connected, 3-$\gamma$-critical graph and $W$ an independent set of $k \geq 2$ vertices in $G$. Then there exists an ordering $F(W) = (w_1, w_2, ..., w_k)$ of the vertices of $W$ and a sequence $Y(W) = (y_1, y_2, ..., y_{k-1})$ of $k-1$ distinct vertices in $G$ such that $[w_i, y_i] \rightarrow w_{i+1}$, for each $i$ with $1 \leq i \leq k-1$.*

Note that we sometimes denote $w_i$ as $f_i(W)$ and $y_i$ as $y_i(W)$ for clarity. The following two lemmas were proved in [5] for $k \geq 3$, but can be extended to $k \geq 2$ as for Lemma 3.

**Lemma 4** *Let $W$ be an independent set of $k \geq 2$ vertices of a connected, 3-$\gamma$-critical graph $G$, such that $W \cup \{x\}$, with $x \notin W$, is also independent. Then any sequence $Y(W)$ defined in Lemma 3 belongs to $N(x)$.*

**Proof.** For each $i$ with $1 \leq i \leq k-1$, $[w_i, y_i] \rightarrow w_{i+1}$. Since $x \notin W$ and $x$ is not adjacent to $w_i$ for each $i$ with $1 \leq i \leq k$, it follows that $x$ is adjacent to $y_i$ for each $i$ with $1 \leq i \leq k-1$. Hence $Y(W) \subseteq N(x)$. ■

**Lemma 5** *Let $W$ be an independent set of $k \geq 2$ vertices of a connected, 3-$\gamma$-critical graph $G$, $W \cup \{x\}$ an independent set with $x \notin W$ and $Y(W)$ the sequence defined in Lemma 3. If $\deg(x) = k-1$ or if every vertex in $N(x) \setminus Y(W)$ is adjacent to every vertex in $W$, then $y_i w_i \in E(G)$ for each $i$ with $2 \leq i \leq k-1$.*

**Proof.** Let $F(W) = (w_1, w_2, ..., w_k)$ be the ordering of the vertices of $W$ described in Lemma 3 and $Y(W) = (y_1, y_2, ..., y_{k-1})$.

For any $i$ with $2 \leq i \leq k-1$, the vertices $w_1$ and $w_{i+1}$ are not adjacent. So, there exists a vertex $z \in V(G) \setminus \{w_1, w_{i+1}\}$ such that

$$[w_{i+1}, z] \to w_1 \quad \text{or} \quad [w_1, z] \to w_{i+1}.$$

We first show that $z \in N(x) \cap Y(W)$. Since $W \cup \{x\}$ is independent, $z$ dominates $x$ in both cases, and so $z \in N(x)$. Also by Lemma 4, $Y(W) \subseteq N(x)$. If $\deg(x) = k-1$, then $Y(W) = N(x)$, hence $z \in N(x) \cap Y(W)$. If $\deg(x) \neq k-1$, then by the hypothesis every vertex in $N(x) \setminus Y(W)$ is adjacent to every vertex in $W$ (including $w_1$ and $w_{i+1}$). Hence $z \in N(x) \cap Y(W)$.

The case $[w_{i+1}, z] \to w_1$ is impossible since the only vertex in $Y(W)$ that possibly does not dominate $w_1$ is $y_1$ and $y_1$ does not dominate $w_2$. Hence $[w_1, z] \to w_{i+1}$. For each $i$ with $2 \leq i \leq k-2$, $z \in \{y_i, y_{i+1}\}$ as these are the only two vertices in $Y(W)$ that possibly do not dominate $w_{i+1}$. However, $z \neq y_{i+1}$ as $y_{i+1}$ does not dominate $w_{i+2}$ for each $i$ with $2 \leq i \leq k-2$. Thus $z = y_i$ and so $y_i w_i \in E(G)$ for each $i$ with $2 \leq i \leq k-2$. Consider $i = k-1$. The only vertex in $Y(W)$ not adjacent to $w_k$ is $y_{k-1}$. Thus $z = y_{k-1}$ and $y_{k-1} w_{k-1} \in E(G)$. It follows that $y_i w_i \in E(G)$ for each $i$ with $2 \leq i \leq k-1$. ■

# 3 Cut-sets, cut-vertices and end-vertices

This section contains results involving cut-sets and end-vertices, which will be used in proving many of the results leading to the proof of Wojcicka's Conjecture. To begin with, Theorem 6(a) was proved in [11] and (b) in [15].

**Theorem 6** *Let $G$ be a connected, 3-$\gamma$-critical graph.*

(a) *Any vertex $v$ of $G$ is adjacent to at most one end-vertex of $G$.*

(b) *If $w$ is an end-vertex and $v$ is adjacent to $w$, then $N(v) \setminus \{w\}$ induces a clique in $G$.*

**Proof.** (a) Suppose to the contrary that $v$ is adjacent to two end-vertices $a$ and $b$. Then since $ab \notin E(G)$, we can assume without loss of generality that there is a vertex $x \in V(G) \setminus \{a, b\}$ such that $[a, x] \to b$. Since $a$ only dominates $v$ (and itself), it follows that $x$ dominates $V(G) \setminus \{a, b, v\}$ and so $\{v, x\}$ dominates $G$, a contradiction.

(b) Suppose to the contrary that there are two vertices $a$ and $b$ in $N(v) \setminus \{w\}$ such that $ab \notin E(G)$. We may assume without loss of generality that there is a vertex $z \in V(G) \setminus \{a, b\}$ such that $[a, z] \rightarrow b$. Since $\{aw, zb\} \cap E(G) = \phi$, it follows that $z = w$, and so $a$ dominates $V(G) \setminus \{b, w\}$. Thus $\{a, v\}$ dominates $G$, contradicting $\gamma(G) = 3$. ■

The following useful result involving cut-sets was proved in [11]. It was, however, improved upon in [6] for $\delta(G) \geq 2$ (see Theorem 2.10).

**Theorem 7** *Let $G$ be a connected, 3-$\gamma$-critical graph. If $T$ is a cut-set of $G$, then $G - T$ has at most $|T| + 1$ components.*

The next major result that we prove is that every cut-vertex of a 3-$\gamma$-critical graph is adjacent to an end-vertex. This will be useful in obtaining connectivity and toughness properties of 3-$\gamma$-critical graphs with $\delta \geq 2$ needed in the proof of Wojcicka's Conjecture. We first prove the following result.

**Lemma 8** *Let $G$ be a connected, 3-$\gamma$-critical graph with a cut-vertex $v$ and let $A$ and $B$ be the vertex sets of the components of $G - v$. Then exactly one of $A$ and $B$ is contained in $N(v)$.*

**Proof.** If $A \cup B \subseteq N(v)$, then $v$ dominates $G$, contradicting $\gamma(G) = 3$.

Now suppose that $A \backslash N(v) \neq \phi$ and $B \backslash N(v) \neq \phi$. We may choose $a' \in A$ and $b' \in B$ such that $d(a', v) = d(b', v) = 2$. Choose $a \in N(v) \cap A$ and $b \in N(v) \cap B$ such that $\{aa', bb'\} \subseteq E(G)$.

We show that $a$ dominates $A$ or $b$ dominates $B$. Suppose to the contrary that there is $a_1 \in A$, such that $aa_1 \notin E(G)$. Then there is a vertex $x$ such that

$$[a, x] \rightarrow a_1 \quad \text{or} \quad [a_1, x] \rightarrow a.$$

Consider the former case. Since $N(a) \cap B = \phi$ and $v$ does not dominate $b_1$, necessarily $x \in B$, and we may assume without loss of generality that $x = b$, since $b$ dominates $b_1$. Similarly, in the latter case $x \in B \cup \{v\}$, and since $v$ dominates $a$, $x \in B$. Again we may assume without loss of generality that $x = b$. Thus we can assume that $b$ dominates $B$.

Note that since $b$ dominates $B \cup \{v\}$, no vertex $a^* \in A$ dominates $A$, for otherwise $\{b, a^*\}$ is a dominating set, contradicting $\gamma(G) = 3$.

Consider the vertices $b'$ and $v$. Since $b'v \notin E(G)$, there exists a vertex $y$ such that

$$[b', y] \rightarrow v \quad \text{or} \quad [v, y] \rightarrow b'.$$

In the former case, $y \in A$ to dominate $a'$ and $y$ dominates $A$, a contradiction. Hence it must be that $[v, y] \rightarrow b'$, and $y \in A$ in order to dominate $a'$. Thus $v$ dominates $B \backslash \{b'\}$.

Now consider the vertices $a'$ and $v$. Since $a'v \notin E(G)$, there exists a vertex $z$ such that

$$[a', z] \to v \quad \text{or} \quad [v, z] \to a'.$$

In the former case, $z = b'$, since $z$ must dominate $B$ and not $v$, and so $a'$ must dominate $A$, a contradiction. Therefore it must be that $[v, z] \to a'$. Since $v$ does not dominate $b'$, $z \in B$, and so $v$ dominates $A \setminus \{a'\}$.

Furthermore, $B$ is complete: Suppose to the contrary that $b_1$ and $b_2$ are two vertices in $B$ and $b_1 b_2 \notin E(G)$. Then, without loss of generality, there exists a vertex $x$ such that $[b_1, x] \to b_2$. Now $x \in A$ to dominate $a'$ and $x$ dominates $A$, a contradiction.

Since $a'$ does not dominate $A$, there exists a vertex $a_1 \in A$ such that $a_1 a' \notin E(G)$. Since $a_1 b' \notin E(G)$, there exists a vertex $x$ such that

$$[a_1, x] \to b' \quad \text{or} \quad [b', x] \to a_1.$$

The former case is impossible, since $x = v$ to dominate $B \setminus \{b'\}$, but then $a'$ is not dominated. Therefore $[b', x] \to a_1$ and $x \in A$ to dominate $a'$. Note that $x$ must also dominate $v$, but not $a_1$. Let $x = a_2$, where $a_2 \in A \setminus \{a', a_1\}$ and $a_2$ dominates $(A \setminus \{a_1\}) \cup \{v\}$.

Since $a_2 b \notin E(G)$, there exists a vertex $x$ such that

$$[a_2, x] \to b \quad \text{or} \quad [b, x] \to a_2.$$

The former case is impossible since no vertex dominates $b'$ and not $b$, and the latter case is impossible since no vertex dominates $a'$ and not $a_2$. Thus either

$$A \subseteq N(v) \quad \text{or} \quad B \subseteq N(v)$$

and the lemma follows. ■

**Theorem 9** *Let $G$ be a connected, 3-$\gamma$-critical graph. If $v$ is a cut-vertex of $G$, then $v$ is adjacent to an end-vertex of $G$.*

**Proof.** By Theorem 7, $G - v$ has exactly two components. Let $A$ and $B$ be the vertex sets of the components of $G - v$, and assume, without loss of generality, that $B \subseteq N(v)$. Let $X = A \setminus N(v)$. Clearly, $|X| \neq \phi$. We show that $|B| = 1$.

Suppose to the contrary that $|B| \geq 2$ and let $\{b_1, b_2\} \subseteq B$. Consider any $a \in A \cap N(v)$. Since $ab_1 \notin E(G)$, there exists a vertex $x$ such that

$$[a, x] \to b_1 \quad \text{or} \quad [b_1, x] \to a.$$

In the former case, $x$ dominates $b_2$ but not $b_1$, and so $x \in B$. But then $a$ dominates $A$ and $v$ dominates $\{v\} \cup B$, contradicting $\gamma(G) = 3$. In the latter case $x$ dominates $A \setminus \{a\}$, again contradicting $\gamma(G) = 3$. ■

**Corollary 10** *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$. Then $G$ is 2-connected.*

**Proof.** If $G$ is not 2-connected, then $G$ has a cut-vertex $v$. By Theorem 9, $v$ is adjacent to an end-vertex, contradicting $\delta(G) \geq 2$. ■

In an attempt to prove Wojcicka's Conjecture, Flandrin, Tian, Wei and Zhang [6] proved a stronger result than the one stated in Theorem 7. We begin with a lemma.

**Lemma 11** *[6] Let $G$ be a connected, 3-$\gamma$-critical graph. If $T_0$ is a cut-set of $G$ such that $\omega(G - T_0) = |T_0| + 1$, then $v$ is a cut-vertex for any $v \in T_0$.*

**Proof.** If $|T_0| = 1$, then the result holds, so we can assume that $|T_0| \geq 2$. Let $|T_0| = k - 1$ and let $T_1, T_2, \ldots, T_k$ be the vertex sets of the components of $G - T_0$.

Consider a vertex $w_i \in T_i$, for each $i$ with $1 \leq i \leq k$, and let $S_i = T_i \setminus \{w_i\}$. Let $W = \{w_1, w_2, \ldots, w_k\}$. Since $W$ is an independent subset of $V(G)$, it follows from Lemma 3 that there is a sequence $Y(W) = (y_1, y_2, \ldots, y_{k-1})$ such that $[w_i, y_i] \to w_{i+1}$ for each $i$ with $1 \leq i \leq k-1$ (re-organising the numbering of the components if necessary). We consider two cases depending on $T_0$ and the sets $S_i$.

**Case 1** $|T_0| \geq 3$ or $|T_0| = 2$ and $S_i \neq \phi$ for each $i$ with $1 \leq i \leq 3$.
Since $y_i$ dominates $S_j$ for each $j \neq i$, it follows that $Y(W) \subseteq T_0$ and so $Y(W) = T_0$. Consider the vertices $w_1$ and $w_k$. Since $w_1 w_k \notin E(G)$, there exists a vertex $y \in V(G) \setminus \{w_1, w_k\}$ such that

$$[w_1, y] \to w_k \quad \text{or} \quad [w_k, y] \to w_1.$$

In either case $y \in Y(W)$ since $y$ dominates $S_k \cup \{w_2\}$ or $S_1 \cup \{w_2\}$ respectively. Since $y_t w_1 \in E(G)$ for each $t$ with $2 \leq t \leq k-1$ and $y_1 w_2 \notin E(G)$, the case $[w_k, y] \to w_1$ is impossible. So we must have that $[w_1, y] \to w_k$. Since $y_t w_k \in E(G)$ for each $t$ with $1 \leq t \leq k-2$, it follows that $y = y_{k-1}$, *i.e.* $[w_1, y_{k-1}] \to w_k$. Hence $T_{k-1} \subseteq N(y_{k-1})$. But we also have $[w_{k-1}, y_{k-1}] \to w_k$ and since $w_{k-1}$ is not adjacent to any vertex in $T = T_1 \cup T_2 \cup \ldots \cup T_{k-2} \cup S_k$, we have $T \subseteq N(y_{k-1})$. Thus $\{y_{k-1}, w_k\}$ dominates $V(G)$, contradicting $\gamma(G) = 3$. Hence this case is impossible.

**Case 2** $|T_0| = 2$ and at least one of $S_1$, $S_2$ and $S_3$ is empty.
Without loss of generality, assume that $S_3 = \phi$, *i.e.* $T_3 = \{w_3\}$. Thus $N(w_3) \subseteq T_0$. We consider two subcases depending on the value of $\deg(w_3)$.

**Case 2.1** $\deg(w_3) = 1$.
Since $[w_1, y_1] \to w_2$, $y_1 w_3 \in E(G)$. Thus $y_1$ is a cut-vertex and $y_1 \in T_0$. By Theorem 6(b), either $N(y_1) \cap T_1 = \phi$ or $N(y_1) \cap T_2 = \phi$. Consequently $x \in T_0 \setminus \{y_1\}$ is also a cut-vertex and the lemma holds.

**Case 2.2** $\deg(w_3) = 2$.
Since $[w_1, y_1] \to w_2$, it follows that $y_1 w_3 \in E(G)$ and $y_1 \in T_0$. Since $[w_2, y_2] \to w_3$, we have $y_2 \in T_1$ and $w_2$ is adjacent to every vertex in $S_2$. Also, $N(w_3) = T_0 = \{y_1, x\}$ for some $x$ with $x \neq y_2$.

Now $T_1 \cup T_2 \not\subseteq N(T_0)$, for otherwise $T_0$ dominates $V(G)$, contradicting $\gamma(G) = 3$. Without loss of generality, we can assume that $w_2 \notin N(T_0)$. By the connectedness of $G$, $S_2 \neq \phi$.

Consider $u_1 \in T_1$ and $u_2 \in T_2$. There is a vertex $z$ such that

$$[u_1, z] \to u_2 \quad \text{or} \quad [u_2, z] \to u_1.$$

In either case, $z$ dominates $w_3$ so $z \in T_0$. Since $w_2 \notin N(u_1) \cup N(z)$, it is impossible that $[u_1, z] \to u_2$. Hence $[u_2, z] \to u_1$.

Suppose $S_1 = \phi$. If $\deg(w_1) = 1$, then using a similar argument to Case 2.1, the lemma holds. Thus we can assume $\deg(w_1) = 2$, *i.e.* $N(w_1) = T_0$. Since $\{u_2, z\}$ dominates $G - u_1$ (note that $u_1 = w_1$) and $z \in T_0$, it follows that $\{u_2, z\}$ dominates $G$, contradicting $\gamma(G) = 3$.

Hence $S_1 \neq \phi$. Again we consider $[u_2, z] \to u_1$, where $z \in T_0$. Without loss of generality, we can assume that $z = y_1$. Thus $T_1 \backslash \{u_1\} \subseteq N(y_1)$. Consider $u_1' \in T_1 \backslash \{u_1\}$. Using a similar argument to the one above, we have $[u_2, z'] \to u_1'$, for some $z' \in T_0$. Since $u_1' y_1 \in E(G)$, $z' = x$. Thus $T_1 \backslash \{u_1'\} \subseteq N(x)$.

Suppose that $|T_1| \geq 3$. Then there is a vertex $u_1'' \in T_1 \backslash \{u_1, u_1'\}$, and $[u_2, z''] \to u_1''$ for some $z'' \in T_0$. But this is impossible since $T_0 \subseteq N(u_1'')$. Thus $|T_1| = 2$.

Now consider the vertices $w_2$ and $w_3$. Since $[w_2, y_2] \to w_3$, $y_2$ dominates $\{w_1, u_1, x, y_1\}$ but not $w_3$. Thus $y_2 \in T_0 \cup T_1$. Since $\{u_1 y_1, w_1 x\} \cap E(G) = \phi$, $[w_2, y_2] \to w_3$ is impossible, contradicting $\gamma(G) = 3$. So this case is also impossible and the lemma holds. ■

**Theorem 12** *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$. Then $G$ is 1-tough.*

**Proof.** By Theorem 7, if $T_0$ is a cut-set of a connected 3-$\gamma$-critical graph, then $\omega(G - T_0) \leq |T_0| + 1$. We now show that if $\delta(G) \geq 2$, then $\omega(G - T_0) \leq |T_0|$.

Suppose to the contrary that $G$ is a connected 3-$\gamma$-critical graph with $\delta(G) \geq 2$ and that $G$ has a cut-set $T_0$ such that $\omega(G - T_0) = |T_0| + 1$. Then by Lemma 11 and Theorem 2.7, each vertex $v \in T_0$ is adjacent to an end-vertex, contradicting $\delta(G) \geq 2$. ■

**Corollary 13** *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$. If $T$ is a cut-set of $G$, then $G$ has at most $|T|$ components.*

## 4 Independence numbers

Favaron, Tian and Zhang established an upper bound for the independence number $\beta(G)$ of a 3-$\gamma$-critical graph $G$ with $\delta(G) \geq 2$ in [5]. Before we present this result, we prove the following lemma.

**Lemma 14** *Let $x$ be a vertex of degree $d \geq 2$ of a 3-$\gamma$-critical graph $G$, $I$ a maximum independent set and let $A = I \backslash N[x]$. If $|I \cap N(x)| = d$, then $|A| \leq 1$.*

**Proof.** Suppose to the contrary that $N(x) \subseteq I$ and that $|A| \geq 2$. Let $\{z,t\} \subseteq A$. Then $N(x) \cup \{z,t\} \subseteq I$, so $N(x) \cup \{z,t\}$ is independent. By Lemma 3, there exists an ordering $(w_1, w_2, ..., w_{d+2})$ of the vertices of $N(x) \cup \{z,t\}$ and a path $y_1 y_2 ... y_{d+1}$ contained in $V(G) \backslash (N(x) \cup \{z,t\})$ such that $[y_i, w_i] \to w_{i+1}$ for each $i$ with $1 \leq i \leq d+1$. Now for some pair $i,j$ with $1 \leq i \neq j \leq d+1$, $\{w_i, w_j\} \cap \{z,t\} \neq \phi$. Thus either $w_i \notin N(x)$ or $w_j \notin N(x)$. Since $\{y_i, y_j\} \subseteq V(G) \backslash (N(x) \cup \{z,t\})$, either $\{w_i, y_i\}$ or $\{w_j, y_j\}$ does not dominate $x$, a contradiction. Hence $|A| \leq 1$. ■

**Theorem 15** *The independence number $\beta(G)$ of a 3-$\gamma$-critical graph $G$ of minimum degree $\delta(G) \geq 2$ satisfies $\beta(G) \leq \delta(G) + 2$. Moreover, if $\beta(G) = \delta(G) + 2$, then every maximum independent set contains all the vertices of degree $\delta(G)$.*

**Proof.** Let $x$ be any vertex of degree $\delta(G)$ in $G$, $I$ any maximum independent set, and $A = I \backslash N[x]$. Note that if $x \in I$, then $I \cap N(x) = \phi$, and so $|I \cap N[x]| \leq \delta(G)$. By the definition of $A$, $\beta(G) = |N[x] \cap I| + |A|$ and therefore $\beta(G) = |A| + 1$ if $x \in I$ and $\beta(G) = |A| + |N(x) \cap I|$ if $x \notin I$. Hence

$$\beta(G) \leq \max\{|A| + 1, |A| + |N(x) \cap I|\}.$$

If $|A| \leq 1$, then $\beta(G) \leq \max\{2, 1 + |N(x) \cap I|\} \leq \delta(G) + 1$.

If $|A| = 2$, then by Lemma 14, $|I \cap N(x)| \leq \delta(G) - 1$ and thus $\beta(G) \leq \max\{3, \delta(G) + 1\} = \delta(G) + 1$ since $\delta(G) \geq 2$.

Now suppose that $|A| \geq 3$, say $|A| = k$. By Lemma 3 there exists an ordering $F(A) = (a_1, a_2, ..., a_k)$ of the vertices of $A$ and a sequence $Y(A) = (y_1, y_2, ..., y_{k-1})$ such that $[a_i, y_i] \to a_{i+1}$ for each $i$ with $1 \leq i \leq k-1$. Since $A \cup \{x\}$ is independent, $Y(A) \subseteq N(x)$ by Lemma 4, implying that $\delta(G) \geq k - 1$. Moreover, $Y(A) \cap I = \phi$ since $A \backslash \{a_i, a_{i+1}\} \subseteq N(y_i)$ for each $i$ with $1 \leq i \leq k-1$ and thus $|N(x) \cap I| \leq \delta(G) - (k-1)$.

If $x \notin I$, then $|I| = |N(x) \cap I| + |A| \leq \delta(G) + 1$. If $x \in I$, then $|I| = |A| + 1 = k + 1 \leq \delta(G) + 2$, since $\delta(G) \geq k - 1$. Thus $\beta(G) = |I| \leq \delta(G) + 2$ in all cases.

Moreover, the only possibility for $\beta(G)$ to be equal to $\delta(G)+2$ is when $x \in I$ and $|A| = \delta(G)+1$. Since $x$ and $I$ are arbitrary, it follows that every maximum independent set contains every vertex of degree $\delta(G)$. ■

We now present a few results regarding 3-$\gamma$-critical graphs with independence number equal to $\delta + 2$ will contribute to the proof of Wojcicka's Conjecture.

**Theorem 16** [5] *Let $G$ be a 3-$\gamma$-critical graph with minimum degree $\delta(G) \geq 2$ and $\beta(G) = \delta(G) + 2$, and let $x$ be a vertex of $G$ of degree $\delta = \delta(G)$. Then $\langle N(x) \rangle$ is a clique and $\Delta(G) \geq 2\delta(G)$.*

**Proof.** By Theorem 15, every maximum independent set $S$ of $G$ is of the form $W \cup \{x\}$. By Lemma 5 there is an ordering $F(W) = (w_1, w_2, \ldots, w_{\delta+1})$ of the vertices of $W$ and an ordering $Y(W) = (y_1, y_2, \ldots, y_\delta)$ of the vertices of $N(x)$ such that

$$[w_i, y_i] \rightarrow w_{i+1} \text{ for each } i \text{ with } 1 \leq i \leq \delta,$$

$$y_1 w_j \in E(G) \text{ for each } j \text{ with } 3 \leq j \leq \delta + 1$$

and

$$y_i w_j \in E(G) \text{ for each pair } i, j \text{ with } 2 \leq i \leq \delta \text{ and } j \neq i + 1.$$

For any pair $i, j$ of vertices with $1 \leq i < j \leq \delta$, there exists a vertex $y$ such that

$$[w_{i+1}, y] \rightarrow w_{j+1} \text{ or } [w_{j+1}, y] \rightarrow w_i.$$

Without loss of generality, assume $[w_{i+1}, y] \rightarrow w_{j+1}$. The vertex $y$ belongs to $N(x)$ by Theorem 15. The only vertex in $N(x)$ not adjacent to $w_{j+1}$ is $y_j$. So $y = y_j$. Since $w_{i+1} y_i \notin E(G)$, $y_j y_i \in E(G)$. Since this holds for any pair $i, j$, it follows that $\langle N(x) \rangle$ is a clique. Moreover, every vertex $y_i \in N(x) \setminus \{y_1\}$ is adjacent to $x$, to every vertex of $N(x) \setminus \{y_i\}$ and to every vertex of $W \setminus \{w_{i+1}\}$. Hence $\deg(y_i) \geq 1 + (\delta - 1) + \delta = 2\delta$ and therefore $\Delta(G) \geq 2\delta(G)$. ■

**Corollary 17** [5] *Every 3-$\gamma$-critical graph $G$ with $\delta(G) \geq 2$ and $\Delta(G) < 2\delta(G)$ satisfies $\beta(G) \leq \delta(G) + 1$.*

**Theorem 18** [13] *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta = \delta(G) \geq 2$ and $\beta = \beta(G) = \delta(G) + 2$. Then $G$ has only one vertex of degree $\delta(G)$.*

**Proof.** Suppose that $W = \{w_1, w_2, \ldots, w_{\delta+2}\}$ is a maximum independent set of $G$. By Theorem 15, $W$ contains all the vertices of degree $\delta$. By Lemma 3 there exists an ordering $F(W) = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\delta+2}})$ and a sequence $Y(W) = (y_1, y_2, \ldots, y_{\delta+1})$ of $\delta + 1$ vertices such that

$$[w_{i_j}, y_j] \to w_{i_{j+1}} \text{ for each } j \text{ with } 1 \leq j \leq \delta + 1.$$

Suppose $\deg(w_{i_j}) = \delta$ for some $j$ with $1 \leq j \leq \delta + 1$. Since $\delta \geq 2$, $\beta = \delta + 2 \geq 4$. We can therefore choose two vertices $w_{i_k}$ and $w_{i_\ell}$ from $W \backslash \{w_{i_j}, w_{i_{j+1}}\}$. Since $w_{i_k} w_{i_\ell} \notin E(G)$, we can assume without loss of generality that there exists a vertex $u \in V(G) \backslash W$ such that $[w_{j_k}, u] \to w_{i_\ell}$. Obviously $\{w_{i_j}, w_{i_{j+1}}\} \subseteq N(u)$, and in particular $u \in N(w_{i_j})$. By Theorem 16, $\langle N(w_{i_j}) \rangle$ is complete and thus $N(w_{i_j}) \backslash \{u\} \subseteq N(u)$. Since

$$V \backslash \left(N(w_{i_j}) \cup \{w_{i_j}, w_{i_{j+1}}, y_j\}\right) \subseteq N(y_j),$$

$\{y_j, u\}$ dominates $G$, contradicting $\gamma(G) = 3$.

Thus $w_{i_{\delta+2}}$ is the only vertex of degree $\delta$, and the proof is complete. ■

# 5 Longest cycles

In this section we state some results regarding longest cycles that will be used in the proof of Wojcicka's Conjecture. The notation defined here will be used throughout the dissertation.

Let $C$ be a longest cycle of a graph $G$ and $H$ a component of $G - V(C)$. We choose an arbitrary orientation on $C$ and use classical notation: the successor (predecessor) of a vertex of $C$ is denoted by $v^+$ $(v^-)$. If $u$ and $v$ are distinct vertices on $C$, then $\overrightarrow{C}[u, v]$ or $u\overrightarrow{C}v$, (whichever is more convenient) is the path from $u$ to $v$ on $C$, following the orientation. $\overrightarrow{C}[u, v]$ can also be considered to be the set of vertices on the path. $\overleftarrow{C}[u, v] = u\overleftarrow{C}v$ is defined similarly. We define the following sets:

$$X = N_C(H) = \{x_1, x_2, \ldots, x_k\},$$

where the indices follow the orientation of $C$;

$$\begin{aligned} A &= X^+ = \{a_1, a_2, \ldots, a_k\}, \text{ where } a_i = x_i^+, \\ B &= X^- = \{b_1, b_2, \ldots, b_k\}, \text{ where } b_{i-1} = x_i^- \\ \text{and } C_i &= \overrightarrow{C}[a_i, b_i], \text{ for each } i \text{ with } 1 \leq i \leq k, \end{aligned}$$

where the indices are taken modulo $k$.

The first two results are classical in the theory of hamiltonicity and are therefore stated without proof.

**Lemma 19** For any vertex $v \in V(H)$, $A \cup \{v\}$ is independent.

**Lemma 20** *For any pair* $i, j$ *with* $1 \leq i \neq j \leq k$, $N(a_i) \cap N(a_j) \cap (V(G) \backslash V(C)) = \phi$.

We obtain symmetric results by replacing $A$ with $B$.

Recall that a cycle $C$ of a connected graph $G$ is called a *dominating cycle* if each component of $G - V(C)$ has only one vertex, that is, if each edge of $G$ is incident with a vertex of $C$. From the definition it is obvious that if $C$ is a dominating cycle of a graph $G$, then $V(C)$ dominates $V(G)$.

The following theorem is important for the proof of Wojcicka's Conjecture. It helps to establish a lower bound for longest cycles of 2-connected, 3-$\gamma$-critical graphs (see Theorem 23), and was proved in [6] by Flandrin, Tian, Wei and Zhang.

**Theorem 21** [6] *Let* $G$ *be a connected,* 3-$\gamma$-*critical graph. Then each longest cycle of* $G$ *is a dominating cycle.*

**Proof.** Suppose to the contrary that there exists a longest cycle $C$ of $G$ and a component $H$ of $G - V(C)$ such that $|V(H)| \geq 2$. Let $C$ have an arbitrary orientation. Let the sets $X$, $A$, $B$ and $C_i$ be defined as above.

Suppose $|X| = 1$. Then $X = \{x_1\}$ and $x_1$ is a cut-vertex. By Theorem 9 there exists an end-vertex $v$ such that $vx_1 \in E(G)$. But $\langle N(x_1) \backslash \{v\} \rangle$ is not a clique since $a_1$ is not adjacent to any vertex of $H$, by Lemma 19. This is contrary to Theorem 6(b) and so we can assume that $|X| \geq 2$.

We denote the longest $(x_i, x_j)$-path with internal vertices in $H$ by $x_i H x_j$. We first prove the following lemma.

**Lemma 21.1** $Y(A) \subseteq X$ *and* $Y(B) \subseteq X$.

**Proof.** By symmetry, we only need to show that $Y(A) \subseteq X$. By Lemma 3, since $A$ is independent, there exists an ordering $F(A) = (a_{j_1}, a_{j_2}, \ldots, a_{j_k})$ of the vertices of $A$ and a sequence $Y(A) = (y_1, y_2, \ldots, y_{k-1})$ such that $[a_{j_i}, y_i] \rightarrow a_{j_{i+1}}$ for each $i$ with $1 \leq i \leq k-1$.

By Lemma 19, $y_t$ dominates $V(H)$ for each $t$ with $1 \leq t \leq k-1$. Therefore $y_t \in V(H) \cup X$. We will show that $y_t \in X$ for each $t$ with $1 \leq t \leq k-1$. Suppose to the contrary that $y_t \in V(H)$ for some $t$ with $1 \leq t \leq k-1$. Then $k = 2$, for otherwise $\{a_{j_t}, y_t\}$ cannot dominate $A \backslash \{a_{j_t}, a_{j_{t+1}}\}$, a contradiction. Without loss of generality, we can assume that $[a_1, y_1] \rightarrow a_2$.

Any $(x_1, x_2)$-path $P$ (also written $x_1 \overrightarrow{P} x_2$) that is internally disjoint from $C$ contains at most one vertex of $V(H)$, for suppose the contrary. Then by the maximality of $C$, $|C_2| \geq 2$. Thus $a_2^+ \neq x_1$, and so $a_1$ dominates $a_2^+$. But then

$$x_1 \overrightarrow{P} x_2 \overleftarrow{C} a_1 a_2^+ \overrightarrow{C} x_1$$

is a cycle longer than $C$, a contradiction.

Thus $N_H(x_1) = N_H(x_2) = \{u\}$ for some vertex $u \in V(H)$. Since $|V(H)| = 2$, $u$ is a cut-vertex of $G$. So by Theorem 9, the vertex $v$ (where $\{v\} = V(H) \setminus \{u\}$) is an end-vertex. By Theorem 6(b), $x_1 x_2 \in E(G)$.

Since $[a_1, y_1] \rightarrow a_2$ and $y_1 = u$, $V(G) \setminus (V(H) \cup X \cup A) \subseteq N(a_1)$. By the maximality of $C$,

$$a_2 \notin N(z) \text{ for any } z \in V(C_1), \tag{1}$$

for otherwise

$$x_1 \overleftarrow{C} a_2 z \overleftarrow{C} a_1 z^+ \overrightarrow{C} x_2 u x_1$$

is a cycle longer than $C$ if $z \neq b_1$ and

$$x_1 \overleftarrow{C} a_2^+ a_1 \overrightarrow{C} b_1 a_2 x_2 u x_1$$

is a cycle longer than $C$ if $z = b_1$. We now consider two cases.

*Case* 1 $V(G) \setminus (V(C) \cup V(H)) \neq \phi$.
Let $H_1 \neq H$ be a component of $G - C$ and choose $u_1 \in V(H_1)$. Then $V(H_1) \subseteq N(a_1)$. By Lemma 20, $a_2 u_1 \notin E(G)$, and so there exists a vertex $y$ such that

$$[u_1, y] \rightarrow a_2 \quad \text{or} \quad [a_2, y] \rightarrow u_1.$$

In either case, $y \in V(H)$ in order to dominate $v$. Consequently, the second case is impossible since $a_1 \notin N(a_2) \cup N(y)$. Hence $[u_1, y] \rightarrow a_2$, and $u_1$ must dominate $V(C) \setminus \{x_1, x_2, a_2\}$. If $u_1$ also dominated $x_1$, then

$$u_1 a_1 \overrightarrow{C} x_1 u_1$$

would be a cycle longer than $C$, a contradiction. Therefore $y$ dominates $\{x_1, x_2, v\}$ and so $y = u$. Again, by the maximality of $C$ $a_1 = b_1$ and $a_2 = b_2$. Now consider the vertices $u$ and $a_1$. Since $ua_1 \notin E(G)$, there exists a vertex $y'$ such that

$$[a_1, y'] \rightarrow u \quad \text{or} \quad [u, y'] \rightarrow a_1.$$

The former case is impossible since $y'$ must belong to $V(H)$ to dominate $v$, but $V(H) = \{u, v\}$ and $v$ is adjacent to $u$. The second case is also impossible since $y'$ must belong to $V(H_1)$ to dominate $u_1$, but $V(H_1) \subseteq N(a_1)$. This contradicts $\gamma(G) = 3$, and so $Y(A) \subseteq X$.

*Case* 2 $V(G - C) = V(H)$.
Since $\gamma(G) = 3$, neither $\{u, x_1\}$ nor $\{u, x_2\}$ can dominate $V(G)$. Thus

$$N_C[x_i] \neq V(C), \quad \text{for} \quad i = 1, 2. \tag{2}$$

Suppose $a_2 = b_2$. Then $a_1 \neq b_1$, by 2. Consider the vertices $v$ and $a_2$. Since $va_2 \notin E(G)$, there is a vertex $y$ such that

$$[a_2, y] \to v \quad \text{or} \quad [v, y] \to a_2.$$

In the former case, $y \in \{x_1, x_2\}$ since $ua_2 \notin E(G)$. But this is impossible by 1 and 2. Thus $[v, y] \to a_2$, and consequently $y \in (N(x_1) \cap N(x_2)) \setminus \{u, a_2\}$, *i.e.*, $y \in C_1$. Since $yu \notin E(G)$, there exists a vertex $y'$such that

$$[y, y'] \to u \quad \text{or} \quad [u, y'] \to y.$$

The former case is impossible since $v$ cannot be dominated without $u$ being dominated. Consider the latter case. Since $y \in N(x_1) \cap N(x_2) \cap C_1$, either $a_2$ or a vertex of $C_1$ will not be dominated by 1 and 2, and so this case is also impossible. This contradicts $\gamma(G) = 3$, and so $Y(A) \subseteq X$.

Suppose $a_2 \neq b_2$. Then $a_1 b_2 \in E(G)$. By the maximality of $C$, $a_1 \neq b_1$, $x_1 b_1 \notin E(G)$ and $a_2 x_1 \notin E(G)$. Thus there exists a vertex $y$ such that

$$[y, x_1] \to b_1 \quad \text{or} \quad [y, b_1] \to x_1.$$

In either case, $y \in V(H)$ to dominate $v$, and so $a_2 \notin N(y)$. The first case is impossible since $a_2 \notin N(x_1)$ and the second case is impossible since $a_2 \notin N(b_1)$ by 1. This contradicts $\gamma(G) = 3$, consequently $Y(A) \subseteq X$ and the proof of Lemma 21.1 is complete. □

By Lemma 21.1, $Y(A) \subseteq X$, and so we write $X(A) = (x_{i_1}, x_{i_2}, ..., x_{i_{k-1}})$ for an ordering of the $k-1$ vertices of $X$ such that $[a_{j_t}, x_{i_t}] \to a_{j_{t+1}}$ for each $t$ with $1 \leq t \leq k-1$, in what follows.

For each $t$ with $1 \leq t \leq k-1$, we have that

$$A \setminus \{a_{j_t}, a_{j_{t+1}}\} \subseteq N(x_{i_t}); \; a_{j_{t+1}} \notin N(x_{i_t}) \tag{3}$$

and

$$V(H) \subseteq N(x_{i_t}). \tag{4}$$

Let $X \setminus X(A) = \{x_{i_k}\}$ and choose $u \in V(H) \cap N(x_{i_k})$. Then

$$u \in \bigcap_{t=1}^{k} N(x_{i_t}). \tag{5}$$

From 4 and the fact that $|V(H)| \geq 2$, we have:

**Lemma 21.2** *Between any two vertices $x_i$ and $x_j$ of $X$, there is an $(x_i, x_j)$-path $P$ internally disjoint from $C$ that contains at least two vertices of $V(H)$.*

By Lemma 21.2 and the maximality of $C$, $|C_i| \geq 2$ for each $i$ with $1 \leq i \leq k$. We now consider two cases depending on $|N(V(H))| = k$.

**Case 1** $k \geq 3$.

(i) Consider the vertices $u$ and $b_{i_{k-1}}$. Since $ub_{i_{k-1}} \notin E(G)$, there exists a vertex $y$ such that

$$[u, y] \to b_{i_{k-1}} \quad \text{or} \quad [b_{i_{k-1}}, y] \to u.$$

Suppose $[b_{i_{k-1}}, y] \to u$. Then $y \notin X$ by 5 and so $y \in V(H)$ to dominate $V(H) \setminus \{u\}$. But no vertex of $B \setminus \{b_{i_{k-1}}\}$ is dominated by $\{y, b_{i_{k-1}}\}$ (Lemma 19 applied to $B$), a contradiction. Thus $[u, y] \to b_{i_{k-1}}$. Obviously $y \neq x_k$ and by 3, $y \neq x_{i_t}$ for any $t$ with $1 \leq t \leq k-1$. Hence $y \in V(C) \setminus X$. Moreover,

$$V(C) \setminus (X \cup \{b_{i_{k-1}}, y\}) \subseteq N(y);\ b_{i_{k-1}} \notin N[y]. \tag{6}$$

We need the following lemma.

**Lemma 21.3** $y \notin A \cup B \cup A^+ \cup B^-$.

**Proof.** By Lemma 19, $A \cup \{u\}$ is independent, and so if $y \in A$, $\{u, y\}$ does not dominate $A \setminus \{y\}$, a contradiction. Thus $y \notin A$, and by a similar argument $y \notin B$. Suppose to the contrary that $y \in A^+$. Let $y = a_i^+$ for some $i$ with $1 \leq i \leq k$ and let $a_j \in A \setminus \{a_i\}$. By 6, $a_j a_i^+ \in E(G)$. Thus

$$x_i \overrightarrow{P} x_j \overleftarrow{C} a_i^+ a_j \overrightarrow{C} x_i$$

is a cycle longer than $C$, a contradiction. Thus $y \notin A^+$. By a symmetric argument $y \notin B^-$. This completes the proof of Lemma 21.3. $\square$

(ii) We next consider the vertices $u$ and $y$. Since $uy \notin E(G)$, there exists some vertex $y_1$ such that

$$[y_1, u] \to y \quad \text{or} \quad [y, y_1] \to u.$$

Suppose $[y, y_1] \to u$. Then $y_1 \notin X$ by 5. Hence $y_1 \in V(H)$ in order to dominate $V(H) \setminus \{u\}$. But then $b_{i_{k-1}}$ cannot be dominated by $\{y, y_1\}$, a contradiction. Thus $[y_1, u] \to y$. By 6, $y_1 \in X \cup \{b_{i_{k-1}}\}$. But since $y_1$ dominates $B$, $y_1 \neq b_{i_{k-1}}$, by Lemma 19. Thus $y_1 \in X$. By 3, $y_1 = x_{i_k}$, that is, $[u, x_{i_k}] \to y$. By Lemma 21.3, $y \notin A \cup B$, so $y^+, y^- \notin X = N_C(u)$. Hence

$$\{y^+, y^-\} \cup A \cup B \subseteq N(x_{i_k}). \tag{7}$$

Now consider the independent set $B$. As in the case of $A$ there is an ordering

$$F(B) = (b_{l_1}, b_{l_2}, \ldots, b_{l_k})$$

of the vertices of $B$ and an ordering

$$X(B) = (x_{m_1}, x_{m_2}, ..., x_{m_{k-1}})$$

of $k-1$ vertices of $X$ such that $[x_{m_t}, b_{l_t}] \to b_{l_{t+1}}$ for each $t$ with $1 \le t \le k-1$. By 7, $B \subseteq N(x_{i_k})$. So $x_{i_k} \notin X(B)$ and thus $\{x_{i_1}, x_{i_2}, ..., x_{i_{k-1}}\} = \{x_{m_1} x_{m_2}, ..., x_{m_{k-1}}\}$. Analogous to 3, we get

$$B \setminus \{b_{l_t}, b_{l_{t+1}}\} \subseteq N(x_{m_t}); \; b_{l_{t+1}} \notin N[x_{m_t}] \text{ for each } t \text{ with } 1 \le t \le k-1. \tag{8}$$

We now prove the following lemma.

**Lemma 21.4** (a) *$A \cup \{y^+\}$ and $B \cup \{y^-\}$ are independent.*

(b) $y^+y^- \notin E(G)$.

**Proof.** (a) Suppose to the contrary that $A \cup \{y^+\}$ is not independent. Then $a_i y^+ \in E(G)$ for some $a_i \in A$.

Suppose $y \in C_i$. By 6, $y a_{i+1} \in E(G)$, so

$$x_i H x_{i+1} \overleftarrow{C} y^+ a_i \overrightarrow{C} y a_{i+1} \overrightarrow{C} x_i$$

is a cycle longer than $C$, a contradiction.

Thus $y \notin C_i$. But then the cycle

$$a_i y^+ \overrightarrow{C} x_i H x_{i+1} \overrightarrow{C} y b_i^- \overleftarrow{C} a_i$$

is a cycle longer than $C$ since $y b_i^- \in E(G)$ by 6 and $x_i H x_{i+1}$ contains at least two vertices. This contradicts the maximality of $C$. Hence $A \cup \{y^+\}$ is independent. Similarly we can show that $B \cup \{y^-\}$ is independent.

(b) Suppose to the contrary that $y^+y^- \in E(G)$. Let $y \in C_i \setminus \{a_i, b_i\}$ for some $i$ with $1 \le i \le k$. By 6, $\{a_i, a_{i+1}\} \subseteq N(y)$, and so the cycle

$$y a_{i+1} \overrightarrow{C} x_i H x_{i+1} \overleftarrow{C} y^+ y^- \overleftarrow{C} a_i y$$

is longer than $C$, a contradiction. This completes the proof of Lemma 21.4. □

(iii) Now consider the vertices $y^+$ and $y^-$. Since $y^+y^- \notin E(G)$, there is a vertex $y_2$ such that

$$[y^+, y_2] \to y^- \quad \text{or} \quad [y^-, y_2] \to y^+.$$

In either case $y_2 \in X \cup V(H)$ to dominate $V(H)$ since $y \notin A \cup B$ by Lemma 21.3. By Lemma 21.4(a), it follows that $y_2 \in X$ to dominate $A \cup B$. More precisely, by 7 it follows that $y_2 \in X \setminus \{x_{i_k}\}$. Let $y_2 = x_{i_t}$ for some $t$

with $1 \le t \le k-1$. By Lemma 21.3, $y^- \notin A$. Thus if $[y^+, x_{i_t}] \to y^-$, $A \subseteq N(y^+) \cup N(x_{i_t})$. But by Lemma 21.4(a), $a_{j_{t+1}} \notin N(y^+)$ and by 3 $a_{j_{t+1}} \notin N(x_{i_t})$, a contradiction.

Similarly $[y^-, y_2] \to y^+$ is impossible as one vertex of $B$ will not be dominated by $N(y^-) \cup N(y_2)$, contradicting $\gamma(G) = 3$. Hence $|V(H)| = 1$ when $k \ge 3$.

**Case 2** $k = 2$.
Assume without loss of generality that $a_{j_1} = a_1$ and $a_{j_2} = a_2$. Then $x_{i_1} = x_1$ and $x_{i_k} = x_{i_2} = x_2$. From the proof of Lemma 21.3 it is easy to see that $y \notin A \cup B \cup A^+ \cup B^-$ except when $y = b_2^-$ and $[u, y] \to b_1$. Hence if $y \ne b_2^-$, then we can proceed in the same way we did in Case 1. Suppose that $y = b_2^-$. Then $y$ is not adjacent to $u$. So there exists a vertex $y_1$ such that

$$[u, y_1] \to y \quad \text{or} \quad [y, y_1] \to u.$$

Using a similar argument as in (ii) when $k \ge 3$, we have that $[u, y_1] \to y$, *i.e.* $[u, y_1] \to b_2^-$. Since 6 holds, $y_1 \in X$.

Now $b_2^- \ne a_2$, for otherwise $\{b_2^-, u\}$ does not dominate $A$, contradicting $\{b_2^-, u\}$ dominating $V(G) \setminus \{b_2\}$. Thus $y_1 = x_2$ to dominate $a_2$. By the proof of Lemma 21.4(b), $b_2 b_2^{-2} \notin E(G)$. Thus there exists some vertex $y_2$ such that

$$[b_2, y_2] \to b_2^{-2} \quad \text{or} \quad [b_2^{-2}, y_2] \to b_2.$$

In either case, $y_2 \in X$ to dominate $A \cup \{u\}$ and by 7, $y_2 = x_1$. But $[b_2^{-2}, x_1] \to b_2$ is impossible since $x_1 b_2 \in E(G)$. Thus $[b_2, x_1] \to b_2^{-2}$.

Further, $b_2^- a_1 \in E(G)$ by 6, and hence $b_2^{-2} \ne a_2$, for otherwise

$$x_1 H x_2 \overleftarrow{C} a_1 b_2^- b_2 x_1$$

is a cycle longer than $C$ by Lemma 21.2, a contradiction. Since $[b_2, x_1] \to b_2^-$, either $b_2 a_2 \in E(G)$ or $x_1 a_2 \in E(G)$. But then

$$x_1 H x_2 \overleftarrow{C} a_1 b_2^- \overleftarrow{C} a_2 b_2 x_1$$

or

$$x_1 H x_2 \overleftarrow{C} a_1 b_2^- \overleftarrow{C} a_2 x_1$$

is a cycle longer than $C$ by Lemma 21.2, a contradiction. This contradicts $\gamma(G) = 3$ and hence $|V(H)| = 1$.

The proof of Theorem 21 is now complete. ■

The following result is an obvious consequence of Theorem 2.19.

**Corollary 22** [13] *Let $G$ be a connected, 3-$\gamma$-critical graph. Then $G$ has a cycle $C$ such that $V(C)$ dominates $V(G)$.*

The next theorem, also proved by Flandrin, Tian, Wei and Zhang in [6], uses Theorem 21 to establish a lower bound for a longest cycle of a 3-$\gamma$-critical graph. This result is crucial in the proof of Wojcicka's Conjecture.

**Theorem 23** [6] *Let $G$ be a connected, 3-$\gamma$-critical graph of order $n$ and $C$ a longest cycle of $G$. If there exists a vertex $u$ in $V(G)\backslash V(C)$ such that $|N(u) \cap V(C)| \geq 2$, then $c(G) \geq n-1$. In particular, if $G$ is 2-connected, then $c(G) \geq n-1$.*

**Proof.** Let $C$ be a longest cycle of $G$, a connected 3-$\gamma$-critical graph and suppose that there is a vertex $y \in V(G)\backslash V(C)$ such that $\deg_C(y) \geq 2$. We will show that $|V(C)| \geq n-1$.

By Theorem 21, $V(G)\backslash V(C)$ is an independent set, so $\deg_C(u) = \deg(u)$. Take $u \in V(G)\backslash V(C)$ such that

$$\deg(u) = \max\{\deg(y) \mid y \in V(G)\backslash V(C)\}.$$

Let $\deg(u) = k$. Then by the hypothesis, $k \geq 2$. Let $X = N(u)$ and define the sets $A$, $B$ and $C_i$ as before. We first prove the following lemma.

**Lemma 23.1** *Let $y \in C_i\backslash\{a_i, b_i\}$ for some $i$ with $1 \leq i \leq k$. If $A \subseteq N(y)$ and $b \in N(y)$ for some $b \in \overrightarrow{C}[x_{i+1}, x_i] \cap B$, then $A \cup \{y^+\}$ is an independent set in $G$.*

**Proof.** Suppose that $b_t y \in E(G)$ for some $t \neq i$. Since $a_{i+1} y \in E(G)$, we have that $a_j y^+ \notin E(G)$ for any $j \neq i+1$, otherwise

$$x_{i+1} u x_j \overleftarrow{C} a_{i+1} y \overleftarrow{C} a_j y^+ \overrightarrow{C} x_{i+1}$$

is a cycle longer than $C$, a contradiction. Now suppose that $a_{i+1} y^+ \in E(G)$. Then

$$x_{i+1} \overleftarrow{C} y^+ a_{i+1} \overrightarrow{C} b_t y \overleftarrow{C} x_{t+1} u x_{i+1}$$

is a cycle longer than $C$, a contradiction. Thus $A \cap N(y^+) = \phi$. Since $A$ is an independent set of $G$, the result follows. □

To continue the proof of Theorem 23, assume to the contrary that $|V(C)| \leq n-2$. Let $v$ be an arbitrary vertex of $V(G-C)\backslash\{u\}$. We prove the following lemma.

**Lemma 23.2** $|N(u) \cap N(v)| \geq k-1$.

**Proof.** Since $A$ is independent, Lemma 3 asserts the existence of an ordering $F(A) = (a_{j_1}, a_{j_2}, ..., a_{j_k})$ of the vertices of $A$ and a sequence $Y(A) = (y_1, y_2, ..., y_{k-1})$ of $k-1$ distinct vertices such that $[a_{j_t}, y_t] \to a_{j_{t+1}}$ for each $t$ with $1 \leq t \leq k-1$. If $v \notin N(A)$, then $Y(A) \subseteq N(v)$. Also, $Y(A) \subseteq X = N(u)$ since $N(u) \cap A = \phi$. Thus $|N(u) \cap N(v)| \geq |Y(A)| = k-1$.

If $v \in N(A)$, then by Lemma 20, $|N(v) \cap A| = 1$. Without loss of generality, let $a_1 v \in E(G)$. Then $a_1^+ v \notin E(G)$ as $C$ is a longest cycle in $G$. Also $a_1^+ u$ and $a_1^+ a_j$ do not belong to $E(G)$ for any $j \neq 1$, for otherwise the cycle

$$C' = x_1 u a_1^+ \overrightarrow{C} x_1$$

or

$$C' = x_j \overleftarrow{C} a_1^+ a_j \overrightarrow{C} x_1 u x_j$$

is a cycle of maximum length with $a_1 v \in E(G - C')$, contradicting Theorem 21. Thus $A_1 = \{a_1^+, a_2, ..., a_k\}$ is an independent set of $G$. There exists an ordering $F(A_1)$ of the vertices of $A_1$ and a sequence $Y(A_1)$ of $k-1$ distinct vertices such that Lemma 3 holds. Since $N(u) \cap A_1 = N(v) \cap A_1 = \phi$, $Y(A_1) \subseteq N(u) \cap N(v)$. Thus $|N(u) \cap N(v)| \geq |Y(A_1)| = k-1$. $\square$

By Lemma 23.2 and the choice of $u$, $k-1 \leq \deg(v) \leq k$ for any vertex $v \in V(G-C) \setminus \{u\}$. We consider the following two cases.

**Case 1** $\deg(v) = k-1$.
Without loss of generality we may assume that $N(v) = X \setminus \{x_1\}$. Since $uv \notin E(G)$, there exists a vertex $y$ such that

$$[y, v] \to u \quad \text{or} \quad [u, y] \to v.$$

We now show that $y \notin X$ in either case.

In the former case, $y \notin X$ since $N(u) = X$.

In the latter case, $y \notin X \setminus \{x_1\}$ since $N(v) = X \setminus \{x_1\}$. Moreover, $y \neq x_1$, for otherwise $V(G) \setminus (X \cup \{v\}) \subseteq N(x_1)$ which implies that $\{v, x_1\}$ dominates $V(G)$, contrary to $\gamma(G) = 3$.

Since $(A \cup B) \cap (N(u) \cup N(v)) = \phi$, $(A \cup B) \subseteq N(y)$. By Lemma 20 and the fact that $A$ and $B$ are independent sets in $G$, $y \in V(C) \setminus (X \cup A \cup B)$. By Lemma 23.1, $A_2 = A \cup \{y^+\}$ is an independent set of $G$. Thus there exists an ordering $F(A_2)$ of $A_2$ and a sequence $Y(A_2)$ of $k$ distinct vertices such that Lemma 3 holds. Since $N(u) \cap A_2 = N(v) \cap A_2 = \phi$, $Y(A_2) \subseteq N(u) \cap N(v)$. Thus $|N(u) \cap N(v)| \geq |Y(A_2)| = k$, contradicting $\deg(v) = k-1$.

**Case 2** $\deg(v) = k$.

Since $v$ is an arbitrary vertex of $V(G-C)-\{u\}$ and $k \geq 2$, we may assume $\delta(G) \geq 2$. We consider two subcases.

**Case 2.1** $N(u) = N(v)$.
Since $uv \notin E(G)$, there exists a vertex $y$ such that

$$[v, y] \rightarrow u \quad \text{or} \quad [u, y] \rightarrow v.$$

Since $N(u) = N(v) = X$, $y \notin X$. By Lemma 19, $A \cup B \subseteq N(y)$. Hence $y \in V(C) \setminus (X \cup A \cup B)$. Since $y \notin A \cup B$, $N(y^+) \cap \{u, v\} = \phi$, and (by Lemma 23.1) $A \cup \{y^+\}$ is independent. Further, $A_3 = A \cup \{y^+, v\}$ is an independent set of $G$. Thus there exists an ordering $F(A_3)$ of the vertices of $A_3$ and a sequence $Y(A_3)$ of $k+1$ distinct vertices that satisfy Lemma 3. Since $N(u) \cap A_3 = \phi$, $Y(A_3) \subseteq N(u)$. Thus $|N(u)| \geq |Y(A_3)| = k+1$, contrary to $\deg(u) = k$.

**Case 2.2** $N(u) \neq N(v)$.
By Lemma 23.2, $|N(u) \cap N(v)| = k-1$, and we assume without loss of generality that $N(u) \cap N(v) = X \setminus \{x_1\}$. Since $uv \notin E(G)$, there exists a vertex $y$ such that

$$[y, v] \rightarrow u \quad \text{or} \quad [y, u] \rightarrow v.$$

By symmetry, we only deal with the case $[y, v] \rightarrow u$. Using similar arguments to those in Case 1, we can show that $y \in V(C) \setminus X$. If $v \notin N(A) \cup N(B)$, then $A \cup B \subseteq N(y)$. Since $A$ and $B$ are independent, $y \in C_i \setminus \{a_i, b_i\}$ for some $i$ with $1 \leq i \leq k$. This implies that $y^+ \notin N(u) \cap N(v)$. By Lemma 23.1, $A \cup \{y^+\}$ is an independent set of $G$. Thus there exists an ordering $F(A \cup \{y^+\})$ of the vertices of $A \cup \{y^+\}$ and a sequence $Y(A \cup \{y^+\})$ of $k$ distinct vertices such that Lemma 3 is satisfied. Since $(A \cup \{y^+\}) \cap N(u) \cap N(v) = \phi$, $Y(A \cup \{y^+\}) \subseteq (N(u) \cap N(v))$. Thus $|N(u) \cap N(v)| \geq |Y(A \cup \{y^+\})| = k$, a contradiction.

Now suppose $v \in N(A) \cup N(B)$. Without loss of generality assume that $va_1 \in E(G)$. Let $A_4 = (A \setminus \{a_1\}) \cup \{a_1^+\}$. Then $A_4$ is independent by the maximality of $C$. Also, $ua_1^+ \notin E(G)$, for otherwise $ua_1^+ \overrightarrow{C} x_1 u$ is a longest cycle that is not dominating, which contradicts Theorem 21. Therefore $A_4 \cup \{u\}$ is independent.

By the maximality of $C$, $a_1^+ v \notin E(G)$ and hence $a_1 \neq b_1$. Since $N(v) = (X \setminus \{x_1\}) \cup \{a_1\}$, $(A_4 \cup B) \cap N(v) = \phi$. Thus $A_4 \cup B \subseteq N(y)$. Since $A_4$ and $B$ are independent sets and $N(u) = X$, $y \in V(C) \setminus (A_4 \cup B \cup X)$.

Let $y \in C_i \setminus \{a_i, b_i\}$ for some $i$ with $1 \leq i \leq k$, and define $A_5 = A_4 \cup \{y^+\}$. We complete the proof by showing that $A_5$ is independent. Since $A_5 \cap (N(u) \cap N(v)) = \phi$, the sequence $Y(A_5)$, as defined in Lemma 3, is contained in $N(u) \cap N(v)$. This implies that $|N(u) \cap N(v)| \geq |Y(A_5)| = k$, a contradiction.

Since $A_4 \cap (N(u) \cap N(v)) = \phi$ and $y^+ \notin X \cup \{a_1\}$, $A_5 \cap (N(u) \cap N(v)) = \phi$. We now show that $A_5$ is independent. Since $A_4$ is independent, we need only show that $N(y^+) \cap A_4 = \phi$, where $y \in C_i \setminus \{a_i, b_i\}$.

Since $b_{i-1}y \in E(G)$, it follows that $y^+a_j \notin E(G)$ for any $j \neq i$, for otherwise

$$x_i \overrightarrow{C} y b_{i-1} \overleftarrow{C} a_j y^+ \overrightarrow{C} x_j v x_i$$

is a cycle longer than $C$. To show that $\{a_i y^+, a_1^+ y^+\} \cap E(G) = \phi$, we consider the cases $i \neq k$ and $i = k$ separately.

Consider $i \neq k$. Since $a_{i+1}y \in E(G)$, $\{a_i y^+, a_1^+ y^+\} \cap E(G) = \phi$, for otherwise

$$x_{i+1} \overleftarrow{C} y^+ a_i \overrightarrow{C} y a_{i+1} \overrightarrow{C} x_i u x_{i+1}$$

is a cycle longer than $C$, or

$$x_{i+1} \overleftarrow{C} y^+ a_1^+ \overrightarrow{C} y a_{i+1} \overrightarrow{C} x_1 u x_{i+1}$$

is a longest cycle in $G$ that is not dominating, a contradiction.

Now consider $i = k$. Since $\{b_{k-1}y, a_1^+ y\} \subseteq E(G)$, $\{a_1^+ y^+, a_k y^+\} \cap E(G) = \phi$, for otherwise

$$x_k \overrightarrow{C} y b_{k-1} \overleftarrow{C} a_1^+ y^+ \overrightarrow{C} x_1 u x_k$$

or

$$x_1 \overleftarrow{C} y^+ a_k \overrightarrow{C} y a_1^+ \overrightarrow{C} x_k u x_1$$

is a longest cycle of $G$ but is not dominating, a contradiction. Thus $A_5$ is independent, which completes the proof. Thus $c(G) \geq n-1$. ■

The next result is a direct consequence of Theorem 2.21 and Corollary 2.8.

**Corollary 24** [13] *Let $G$ be a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$. Then $c(G) \geq n-1$.*

As indicated in [6], we can use Theorems 21 and 23 to prove Theorems 1 and 2. We present the proofs below for the sake of completeness. Note that by Theorem 21, we can choose a longest dominating cycle $C$ of $G$.

**Proof of Theorem 1.** If $|C| \geq n-1$, then $G$ contains a hamiltonian path. Assume that $|C| \leq n-2$. Then by Theorem 23, $\deg_C(u) = 1$ for any vertex $u \in V(G-C)$. Since $G$ is 3-$\gamma$-critical, we can easily prove that $|C| = n-2$

when $n > 6$. Let $\{u,v\} = V(G-C)$, $\{x\} = N_C(u)$ and $\{y\} = N_C(v)$. Since $C$ is a dominating cycle, $u$ and $v$ are end-vertices. By Theorem 6(a), $x \neq y$. When $x = y^+$ or $x = y^-$, then $G$ contains a hamiltonian path. When $x \neq y^+$ and $x \neq y^-$, then $x^+y^+ \in E(G)$, for suppose this is not the case. Then there is a vertex $z$ such that $[x^+, z] \to y^+$ or $[y^+, z] \to x^+$. In either case, $\{u,v\} \subseteq N(z)$, which is impossible. Thus $x^+y^+ \in E(G)$ and $ux\overleftarrow{C}y^+x^+\overrightarrow{C}yv$ is a hamiltonian path. ■

**Proof of Theorem 2.** If $c(G) = n-1$, then clearly $G - V_1(G)$ is hamiltonian. If $c(G) \leq n-2$, then by Theorem 23, $\deg(u) = 1$ for any vertex $u \in V(G-C)$. So $G - V_1(G)$ is hamiltonian. ■

The proof of Wojcicka's Conjecture, which is given in Sections 6 and 7, is obtained through contradiction. The results that follow apply to the longest cycles of a 3-$\gamma$-critical graph $G$ of order $n$ with minimum degree $\delta(G) \geq 2$, and which we assume is not hamiltonian.

Let $C$ be a longest cycle of such a graph $G$. Then by Theorem 23, $|C| = n-1$ and so $G - V(C)$ has only one component $H$ which consists of a single vertex. Let $V(H) = \{x_0\}$ and let the sets $X$, $A$, $B$ and $C_i$ be defined as earlier in the section. In this instance we let $\deg(x_0) = r$, and so

$$\begin{aligned}
X &= \{x_1, x_2, \ldots, x_r\} = N(x_0), \\
A &= \{a_1, a_2, \ldots, a_r\} = X^+, \quad i.e.\ a_i = x_i^+, \\
B &= \{b_1, b_2, \ldots, b_r\} = X^-; \quad i.e.\ b_{i-1} = x_i^- \\
\text{and } C_i &= \overrightarrow{C}[a_i, b_i] \quad \text{for each } i \text{ with } 1 \leq i \leq r,
\end{aligned}$$

where the indices are taken modulo $r$. We observe the following as a result of the maximality of $C$:

- $A$ and $B$ are independent sets.
- $A \cap X = \phi$, which implies that $A \cup \{x_0\}$ is an independent set.
- $B \cap X = \phi$, which implies that $B \cup \{x_0\}$ is an independent set.

We say that a vertex $v$ of $C_i$ is an *A-vertex* if $v^+a_i \in E(G)$ and a *B-vertex* if $v^-b_i \in E(G)$. It is easy to see that each $a_i \in A$ (each $b_i \in B$, respectively), for each $i$ with $1 \leq i \leq r$, is an $A$-vertex (a $B$-vertex, respectively).

The following lemmas result from the maximality of $C$.

**Lemma 25** [5] *Let $u_i \in C_i$ and $u_j \in C_j$ be two A-vertices (or B-vertices) with $i \neq j$. Then $u_i u_j \notin E(G)$.*

**Proof.** Suppose to the contrary that $u_i u_j \in E(G)$. Then the cycle

$$u_i u_j \overleftarrow{C} a_j u_j^+ \overrightarrow{C} x_i x_0 x_j \overleftarrow{C} u_i^+ a_i \overrightarrow{C} u_i$$

is longer than $C$, a contradiction. ■

**Lemma 26** [5] (a) *Let $u_i \in C_i$ and $u_j \in C_j$ be two A-vertices (or B-vertices) with $i \neq j$. For any vertex $v \in \overrightarrow{C}\left[u_i^+, u_j^-\right]$, if $vu_i \in E(G)$ then $v^- u_j \notin E(G)$.*

(b) *Let $a_i \in A$ and $b_j \in B$ such that $i \neq j+1$. For any vertex $v \in \overrightarrow{C}[a_{j+1}, x_i]$, if $vb_j \in E(G)$, then $v^- a_i \notin E(G)$.*

**Proof.** (a) Suppose to the contrary that $\{vu_i, v^- u_j\} \subseteq E(G)$. If $v = a_j$ then $vu_i \notin E(G)$, and if $v = u_i^+$ then $v^- u_j \notin E(G)$ by Lemma 25. So we can assume that $v \notin \{a_j, u_i^+\}$.

If $v \in \overrightarrow{C}\left[u_i^{+2}, x_j\right]$ then the cycle

$$x_j x_0 x_i \overleftarrow{C} u_j^+ a_j \overrightarrow{C} u_j v^- \overleftarrow{C} u_i^+ a_i \overrightarrow{C} u_i v \overrightarrow{C} x_j$$

is longer than $C$, a contradiction. If $v \in \overrightarrow{C}\left[a_j^+, u_j^-\right]$ then the cycle

$$x_j x_0 x_i \overleftarrow{C} u_j^+ a_j \overrightarrow{C} v^- u_j \overrightarrow{C} v u_i \overleftarrow{C} a_i u_i^+ \overrightarrow{C} x_j$$

is longer than $C$, a contradiction.

(b) Suppose to the contrary that $\{vb_j, v^- a_i\} \subseteq E(G)$. Then the cycle

$$v b_j \overleftarrow{C} a_i v^- \overleftarrow{C} x_{j+1} x_0 x_i \overleftarrow{C} v$$

is longer than $C$, a contradiction. ■

**Lemma 27** [13] *Let $a_i \in A$ and $b_j \in B$ such that $i \neq j+1$. For any $v \in \overrightarrow{C}[x_{j+1}, b_{i-1}]$, if $vb_j \in E(G)$, then $v^+ a_i \notin E(G)$.*

**Proof.** Suppose to the contrary that $\{vb_j, v^+ a_i\} \subseteq E(G)$. Then the cycle

$$x_{j+1} \overrightarrow{C} v b_j \overleftarrow{C} a_i v^+ \overrightarrow{C} x_i x_0 x_{j+1}$$

is longer than $C$, a contradiction. ■

**Lemma 28** [5] *Suppose $b_{j-1}a_j \in E(G)$. If $v \in C_i$ is an $A$-vertex with $i \neq j$, then $vx_j \notin E(G)$. If $v \in C_i$ is a $B$-vertex with $i \neq j-1$, then $vx_j \notin E(G)$.*

**Proof.** Suppose that $b_{j-1}a_j \in E(G)$ and that $v \in C_i$ is an $A$-vertex with $i \neq j$. Also suppose, to the contrary, that $vx_j \in E(G)$. Now $v$ and $a_j$ are $A$-vertices and $x_j \in \overrightarrow{C}\left[v^+, a_j^-\right]$. Since $b_{j-1}a_j \in E(G)$, $x_j v \notin E(G)$ by Lemma 26(a). The proof for the case where $v$ is a $B$-vertex follows similarly from Lemma 26. ■

**Lemma 29** [13] *Suppose $a_i b_j \in E(G)$ for some pair $i,j$ with $i \neq j+1$. $\{u, u^+\} \subseteq \overrightarrow{C}[a_i, b_j]$ and $\{v, v^+\} \subseteq \overrightarrow{C}\left[b_j^+, a_i^-\right]$. Then*

$$\left|\left\{uv, u^+v^+\right\} \cap E(G)\right| \leq 1 \text{ and } \left|\left\{u^+v, v^+u\right\} \cap E(G)\right| \leq 1.$$

**Proof.** Suppose to the contrary that $\{a_i b_j, uv, u^+v^+\} \subseteq E(G)$. Then the cycle

$$x_{j+1} \overrightarrow{C} vu \overleftarrow{C} a_i b_j \overleftarrow{C} u^+ v^+ \overrightarrow{C} x_i x_0 x_{j+1}$$

is longer than $C$, a contradiction. Similarly we can show that

$$\left|\left\{uv^+, u^+v\right\} \cap E(G)\right| \leq 1.$$

■

The following three results are obtained using the maximality of $C$ and the assumption that $\beta(G) \leq r+1$.

**Lemma 30** [5] (a) *If $v \in C_i$ is an $A$-vertex, then all the vertices of $\overrightarrow{C}[a_i, v]$ are $A$-vertices.*

(b) *If $v \in C_i$ is a $B$-vertex, then all the vertices of $\overrightarrow{C}[v, b_i]$ are $B$-vertices.*

*Moreover, if $a_i b_i \in E(G)$, then all the vertices in $C_i \backslash \{a_i, b_i\}$ are $A$-vertices and $B$-vertices.*

**Proof.** (a) Suppose that $v \in C_i$ is an $A$-vertex and that, to the contrary, there are vertices in $\overrightarrow{C}\left[a_i^+, v^+\right]$ that are not adjacent to $a_i$. Let $y$ be the last vertex in $\overrightarrow{C}\left[a_i^+, v^+\right]$ that is not adjacent to $a_i$. Then $y$ is an $A$-vertex, and by Lemma 25 $A \cup \{x_0, y\}$ is independent. But $|A \cup \{x_0, y\}| = r+2$, contradicting $\beta(G) \leq r+1$.

(b) The proof here is similar to (a).

Now suppose $a_ib_i \in E(G)$. Then $b_i^-$ is an $A$-vertex and so by (a), all the vertices in $\overrightarrow{C}\left[a_i, b_i^-\right]$ are $A$-vertices. Similarly all the vertices in $\overrightarrow{C}\left[a_i^+, b_i\right]$ are $B$-vertices. Thus all the vertices in $C_i \backslash \{a_i, b_i\}$ are both $A$-vertices and $B$-vertices. ∎

**Lemma 31** [5] *Let $u_i \in C_i$ be an $A$-vertex. If $N(u_i) \cap C_{i-1} \neq \phi$, then $u_ib_{i-1} \in E(G)$. Similarly, let $v_i \in C_i$ be a $B$-vertex. If $N(v_i) \cap C_{i+1} \neq \phi$, then $v_ia_{i+1} \in E(G)$.*

**Proof.** Suppose that $N(u_i) \cap C_{i-1} \neq \phi$, and that to the contrary $u_ib_{i-1} \notin E(G)$. Let $y$ be the last vertex in $C_{i-1}$ that is adjacent to $u_i$. By Lemma 26(a) $y^+$ is not adjacent to $a_j$ for $j \neq i$ and $y^+$ is not adjacent to $u_i$ by the choice of $y$. Also $N(u_i) \cap (A \backslash \{a_i\}) = \phi$ by Lemma 25. Hence $\{y^+, x_0, u_i\} \cup (A \backslash \{a_i\})$ is an independent set of $r+2$ vertices, contradicting $\beta(G) \leq r+1$. The proof is similar if $v_i$ is a $B$-vertex. ∎

**Lemma 32** [5] *For each $a_i \in A \backslash B$, $N(a_i) \cap B \neq \phi$ and for each $b_j \in B \backslash A$, $N(b_j) \cap A \neq \phi$.*

**Proof.** Let $a_i \in A \backslash B$ and suppose to the contrary that $N(a_i) \cap B = \phi$. Then $B \cup \{a_i, x_0\}$ is an independent set of $r+2$ vertices, contradicting $\beta(G) \leq r+1$. Similarly, $N(b_j) \cap A \neq \phi$ for each $b_j \in B \backslash A$. ∎

The following lemmas result from the maximality of $C$ and the assumption that $\deg(x_0) = \delta(G)$ and $\beta(G) = \delta(G) + 2$.

**Lemma 33** [13] *For each $a_i \in A$ (or $b_j \in B$), there exists a vertex $y$ such that $[a_i, y] \rightarrow x_0$ (or $[b_j, y] \rightarrow x_0$), where $y \notin X$.*

**Proof.** Since $a_ix_0 \notin E(G)$, there exists a vertex $y$ such that $[a_i, y] \rightarrow x_0$ or $[x_0, y] \rightarrow a_i$. Suppose that $[x_0, y] \rightarrow a_i$. Then $\{x_i, y\}$ dominates $G$ since $\langle X \rangle$ is a clique (by Theorem 16), contradicting $\gamma(G) = 3$. Thus $[a_i, y] \rightarrow x_0$ and obviously $y \notin X$. ∎

**Lemma 34** [13] (a) *If $a_ib_j \in E(G)$ for some pair i,j with $i \notin \{j, j+1\}$, then*

$$N(b_{i-1}) \cap \{a_{i+1}, a_{i+2}, \ldots, a_j\} = \phi.$$

(b) *If $a_ib_j \in E(G)$ for some pair i,j with $i \notin \{j, j-1\}$, then*

$$N(a_{j+1}) \cap \{b_i, b_{i+1}, \ldots, b_{j-1}\} = \phi.$$

**Proof.** (a) Suppose to the contrary that

$$a_\ell \in N(b_{i-1}) \cap \{a_{i+1}, a_{i+2}, \ldots, a_j\}.$$

By Theorem 16, $\langle X \rangle$ is a clique and so

$$b_{i-1} a_\ell \overrightarrow{C} b_j a_i \overrightarrow{C} x_\ell x_i x_0 x_{j+1} \overrightarrow{C} b_{i-1}$$

is a cycle longer than $C$, a contradiction.

The second case follows a similar proof. ■

# 6 The case $\beta \leq \delta + 1$

In this and the next section we complete the proof of Wojcicka's Conjecture, restated below for emphasis. To summarise, the following properties of a connected 3-$\gamma$-critical graph $G$ with $\delta(G) \geq 2$ have contributed towards the proof of Wojcicka's Conjecture:

| | | | |
|---|---|---|---|
| (i) | Theorem 15 | [5] | $\beta(G) \leq \delta(G) + 2$. |
| (ii) | Theorem 12 | [6] | $G$ is 1-tough. |
| (iii) | Theorem 21 | [6] | Each longest cycle of $G$ is a dominating cycle. |
| (iv) | Theorem 23 | [6] | $c(G) \geq n - 1$. |
| (v) | Theorem 15 | [5] | If $\beta(G) = \delta(G) + 2$, then every maximum independent set contains all vertices of degree $\delta(G)$. |
| (vi) | Theorem 16 | [5] | If $\beta(G) = \delta(G) + 2$ and $\deg(x) = \delta(G)$, then $N[x]$ induces a clique. |
| (vii) | Theorem 18 | [13] | If $\beta(G) = \delta(G) + 2$, then $G$ has only one vertex of degree $\delta(G)$. |

Result (i) helped to divide the proof of the conjecture into the two cases $\beta \leq \delta + 1$ and $\beta = \delta + 2$. Results (ii) and (iii) were used to obtain (iv), which is central to the proof of the conjecture. Further, (v) was used to obtain (vi) which was in turn used to prove (vii). Finally, (vi) and (vii) are crucial results in the proof of the case $\beta = \delta + 2$. Proofs of all these results have been given in the previous sections, as indicated.

**Conjecture 1** (Wojcicka's Conjecture) Every connected, 3-$\gamma$-critical graph $G$ with $\delta(G) \geq 2$ is hamiltonian.

Let $G$ be a 3-$\gamma$-critical graph of order $n$ and with $\delta(G) \geq 2$. If $C$ is a longest cycle of $G$, and $G$ is not hamiltonian, then it follows from (iv) that $|V(C)| = n - 1$. Thus $V(G - C)$ contains one vertex, which we will call $x_0$.

To prove the conjecture for $\beta(G) \leq \delta(G) + 1$, we first show that if $G$ is non-hamiltonian, then $\beta(G) \geq \deg(x_0) + 2$ (Theorem 35). Since

$\deg(x_0) \geq \delta(G)$, it is an obvious consequence of Theorem 35 that $G$ is hamiltonian when $\beta(G) \leq \delta(G) + 1$. Another consequence of Theorem 35 is that if $\beta(G) = \delta(G) + 2$, then $\deg(x_0) = \delta(G)$, an observation which is crucial to the proof of the case $\beta(G) = \delta(G) + 2$.

The proofs for both cases make extensive use of the results involving longest cycles, which occur in Section 5.

The following theorem will help to establish the conjecture for $\beta \leq \delta + 1$.

**Theorem 35** [5] *Let $G$ be a non-hamiltonian, 3-$\gamma$-critical graph with $\delta(G) \geq 2$ and let $C$ be a longest cycle of $G$ with $x_0$ the only vertex not on $C$. Then $\beta(G) \geq \deg(x_0) + 2$.*

**Proof.** Let the sets $X$, $A$, $B$ and $C_i$ be defined as in Section 5 and let $\deg(x_0) = r$. Suppose to the contrary that $\beta(G) \leq r + 1$. We consider three cases, depending on the value of $r$.

**Case 1** $r = 2$.
We first show that $A \cap B = \phi$. Suppose to the contrary that $a_1 = b_1$. Then $a_1 b_2 \notin E(G)$ and $b_1 a_2 \notin E(G)$ by Lemma 25. By Lemma 31, $E(C_1, C_2) = \phi$. Thus $G - \{x_1, x_2\}$ consists of three components, contradicting Corollary 13. Thus $a_1 \neq b_1$ and $a_2 \neq b_2$. By Lemma 32 either $\{a_1 b_1, a_2 b_2\} \subseteq E(G)$ or $\{a_1 b_2, a_2 b_1\} \subseteq E(G)$. We will show that in either case

$$\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\} \subseteq E(G).$$

**Case 1.1** $\{a_1 b_1, a_2 b_2\} \subseteq E(G)$.
By Lemma 30, the vertices in $C_i \backslash \{a_i, b_i\}$ are both $A$-vertices and $B$-vertices for $i = 1, 2$, and so by Lemma 25, $E(C_1, C_2) \subseteq \{a_1 b_2, a_2 b_1\}$. By the 1-toughness of $G$, $E(C_1, C_2) \neq \phi$. Therefore, at least one of $a_1 b_2$ and $a_2 b_1$ belongs to $E(G)$. Suppose, without loss of generality, that $a_1 b_2 \in E(G)$ and $a_2 b_1 \notin E(G)$. Since $a_1 b_2 \in E(G)$, $|C_i| \geq 3$ for $i = 1, 2$, for otherwise $\{b_2, x_2\}$ dominates $G$ if $|C_1| = 2$ and $\{a_1, x_2\}$ dominates $G$ if $|C_2| = 2$, contradicting $\gamma(G) = 3$.

Since $a_2 b_1 \notin E(G)$, there is a vertex $x \in V(G) \backslash \{a_2, b_1\}$ such that

$$[a_2, x] \to b_1 \text{ or } [b_1, x] \to a_2.$$

In either case, $x = x_1$ since $x$ must dominate $\{x_0, a_1, b_2\}$ but must not be adjacent to $b_1$ or $a_2$.

Now, $[a_2, x_1] \to b_1$ is impossible since for each $v \in C_1 \backslash \{a_1, b_1\}$, $v \notin N(a_2)$ by Lemma 25 and $v \notin N(x_1)$, for otherwise the cycle

$$a_1 b_2 \overleftarrow{C} x_2 x_0 x_1 v \overrightarrow{C} b_1 v^- \overleftarrow{C} a_1$$

is longer than $C$, a contradiction. So $[b_1, x_1] \to a_2$. But this is not possible because for each $v \in C_2 \backslash \{a_2, b_2\}$, $v \notin N(b_1)$ by Lemma 25 and $v \notin N(x_1)$, for otherwise the cycle

$$vx_1x_0x_2\overleftarrow{C}a_1b_2\overleftarrow{C}v^+a_2\overrightarrow{C}v$$

is longer than $C$, a contradiction. Hence $a_2b_1 \in E(G)$.

**Case 1.2** $\{a_1b_2, a_2b_1\} \subseteq E(G)$.
We show that $\{a_1b_1, a_2b_2\} \subseteq E(G)$. Suppose that $a_1b_1 \notin E(G)$. Then there exists a vertex $x \in X$ such that

$$[a_1, x] \to b_1 \text{ or } [b_1, x] \to a_1,$$

since $x$ must dominate $x_0$.

Suppose $[a_1, x] \to b_1$. Then $x = x_1$ as $x_2$ dominates $b_1$. But $a_2x_1 \notin E(G)$, for otherwise

$$a_2x_1x_0x_2\overleftarrow{C}a_1b_2\overrightarrow{C}a_2$$

is a cycle longer than $C$, a contradiction. Thus $[b_1, x] \to a_1$. In this case $x = x_2$ since $x_1$ dominates $a_1$. But $b_2x_2 \notin E(G)$ by Lemma 28. Thus $a_1b_1 \in E(G)$.

Similarly we can show that $a_2b_2 \in E(G)$.

Hence $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\} \subseteq E(G)$ and as shown in Case 1.1, $|C_i| \geq 3$, and the vertices in $C_i \backslash \{a_i, b_i\}$ are both $A$- and $B$-vertices, for $i = 1, 2$. Consider the vertices $v_1 \in C_1 \backslash \{a_1, b_1\}$ and $v_2 \in C_2 \backslash \{a_2, b_2\}$. By Lemma 25, there is an $x \in V(G) \backslash \{v_1, v_2\}$ such that

$$[v_1, x] \to v_2 \text{ or } [v_2, x] \to v_1.$$

Suppose, without loss of generality, that $[v_1,x] \to v_2$. Then $x \in X$ as $x$ must dominate $\{x_0, a_2, b_2\}$. Note that $x \neq x_1$, for otherwise $a_2x_1 \in E(G)$ and the cycle

$$x_1x_0x_2\overleftarrow{C}a_1b_2\overleftarrow{C}a_2x_1$$

is longer than $C$, a contradiction. So $x = x_2$. But then $b_2x_2 \in E(G)$, which contradicts Lemma 28. Hence $\{v_1, x\}$ does not dominate $V(G) \backslash \{v_2\}$ for any $x \in X$, contradicting $\gamma(G) = 3$. Therefore $\beta(G) \geq r + 2$ if $r = 2$.

**Case 2** $r = 3$.
For convenience, we define:

$$\begin{aligned} p &= |\{a_1b_3, a_2b_1, a_3b_2\} \cap E(G)| \\ q &= |\{a_1b_2, a_2b_3, a_3b_1\} \cap E(G)|. \end{aligned}$$

We first show that $|C_i| \geq 2$ for each $i$ with $1 \leq i \leq 3$. Suppose to the contrary that $|C_i| \leq 1$, for some $i$, say $i = 1$. Then $a_1 = b_1$, and by Lemma 25,

$$\{a_1a_2, a_1a_3, a_1b_2, a_1b_3\} \cap E(G) = \phi.$$

So by Lemma 31, $E(C_1, C_3) = E(C_1, C_2) = \phi$. Thus $a_1x_3 \in E(G)$ since

$$x_1x_0x_2\overrightarrow{C}x_1$$

is a cycle as long as $C$ but does not contain $a_1$, and so

$$\deg(a_1) = \deg(x_0) = 3.$$

By Lemma 28, $a_3b_2 \notin E(G)$.

Now consider the vertices $a_1$ and $x_0$. Since $a_1$ is not adjacent to $x_0$, there exists a vertex $y \in V(G) \setminus \{x_0, a_1\}$ such that

$$[a_1, y] \rightarrow x_0 \text{ or } [y, x_0] \rightarrow a_1.$$

In either case, since $N(a_1) = N(x_0) = X$ and $a_2a_3$ and $b_2b_3$ do not belong to $E(G)$, it follows that $y \in C_1 \cup C_2 \setminus \{a_2, a_3, b_2, b_3\}$. The vertex $y$ must dominate $C_2$ and $C_3$, and thus $a_3b_2 \in E(G)$ by Lemma 31, a contradiction.

We now prove two results regarding the numbers $p$ and $q$.

**Lemma 35.1** *If $p \geq 2$, then $p = 3$.*

**Proof.** Without loss of generality, assume to the contrary that $\{a_1b_3, a_2b_1\} \subseteq E(G)$ and that $a_3b_2 \notin E(G)$. Then there exists $x \in X \setminus \{x_3\}$ such that

$$[a_3, x] \rightarrow b_2 \text{ or } [b_2, x] \rightarrow a_3.$$

The first case is impossible since $a_2x_1$ and $a_1x_2$ do not belong to $E(G)$ by Lemma 29. Similarly the second case is impossible, contradicting $\gamma(G) = 3$. Hence $a_3b_2 \in E(G)$ and $p = 3$. □

**Lemma 35.2** *If $q \geq 2$, then $q = 3$ and $p = 3$.*

**Proof.** Assume, without loss of generality, that $\{a_1b_2, a_2b_3\} \subseteq E(G)$, and suppose, to the contrary, that $a_3b_1 \notin E(G)$. Then there exists some $x \in X$ such that

$$[a_3, x] \rightarrow b_1 \quad \text{or} \quad [b_1, x] \rightarrow a_3.$$

If $[a_3, x] \rightarrow b_1$, then $x \in \{x_1, x_3\}$. But $a_2 \notin N(x_1) \cup N(x_3)$ by Lemma 26(b) and (1) respectively, and so this case is impossible. So, $[b_1, x] \rightarrow a_3$, where $x \in \{x_1, x_2\}$. But $b_2 \notin N(x_1) \cup N(x_2)$ by Lemma 26(b) and (1) respectively, a contradiction. Thus $a_3b_1 \in E(G)$ and $q = 3$.

We now prove that $a_2b_1 \in E(G)$. Suppose to the contrary that $a_2b_1 \notin E(G)$. Then

$$[a_2, x] \to b_1 \quad \text{or} \quad [b_1, x] \to a_2$$

for some $x \in \{x_1, x_3\}$. By Lemma 26, $a_2 \notin N(x_1) \cup N(x_3)$ and $b_1 \notin N(x_1) \cup N(x_3)$. So in either case $x_1$ must be adjacent to $x_3$. But this results in the cycle

$$x_1 x_3 \overrightarrow{C} b_3 a_2 \overrightarrow{C} b_2 a_1 \overrightarrow{C} x_2 x_0 x_1$$

which is longer than $C$, a contradiction. So $a_2b_1 \in E(G)$. Since $q = 3$, we can use symmetric arguments to show that $a_1b_3$ and $a_3b_2$ belong to $E(G)$, and hence $p = 3$. □

To prove the theorem for $r = 3$, we consider the following two cases:

$$p \geq 2 \text{ and } p \leq 1.$$

**Case 2.1** $p \geq 2$.
By Lemma 35.1 this implies that $p = 3$, and so $b_i a_{i+1} \in E(G)$ for each $i$ with $1 \leq i \leq 3$. We also know that $|C_i| \geq 2$ for each $i$ with $1 \leq i \leq 3$, and we now show that $|C_i| \geq 3$. Suppose, to the contrary, that $C_1 = \{a_1, b_1\}$, for example. Then the cycle

$$x_1 a_1 b_3 \overleftarrow{C} a_2 b_1 x_2 x_0 x_1$$

is longer that $C$, a contradiction. Thus $|C_i| \geq 3$ for each $i$ with $1 \leq i \leq 3$.

Now suppose $a_1b_1 \notin E(G)$. Then

$$[a_1, x] \to b_1 \quad \text{or} \quad [b_1, x] \to a_1$$

for some $x \in X$. Suppose $[a_1, x] \to b_1$. Then $\{a_2, a_3\} \subseteq N(x)$, where $x \in \{x_1, x_3\}$. But $a_2 \notin N(x_3)$ and $a_3 \notin N(x_1)$ by Lemma 28, a contradiction. By a symmetrical argument, we can show that the second case is impossible. Hence $a_1b_1 \in E(G)$. Similarly, $a_2b_2$ and $a_3b_3$ belong to $E(G)$. By Lemma 30, all the vertices in $C_i \setminus \{a_i, b_i\}$, for each $i$ with $1 \leq i \leq 3$, are $A$-vertices and $B$-vertices.

Since $|C_i| \geq 3$ for each $i$ with $1 \leq i \leq 3$, we can take $v_1 \in C_1 \setminus \{a_1, b_1\}$ and $v_2 \in C_2 \setminus \{a_2, b_2\}$. By Lemma 25 there exists a vertex $x \in X$ such that

$$[v_1, x] \to v_2 \quad \text{or} \quad [v_2, x] \to v_1.$$

In either case $x$ must dominate $\{a_3, b_3\}$, but $b_3 \notin N(x_2) \cup N(x_3)$ and $a_3 \notin N(x_1)$ by Lemma 28, a contradiction. Hence the theorem holds when $p \geq 2$.

**Case 2.2** $p \leq 1$.
By Lemma 35.2 this implies that $q \leq 1$ too. Assume without loss of generality that $a_2b_1$ and $a_3b_2$ do not belong to $E(G)$. By Lemma 31,

$$N(a_2) \cap C_1 = N(b_1) \cap C_2 = N(a_3) \cap C_2 = N(b_2) \cap C_3 = \phi.$$

Now consider the vertices $x_0$ and $a_1$. Since $x_0a_1 \notin E(G)$, there is a vertex $y \in V(G)\setminus\{x_0, a_1\}$ such that

$$[a_1, y] \rightarrow x_0 \quad \text{or} \quad [x_0, y] \rightarrow a_1.$$

In both cases $y$ must dominate $\{a_2, a_3\}$. So $y \notin C_2 \cup C_1$ and $y$ is not an $A$-vertex.

If $[a_1, y] \rightarrow x_0$, then $y \notin X$ and thus $y \in C_3\setminus\{a_3\}$. Since $N(b_2) \cap C_3 = \phi$, $a_1b_2 \in E(G)$, and so $a_3b_1$ and $a_2b_3$ do not belong to $E(G)$, since $q \leq 1$. Since $a_2b_3 \notin E(G)$, $y \in C_3\setminus\{a_3, b_3\}$. Since $y$ is not an $A$-vertex, $a_3b_3 \notin E(G)$ by Lemma 30. Thus $N(a_3) \cap B = \phi$, contradicting Lemma 32.

Thus $[x_0, y] \rightarrow a_1$ and $y$ must dominate $A \cup B\setminus\{a_1\}$. Since

$$N(a_2) \cap C_1 = N(b_1) \cap C_2 = N(b_2) \cap C_3 = \phi,$$

it follows that $y \in X\setminus\{x_1\}$. Suppose $y = x_2$. Then $[x_0, x_2] \rightarrow a_1$ and therefore $\{x_2a_3, x_2b_2, x_2b_3\} \subseteq E(G)$. Hence by Lemma 26, $b_3a_2$ and $a_3b_1$ do not belong to $E(G)$.

Applying Lemma 32 to $b_1$, $a_2$ and $a_3$, we get $a_ib_i \in E(G)$ for each $i$ with $1 \leq i \leq 3$. Thus by Lemma 30 all the vertices in $C_i\setminus\{a_i, b_i\}$ are $A$- and $B$-vertices for each $i$ with $1 \leq i \leq 3$. By Lemma 25,

$$E(C_i\setminus\{a_i, b_i\}, C_j\setminus\{a_j, b_j\}) = \phi \text{ for } 1 \leq i < j \leq 3.$$

But there must be at least one edge between the vertex sets $C_1, C_2, C_3$ since $G - X$ has at most three components as $G$ is 1-tough. The only possible edges are $a_1b_3$ and $a_1b_2$.

Suppose $a_1b_3 \in E(G)$, *i.e.* $p = 1$, and consider the vertices $a_3$ and $b_2$. Since $a_3b_2 \notin E(G)$,

$$[a_3, x] \rightarrow b_2 \quad \text{or} \quad [b_2, x] \rightarrow a_3$$

for some $x \in X$. In either case $x \notin \{x_3, x_2\}$ since $\{b_2, a_3\} \subseteq N(x_2) \cap N(x_3)$. Hence $x = x_1$ and thus $x_1a_2 \in E(G)$ in the first case and $x_1b_1 \in E(G)$ in the second case. In each case, this contradicts Lemma 29.

Therefore $a_1b_3 \notin E(G)$, *i.e.* $p = 0$, and $a_1b_2 \in E(G)$. By Lemma 26, $x_3b_1$ and $x_1b_1$ do not belong to $E(G)$. Now consider the vertices $b_2$ and $b_3$. Since $b_2b_3 \notin E(G)$, there is a vertex $x$ such that

$$[b_2, x] \rightarrow b_3 \quad \text{or} \quad [b_3, x] \rightarrow b_2.$$

In the first case, $x = x_3$, since $\{x_1b_3, x_2b_3\} \subseteq E(G)$, which implies that $x_3b_1 \in E(G)$, a contradiction. In the second case, $x = x_1$, since $\{x_2b_2, x_3b_2\} \subseteq E(G)$, implying $x_1b_1 \in E(G)$, a contradiction. So $y \neq x_2$. In a similar way we can show that $y \neq x_3$, and thus $\gamma(G) = 3$ is contradicted.

Therefore $\beta(G) \geq r+2$ when $r = 3$.

**Case 3** $r \geq 4$.
Since the set $A$ is contained in the independent set $A \cup \{x_0\}$, it follows from Lemma 4 that there is an ordering $F(A) = (a_{j_1}, a_{j_2}, \ldots, a_{j_r})$ of the vertices of $A$ and a sequence $Y(A) = (x_{i_1}, x_{i_2}, \ldots, x_{i_{r-1}})$ of $r-1$ vertices of $N(x_0) = X$ such that $[a_{j_t}, x_{i_t}] \to a_{j_{t+1}}$ for each $t$ with $1 \leq t \leq r-1$. Let $X \backslash V(Y(A)) = \{x_{i_r}\}$. We have the following:

$$\begin{aligned} Y(A) \backslash \{x_{i_1}\} &\subseteq N(a_{j_1}) \\ Y(A) \backslash \{x_{i_{r-1}}\} &\subseteq N(a_{j_r}) \\ Y(A) \backslash \{x_{i_{l-1}}, x_{i_l}\} &\subseteq N(a_{j_l}) \text{ for each } l \text{ with } 2 \leq l \leq r-1 \end{aligned} \tag{9}$$

and

$$r - 2 \leq |N(x_{i_t}) \cap A| \leq r-1 \text{ for each } t \text{ with } 1 \leq t \leq r-1. \tag{10}$$

Since $r \geq 4$,

$$N(x_{i_t}) \cap (A \backslash \{a_{j_t}\}) \neq \phi \text{ for each } t \text{ with } 1 \leq t \leq r-1. \tag{11}$$

The following lemma follows from 11 and Lemma 28.

**Lemma 35.3** *For every index $t \neq i_r$, $b_{t-1}a_t \notin E(G)$.*

Now consider the vertices $x_0$ and $a_{j_r}$. Since $x_0a_{j_r} \notin E(G)$, either

$$[a_{j_r}, y_a] \to x_0 \quad \text{or} \quad [x_0, y_a] \to a_{j_r}$$

for some vertex $y_a \in V(G) \backslash \{x_0, a_{j_r}\}$.

Suppose $[a_{j_r}, y_a] \to x_0$. Clearly, $y_a \notin X \cup A$. Let $y_a \in C_k \backslash \{a_k\}$ for some $k$ with $1 \leq k \leq r$. Then $|C_k| \geq 2$. We first prove the following five lemmas.

**Lemma 35.4** $k = i_{r-1}$.

**Proof.** Suppose not. Since $[a_{i_r}, y_a] \to x_0$, it follows that $y_a a_{k+1} \in E(G)$ and thus by Lemma 31, $a_{k+1}b_k \in E(G)$ for some $k+1 \neq i_r$. But this contradicts Lemma 35.3. Hence $y_a \in C_{i_{r-1}}$. □

**Lemma 35.5** $a_{i_r}b_{i_{r-2}} \in E(G)$.

**Proof.** Suppose not. Then $y_a b_{i_{r-2}} \in E(G)$ which implies that $a_{i_{r-1}}b_{i_{r-2}} \in E(G)$, by Lemma 31. But this, once again, contradicts Lemma 35.3. □

**Lemma 35.6** $N(x_{i_{r-1}}) \cap A = \{a_{i_r}, a_{i_{r-1}}\}$ *and thus* $r = 4$.

**Proof.** By Lemma 35.5 and Lemma 26(a), $N(x_{i_{r-1}}) \cap (A \setminus \{a_{i_r}, a_{i_{r-1}}\}) = \phi$. The lemma therefore holds by 10. □

**Lemma 35.7** $a_{i_{r-1}} b_{i_{r-1}} \in E(G)$.

**Proof.** Since $a_{i_r} x_{i_{r-1}} \in E(G)$ by Lemma 35.6, $N(a_{i_{r-1}}) \cap (B \setminus \{b_{i_{r-1}}\}) = \phi$ by Lemma 26(b). Thus $a_{i_{r-1}} b_{i_{r-1}} \in E(G)$ by Lemma 32. □

**Lemma 35.8** $y_a = b_{i_{r-1}}$, *i.e.* $[a_{i_r}, b_{i_{r-1}}] \to x_0$.

**Proof.** Suppose not. Then by Lemma 35.7 and Lemma 30, $y_a$ is an $A$-vertex, and by Lemma 25, $\{y_a, a_{i_r}\}$ does not dominate $A \setminus \{a_{i_r}, a_{i_{r-1}}\}$, a contradiction. □

Using $r = 4$ and 9, we find that there are at least two vertices in $A$ that are adjacent to at least two vertices in $Y(A)$. So we can choose a vertex $a_j \neq a_{i_r}$ and one neighbour $x_s$ of $a_j$ in $X \setminus \{x_{i_r}\}$ such that $x_s \neq x_j$. If $x_s \in \overrightarrow{C}[x_{j+1}, x_{i_{r-1}}]$, then by Lemma 26(a), $\{a_{i_r}, b_{i_{r-1}}\}$ does not dominate $b_{s-1}$. If $x_s \in \overrightarrow{C}[x_{i_r}, x_{j-1}]$, then by Lemma 26(b), $\{a_{i_r}, b_{i_{r-1}}\}$ does not dominate $a_s$. Thus it is impossible that $[a_{i_r}, b_{i_{r-1}}] \to x_0$.

Therefore $[x_0, y_a] \to a_{i_r}$. Since $N(x_0) \cap (A \cup B) = \phi$, we get

$$(A \cup B) \setminus \{a_{i_r}\} \subseteq N(y_a). \tag{12}$$

Suppose that $y_a \in C_t$ for some $t$ with $1 \leq t \leq r$. If $t = i_{r-1}$, then $y_a b_{i_{r-2}} \in E(G)$ by 12, and thus $a_{i_{r-1}} b_{i_{r-2}} \in E(G)$ by Lemma 31. But this contradicts Lemma 35.3. If $t \neq i_{r-1}$, then $y_a a_{t+1} \in E(G)$, again by 12, and thus $a_{t+1} b_t \in E(G)$ by Lemma 31. Again, this contradicts Lemma 35.3. Thus $y_a \in X$. Since $x_{i_r} a_{i_r} \in E(G)$, $y_a \in Y(A)$.

Now consider the independent set $B$. Since $B$ is contained in the independent set $B \cup \{x_0\}$, we obtain an ordering $F(B) = (b_{l_1}, b_{l_2}, \ldots, b_{l_r})$ of the vertices of $B$ and a sequence $Y(B) = (x_{k_1}, x_{k_2}, \ldots, x_{k_{r-1}})$ of $r-1$ vertices of $X$ such that $[b_{l_t}, x_{k_t}] \to b_{l_{t-1}}$ for each $t$ with $1 \leq t \leq r-1$. Let $X \setminus Y(B) = \{x_{k_r}\}$.

We can show by a symmetric argument that $[b_{k_r}, y_b] \to x_0$ is impossible for any vertex $y_b \in V(G) \setminus \{b_{k_r}, x_0\}$ and that $[x_0, y_b] \to b_{k_r}$ for some $y_b \in X$. Analogous to 12, we also have the following:

$$A \cup B \setminus \{b_{k_r}\} \subseteq N(y_b). \tag{3.4'}$$

By the second inequality of 10, $A \setminus N(x) \neq \phi$ for each $x \in Y(A)$. Thus by (3.4′), $y_b = x_{i_r}$. Similarly $y_a = x_{k_r}$.

Now, $y_b = x_{i_r}$ implies that $A \subseteq N(x_{i_r})$. By Lemma 5, $a_{j_t}x_{i_t} \in E(G)$ for each $t$ with $2 \leq t \leq r-1$. Thus

$$|N(a_i) \cap X| \geq r-1 \text{ for any } a_i \in A. \tag{13}$$

Also by Lemma 28, we have $b_{i_{r-1}}a_{i_r} \notin E(G)$, and so, using Lemma 3.13, the following now holds:

$$\text{For every } t \text{ with } 1 \leq t \leq r,\ b_{t-1}a_t \notin E(G). \tag{14}$$

We are now ready to conclude the proof. Suppose that $a_ib_j \in E(G)$ for some $i \neq j$. By 14, $j \neq i-1$. By Lemma 26, $a_{i+1}x_i$ and $a_{i+1}x_{j+1}$ do not belong to $E(G)$ and thus $|N(a_{i+1}) \cap X| \leq r-2$, which contradicts 13. Therefore

$$a_ib_j \notin E(G) \text{ for all } i \neq j. \tag{15}$$

For each $a_i \neq b_i$, $a_ib_i \in E(G)$ by Lemma 32 and 15. Hence if $a_i \neq b_i$, all the vertices in $C_i \backslash \{a_i, b_i\}$ are $A$-vertices and $B$-vertices by Lemma 30. Thus by Lemma 25, $E(C_i, C_j) = \phi$ for all $i \neq j$. This contradicts the fact that $G$ is 1-tough, and hence $\beta(G) \geq r+2$ when $r \geq 4$. ■

As a direct consequence of Theorem 35, we have the following result:

**Corollary 36** [5] *If $G$ is a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$ and $\beta(G) \leq \delta(G)+1$, then $G$ is hamiltonian.*

This leaves open only the case $\beta = \delta + 2$, which is considered in the next section.

# 7 The case $\beta = \delta + 2$

Theorem 37 which was proved by Tian, Wei and Zhang in [13] is the final result needed to settle Wojcicka's Conjecture.

**Theorem 37** [13] *If $G$ is a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$ and $\beta(G) = \delta(G) + 2$, then $G$ is hamiltonian.*

**Proof.** Suppose to the contrary that $G$ is not hamiltonian and let $C$ be a longest cycle of $G$. By Theorem 23, $|V(C)| = n-1$.

Let $x_0$ be the only vertex not on $C$ and define the sets $X$, $A$, $B$ and $C_i$ as in Section 5. By Theorem 35, $\beta(G) \geq \deg(x_0)+2$. Since $\beta(G) = \delta(G)+2$ by the hypothesis, and $\deg(x_0) \geq \delta(G)$, it follows that $\deg(x_0) = \delta(G) = r$.

It follows from Theorem 18 that $\deg(v) > r$ for any vertex $v \in V(G) \setminus \{x_0\}$. Thus

$$V(C') = V(C) \text{ for each longest cycle } C' \text{ of } G. \tag{16}$$

Before continuing with the proof of the theorem, we prove the following lemma.

**Lemma 37.1** $|C_i| \geq 2$ *for each* $i$ *with* $1 \leq i \leq r$.

**Proof.** Suppose to the contrary that $a_i = b_i$ for some $i$ with $1 \leq i \leq r$. Then

$$x_i x_0 x_{i+1} \overrightarrow{C} x_i$$

is a cycle as long as $C$ which does not contain $a_i$, contradicting 16. Therefore the result holds. □

We now proceed with the proof of Theorem 37 and consider two cases depending on the value of $r$.

**Case 1** $r = 2$.
We define

$$p = |E(\{a_1, a_2\}, \{b_1, b_2\})|.$$

Clearly $p \leq 4$, and furthermore $p \geq 1$, otherwise $\{x_0, a_1, a_2, b_1, b_2\}$ is an independent set consisting of five vertices, contradicting $\beta(G) = 4$. We first prove two lemmas before proceeding with the proof of this case.

**Lemma 37.2** $|C_i| \geq 3$ for $i = 1, 2$.

**Proof.** Suppose to the contrary that $|C_1| = 2$. Then $|C_2| \geq 3$, for otherwise $\{x_1, x_2\}$ is a dominating set, contradicting $\gamma(G) = 3$. We now show that $E(\{a_1, a_2\}, \{b_1, b_2\}) = \{a_1 b_1\}$, *i.e.* $p = 1$.

Suppose $a_2 b_1 \in E(G)$. Then

$$a_2 b_1 x_2 x_0 x_1 \overleftarrow{C} a_2$$

is a cycle as long as $C$ which does not contain $a_1$. This contradicts 16, so $a_2 b_1 \notin E(G)$. Similarly $a_1 b_2 \notin E(G)$.

Suppose $a_2 b_2 \in E(G)$ and recall that $E(C_1, C_2) \neq \phi$ by Theorem 12. We can, therefore, assume without loss of generality that $N(b_1) \cap C_2 \neq \phi$. Let $u \in C_2$ such that $ub_1 \in E(G)$. Then $\{u^- b_2, u^- a_1\} \cap E(G) = \phi$ by Lemma 26, and $u^- x_2 \notin E(G)$ by Lemma 29. Now consider the vertices $a_1$ and $b_2$. Since $a_1 b_2 \notin E(G)$, either

$$[a_1, x_2] \rightarrow b_2 \quad \text{or} \quad [b_2, x_2] \rightarrow a_1.$$

In both cases $u^-$ cannot be dominated, contradicting $\gamma(G) = 3$. Thus $a_2b_2 \notin E(G)$. Therefore either

$$[a_2, x_2] \to b_2 \quad \text{or} \quad [b_2, x_1] \to a_2.$$

Without loss of generality assume that $[a_2, x_2] \to b_2$, hence $x_2a_1 \in E(G)$ and $x_2b_2 \notin E(G)$. Also, note that $|C_2| \geq 4$, otherwise $\{x_2, b_2\}$ dominates $V(G)$, contradicting $\gamma(G) = 3$.

Consider the vertices $a_1$ and $a_2$. Since $a_1a_2 \notin E(G)$, either

$$[a_1, x] \to a_2 \quad \text{or} \quad [a_2, x] \to a_1$$

for some $x \in X \cup \{x_0\}$. Consider the latter case and note that $x \notin X$ since $X \subseteq N(a_1)$ and $x \neq x_0$ as $\{a_2, x_0\}$ does not dominate $\{b_1, b_2\}$. Therefore $[a_2, x] \to a_1$ is impossible and thus $[a_1, x_1] \to a_2$, which implies that $a_2x_1 \notin E(G)$.

We will now prove that $C_2 \setminus \{a_2, b_2\} \subseteq N(a_1)$. Suppose to the contrary that there is a vertex $v \in C_2 \setminus \{a_2, b_2\}$ such that $va_1 \notin E(G)$. Then either

$$[v, x] \to a_1 \quad \text{or} \quad [a_1, x] \to v$$

for some $x \in X$. The former is impossible since $X \subseteq N(a_1)$. The latter is impossible since $\{a_1, x_1\}$ does not dominate $a_2$ and $\{a_1, x_2\}$ does not dominate $b_2$. Hence $C_2 \setminus \{a_2, b_2\} \subseteq N(a_1)$. Since $|C_2| \geq 4$, we have the following cycle:

$$x_1x_0x_2a_2a_2^+a_1a_2^{+2}\overrightarrow{C}x_1,$$

which is as long as $C$, but excludes $b_1$, contradicting 16. □

**Lemma 37.3** $p \leq 2$.

**Proof.** We first show that $\{a_2b_1, a_1b_2\} \not\subseteq E(G)$. Suppose to the contrary that $a_2b_1$ and $a_1b_2$ are edges in $G$. Then, by Lemma 28, $a_2x_1$ and $b_2x_2$ are not edges in $G$. Let $u \in C_1 \setminus \{a_1, b_1\}$ such that $u \notin N(b_2)$ but $\overrightarrow{C}[x_1, u^-] \subseteq N(b_2)$, and let $v \in C_2 \setminus \{a_2, b_2\}$ such that $v \notin N(b_1)$ but $\overrightarrow{C}[x_2, v^-] \subseteq N(b_1)$. Then $uv \notin E(G)$, for otherwise

$$uv\overrightarrow{C}b_2u^-\overleftarrow{C}x_1x_0x_2\overrightarrow{C}v^-b_1\overleftarrow{C}u$$

is a cycle longer than $C$, a contradiction.

Without loss of generality we may assume that there exists a vertex $x \in X$ such that $[u, x] \to v$. Then $ua_2 \notin E(G)$, for otherwise

$$x_1x_0x_2\overleftarrow{C}ua_2\overrightarrow{C}b_2u^-\overleftarrow{C}x_1$$

is a cycle longer than $C$, and (by our choice of $u$) $ub_2 \notin E(G)$. Thus $x$ must dominate $\{a_2, b_2\}$. If $x = x_1$, then $a_2$ is not dominated and if $x = x_2$, then $b_2$ is not dominated, a contradiction.

Thus $\{a_2b_1, a_1b_2\} \not\subseteq E(G)$, and so $p \leq 3$. Suppose to the contrary that $p = 3$. Without loss of generality, we may assume that $a_2b_1 \notin E(G)$. Therefore, either

$$[a_2, x_1] \to b_1 \quad \text{or} \quad [b_1, x_1] \to a_2.$$

Consider the former case. Since $a_1b_1 \in E(G)$, $b_1^- a_2 \notin E(G)$ by Lemma 26(a). Therefore $b_1^- x_1 \in E(G)$. But this results in the cycle

$$b_1^- x_1 x_0 x_2 \overrightarrow{C} b_2 a_1 \overrightarrow{C} b_1^-,$$

which is as long as $C$ but excludes $b_1$, contradicting 16. Similarly, if we assume $[b_1, x_1] \to a_2$, we get a contradiction. Thus $p \leq 2$. □

We are now ready to prove the theorem for $\delta(G) = 2$ by considering four subcases.

**Case 1.1** $\{a_1b_1, a_2b_1\} \subseteq E(G)$.
Since $a_2b_2 \notin E(G)$, either

$$[a_2, x_2] \to b_2 \quad \text{or} \quad [b_2, x_1] \to a_2.$$

Consider the former case. Since $a_2a_1 \notin E(G)$, $a_1x_2 \in E(G)$. But this results in the cycle

$$x_2 x_0 x_1 \overleftarrow{C} a_2 b_1 \overleftarrow{C} a_1 x_2$$

which is longer than $C$, a contradiction. Therefore $[a_2, x_2] \to b_2$ is impossible, and thus $[b_2, x_1] \to a_2$ and $x_1b_1 \in E(G)$.

Now consider the vertices $a_1$ and $b_2$. Since $a_1b_2 \notin E(G)$, either

$$[a_1, x_2] \to b_2 \quad \text{or} \quad [b_2, x_2] \to a_1.$$

Consider the latter case and note that $b_2a_1^+ \notin E(G)$ by Lemma 26(a): therefore $x_2a_1^+ \in E(G)$. But this results in the cycle

$$x_2 a_1^+ \overrightarrow{C} b_1 a_2 \overrightarrow{C} x_1 x_0 x_2,$$

which is as long as $C$ but excludes $a_1$, contradicting 16. Therefore $[b_2, x_2] \to a_1$ is impossible and so $[a_1, x_2] \to b_2$. Now $b_2^- a_1 \notin E(G)$, for otherwise

$$b_2^- a_1 \overrightarrow{C} b_1 x_1 x_0 x_2 \overrightarrow{C} b_2^-$$

is a cycle as long as $C$ excluding $b_2$, contradicting 16. Also, $b_2^- x_2 \notin E(G)$, for otherwise

$$b_2^- x_2 x_0 x_1 \overrightarrow{C} b_1 a_2 \overrightarrow{C} b_2^-$$

is a cycle as long as $C$ which excludes $b_2$, again contradicting 16.

We get a contradiction in a similar way if we assume that $\{a_1 b_2, a_2 b_2\} \subseteq E(G)$.

**Case 1.2** $\{a_1 b_1, a_2 b_2\} \subseteq E(G)$.
Consider the vertices $a_1$ and $x_0$. By Lemma 33 there exists a vertex $y$ such that $[a_1, y] \rightarrow x_0$. Since $\{a_1 b_2, a_1 a_2\} \cap E(G) = \phi$, it follows that $\{y b_2, y a_2\} \subseteq E(G)$. Suppose that $y \in C_1$. Then clearly $y \notin \{a_1, b_1\}$. Also $y \neq b_1^-$, otherwise the cycle

$$b_1^- b_2 \overleftarrow{C} x_2 x_0 x_1 \overleftarrow{C} b_1^-$$

is as long as $C$ and excludes $b_1$, contradicting 16.

We now show that

$$N(y^+) \cap \{x_1, x_2, a_1, a_2\} = \phi.$$

By Lemma 27 and the fact that $y b_2 \in E(G)$, it follows that $y^+ a_2 \notin E(G)$. Next, $y^+ x_1 \notin E(G)$, for otherwise the cycle

$$y^+ x_1 x_0 x_2 \overrightarrow{C} b_2 y \overleftarrow{C} a_1 b_1 \overleftarrow{C} y^+$$

results, which is longer than $C$. Further, $y^+ x_2 \notin E(G)$, for otherwise the cycle

$$x_2 y^+ \overrightarrow{C} b_1 a_1 \overleftarrow{C} y a_2 \overleftarrow{C} x_1 x_0 x_2$$

results, which is also longer that $C$. Finally, $y^+ a_1 \notin E(G)$ as a result of Lemma 26(a) and the fact that $y a_2 \in E(G)$.

Now consider the vertices $a_1$ and $a_2$. Since $a_1 a_2 \notin E(G)$, either

$$[a_1, x_1] \rightarrow a_2 \quad \text{or} \quad [a_2, x_2] \rightarrow a_1.$$

But in either case $y^+$ cannot be dominated, a contradiction. Hence $y \in C_2$.

We now show that $C_1 \subseteq N[a_1]$. Suppose the contrary and let $u$ be the first vertex in $\overleftarrow{C}[b_1, a_1]$ not adjacent to $a_1$. Then $yu \in E(G)$ and so $y \neq b_2^-$, for otherwise

$$b_2 x_1 x_0 x_2 \overleftarrow{C} u^+ a_1 u b_2^- \overleftarrow{C} a_2 b_2$$

is a cycle longer than $C$. By Lemma 26(a), $y^+b_1 \notin E(G)$ because $yb_2 \in E(G)$. Next, $y^+x_1 \notin E(G)$, for otherwise

$$y^+x_1x_0x_2\overleftarrow{C}u^+a_1\overrightarrow{C}uy\overleftarrow{C}a_2b_2\overleftarrow{C}y^+$$

is a cycle longer than $C$. Further, $y^+a_2 \notin E(G)$, for otherwise the cycle

$$y^+\overrightarrow{C}x_1x_0x_2\overleftarrow{C}u^+a_1\overleftarrow{C}uy\overleftarrow{C}a_2y^+$$

is longer than $C$. Thus $N(y^+) \cap \{b_1, x_1, a_2\} = \phi$. Now consider the vertices $a_2$ and $b_1$. Since $a_2b_1 \notin E(G)$, either

$$[a_2, x_1] \to b_1 \quad \text{or} \quad [b_1, x_1] \to a_2.$$

But in either case $y^+$ cannot be dominated; a contradiction. Therefore $C_1 \subseteq N[a_1]$.

Again, by Lemma 33, there exists a vertex $y' \in X$ such that $[a_2, y'] \to x_0$. By a similar argument, we can show that $C_2 \subseteq N[a_2]$. Thus $E(C_1, C_2) = \phi$ by Lemma 25, contradicting the 1-toughness of $G$.

**Case 1.3** $a_1b_2 \in E(G)$; $\{a_1b_1, a_2b_1, a_2b_2\} \cap E(G) = \phi$.
Since $a_1b_1 \notin E(G)$, either $[a_1, x_1] \to b_1$ or $[b_1, x_2] \to a_1$. If $[a_1, x_1] \to b_1$, then $a_2x_1 \in E(G)$ since $a_1a_2 \notin E(G)$. But this results in the cycle

$$x_1x_0x_2\overleftarrow{C}a_1b_2\overleftarrow{C}a_2x_1,$$

which is longer than $C$, a contradiction. Thus $[b_1, x_2] \to a_1$ and so $b_2x_2 \in E(G)$.

Since $a_2b_2 \notin E(G)$, either

$$[a_2, x] \to b_2 \quad \text{or} \quad [b_2, x] \to a_2$$

for some $x \in X$. The former case is impossible since $X \subseteq N(b_2)$. Thus $[b_2, x] \to a_2$ and $x = x_1$. But the cycle

$$a_2\overrightarrow{C}b_2a_1\overleftarrow{C}b_1x_1x_0x_2a_2$$

results, which is longer than $C$, a contradiction.

We obtain a similar contradiction if $a_2b_1 \in E(G)$ and $\{a_1b_1, a_1b_2, a_2b_2\} \cap E(G) = \phi$.

**Case 1.4** $a_1b_1 \in E(G)$; $\{a_1b_2, a_2b_1, a_2b_2\} \cap E(G) = \phi$.
Since $a_2b_2 \notin E(G)$, either $[a_2, x_2] \to b_2$ or $[b_2, x_1] \to a_2$. Without loss of generality, assume that $[a_2, x_2] \to b_2$. This implies that $x_2a_1 \in E(G)$ and that $x_2b_2 \notin E(G)$.

Since $a_1a_2 \notin E(G)$ and $X \subseteq N(a_1)$, it follows that $[a_1, x_1] \to a_2$, implying that $a_2x_1 \notin E(G)$.

Now consider the vertices $a_2^+$ and $a_1$. Note that $a_2^+ a_1 \notin E(G)$, for otherwise the cycle

$$a_2^+ \overrightarrow{C} x_1 x_0 x_2 \overleftarrow{C} a_1 a_2^+$$

results, which is as long as $C$ but excludes $a_2$, a contradiction. Therefore

$$[a_2^+, x] \to a_1 \quad \text{or} \quad [a_1, x] \to a_2^+$$

for some $x \in X$. Since $X \subseteq N(a_1)$, it follows that $[a_1, x] \to a_2^+$. But $\{a_1, x_1\}$ does not dominate $a_2$ and $\{a_1, x_2\}$ does not dominate $b_2$, a contradiction.

Similarly, if $a_2 b_2 \in E(G)$ and $\{a_1 b_1, a_1 b_2, a_2 b_1\} \cap E(G) = \phi$, we obtain a contradiction.

This proves the theorem for $r = 2$.

**Case 2** $r \geq 3$.
Two additional lemmas are required for the proof of this case, the first of which is given below.

**Lemma 37.4** *There exists a maximum independent set of the form*

$$A \cup \{x_0, b\} \text{ for some } b \in B$$

*or*

$$B \cup \{x_0, a\} \text{ for some } a \in A.$$

**Proof.** Suppose the contrary. Then

$$N[a_i] \cap B \neq \phi \text{ for each } a_i \in A$$

and

$$N[b_j] \cap A \neq \phi \text{ for each } b_j \in B. \tag{17}$$

We will consider two cases.

*Case* 1 There exists a maximum independent set $I$ such that $A \subseteq I$ or $B \subseteq I$.
Without loss of generality, assume that $A \subseteq I$. Since $\deg(x_0) = \delta(G)$ $(= r)$, it follows from Theorem 15 that $x_0 \in I$. So we may suppose that

$$I = A \cup \{x_0, u\},$$

where $u \in V(G) \setminus (\{x_0\} \cup X \cup A \cup B)$. Let $S \cup \{x_0\} = I$. Then $S$ is an independent set consisting of $r + 1$ vertices. By Lemma 4 there exists an ordering

$$F(S) = (s_1, s_2, \ldots, s_{r+1})$$

of the vertices of $S$, and the sequence

$$Y(S) = (y_1, y_2, \ldots, y_r) \subseteq N(x_0) = X$$

such that

$$[s_i, y_i] \to s_{i+1} \text{ for each } i \text{ with } 1 \leq i \leq r.$$

We will first show the following:

(i) $\{y_1, y_2, \ldots, y_r\} = X$.

(ii) $N(s_i) \cap X \geq r - 1$ for each $i$ with $1 \leq i \leq r + 1$.

(iii) If $x_i = y_1$, then $|N(x_i) \cap A| \geq r - 2$; otherwise $|N(x_i) \cap A| \geq r - 1$.

(iv) $a_{i+1}b_i \notin E(G)$ for each $i$ with $1 \leq i \leq r$.

(i) By Lemma 4, $Y(S) \subseteq N(x_0) = X$. Since $|Y(S)| = |X| = r$, (i) follows.

(ii) Since $[s_i, y_i] \to s_{i+1}$ for each $i$ with $1 \leq i \leq r$, it follows that

$$\begin{aligned} Y \setminus \{y_1\} &\subseteq N(s_1), \\ Y \setminus \{y_r\} &\subseteq N(s_{r+1}) \quad \text{and} \\ Y \setminus \{y_i, y_{i+1}\} &\subseteq N(s_i) \text{ for each } i \text{ with } 2 \leq i \leq r. \end{aligned}$$

By Lemma 5, $y_i s_i \in E(G)$ for each $i$ with $2 \leq i \leq r$, and so (ii) follows.

(iii) Since $S = A \cup \{u\}$,

$$|A \cap N(x_i)| \geq |S \cap N(x_i)| - 1 \quad \text{for each } i \text{ with } 1 \leq i \leq r.$$

Suppose that $x_i = y_1$. Since $S \setminus \{s_1, s_2\} \subseteq N(x_i)$,

$$|A \cap N(x_i)| \geq ((r+1) - 2) - 1 = r - 2.$$

Suppose that $x_i = y_j$, where $j > 1$. Then $S \setminus \{s_{j+1}\} \subseteq N(x_i)$ by Lemma 5. Therefore

$$\begin{aligned} |A \cap N(x_i)| &\geq ((r+1) - 1) - 1 \\ &= r - 1. \end{aligned}$$

Hence (iii) holds.

(iv) Suppose that $r \geq 4$. Then by (iii),

$$|N(x_i) \cap A| \geq r - 2 \geq 2 \quad \text{for each } i \text{ with } 1 \leq i \leq r.$$

So $x_{i+1}$ is adjacent to some $a_j \neq a_{i+1}$. If $a_{i+1}b_i \in E(G)$, then

$$x_0 x_{i+1} a_j \overrightarrow{C} b_i a_{i+1} \overrightarrow{C} x_j x_0$$

is a cycle longer than $C$.

Suppose $r = 3$ and suppose to the contrary that $a_1 b_3 \in E(G)$. By Lemma 28, $a_3 x_1 \notin E(G)$, and $a_2 x_1 \notin E(G)$, for otherwise the cycle

$$x_1 x_0 x_2 \overleftarrow{C} a_1 b_3 \overleftarrow{C} a_2 x_1$$

results, which is longer than $C$. It follows from (iii) that $y_1 = x_1$, and that $|N(x_2) \cap A| \geq 2$ and $|N(x_3) \cap A| \geq 2$. Since $\{a_1, a_3\} \cap N(x_2) \neq \phi$, $a_2 b_1 \notin E(G)$, otherwise a cycle longer than $C$ results. Similarly $a_3 b_2 \notin E(G)$.

Suppose $s_1 = a_1$. Then $[a_1, x_1] \to s_2$. Since $\{a_1a_2, a_1a_3, x_1a_2, x_1a_3\} \cap E(G) = \phi$, $\{a_1, x_1\}$ does not dominate at least one of $s_3$ and $s_4$, a contradiction.

Suppose that $s_1 = a_2$. Then $[a_2, x_1] \to s_2$. Now $b_1 a_2 \notin E(G)$ and $b_1 x_1 \notin E(G)$ by Lemma 28 and our assumption that $a_1 b_3 \in E(G)$, so $\{a_2, x_1\}$ does not dominate $b_1$, a contradiction.

Suppose that $s_1 = a_3$. Then $[a_3, x_1] \to s_2$. Now $b_2 a_3 \notin E(G)$ and $b_2 x_1 \notin E(G)$ by Lemma 28 and our assumption that $a_1 b_3 \in E(G)$. Hence $\{a_3, x_1\}$ does not dominate $b_2$, a contradiction.

Suppose that $s_1 = u$. Then $[u, x_1] \to s_2$ and $\{a_1, a_2, a_3\} = \{s_2, s_3, s_4\}$. This implies that at least one of $x_1 a_2$ and $x_1 a_3$ are in $G$, a contradiction.

Hence (iv) holds.

We define

$$h = \max \{ j - i \mid a_i b_j \in E(G),\ a_i \in A,\ b_j \in B \},$$

where all operations are modulo $r$, and show that

$$h = 0. \tag{18}$$

Suppose to the contrary that $h \geq 1$. Then we may suppose without loss of generality that

$$a_1 b_{h+1} \in E(G) \ \text{ for some } h > 0.$$

By (iv), $a_1 \neq a_{h+2}$, which implies that $x_1 \neq x_{h+2}$. Moreover, by Lemma 26, $\{a_{h+1}x_1, a_{h+1}x_{h+2}\} \cap E(G) = \phi$ and so $|N(a_{h+1}) \cap X| \leq r - 2$, contradicting (ii).

We now show that

$$C_i \backslash \{a_i, b_i\} \subseteq N(a_i) \cap N(b_i) \ \text{ for each } i \text{ with } 1 \leq i \leq r.$$

Suppose the contrary. Note that it follows from 17 and 18 that

$$a_ib_i \in E(G) \text{ for each } i \text{ with } 1 \le i \le r.$$

We can therefore choose, without loss of generality, a vertex $w \in C_1$ such that $wb_1 \notin E(G)$ and $\overrightarrow{C}[a_1, w^-] \subseteq N(b_1)$. Since $w$ is a $B$-vertex, $B \cup \{w\}$ is independent. Also,

$$C_i \backslash \{b_i\} \subseteq N(b_i), \text{ for each } i \text{ with } 2 \le i \le r. \tag{19}$$

for otherwise there will be a vertex $z \in C_i \backslash \{a_i, b_i\}$, where $i \neq 1$, such that $zb_i \notin E(G)$ and $\overrightarrow{C}[a_i, z^-] \subseteq N(b_i)$. Thus $z$ is a $B$-vertex, and by Lemma 25, $B \cup \{z, w\}$ is an independent set, hence $B \cup \{z, w, x_0\}$ is an independent set of $r+3$ vertices, a contradiction.

By Lemma 33 there exists $y \notin X$ such that $[b_1, y] \to x_0$. Since $b_1a_2 \notin E(G)$, $y \notin A \backslash \{a_2\}$. Furthermore $y \neq a_2$, for otherwise $a_2b_r \in E(G)$, contradicting 18. Clearly $y \notin B \backslash \{b_1\}$, otherwise $B \backslash \{b_1, y\}$ cannot be dominated (by Lemma 25). Moreover, $y \notin \cup_{i=2}^r C_i \backslash \{a_i, b_i\}$, otherwise $B$ cannot be dominated (by 19 and Lemma 25). Hence $y \in C_1 \backslash \{a_1, b_1\}$. Since $\{b_1a_2, b_1b_r\} \cap E(G) = \phi$, $\{ya_2, yb_r\} \subseteq E(G)$. It follows that $y \neq b_1^-$, for otherwise the cycle

$$x_1x_0x_2b_1a_1\overrightarrow{C}b_1^-a_2\overrightarrow{C}x_1$$

results, which is longer than $C$.

Let $A_1 = A \cup \{y^+\}$. Since $yb_r \in E(G)$, it follows from Lemma 27 that $A_1 \backslash \{a_1\}$ is independent. Moreover, since $ya_2 \in E(G)$, it follows from Lemma 26(a) that $y^+a_1 \notin E(G)$ and so $A_1$ is independent. Thus there is an ordering $F(A_1)$ of the vertices of $A_1$ and a sequence $Y(A_1)$ such that Lemma 3 is satisfied. Since $A_1 \cup \{x_0\}$ is independent, it follows from Lemma 4 that $Y(A_1) \subseteq X$, and by applying (ii) to $A_1$, it follows that $|N(y^+) \cap X| \ge r-1$. This implies that at least one of $y^+x_1$ and $y^+x_2$ is in $E(G)$, contradicting Lemma 29.

Hence $C_i \backslash \{a_i, b_i\} \subseteq N(b_i)$ for each $i$ with $1 \le i \le r$. Similarly, we can show that $C_i \backslash \{a_i, b_i\} \subseteq N(a_i)$ for each $i$ with $1 \le i \le r$.

It therefore follows from Lemma 25 and 18 that

$$E(C_i, C_j) = \phi, \text{ for } 1 \le i \neq j \le r,$$

contradicting the 1-toughness of $G$.

*Case* 2 For each maximum independent set $I$, $A \not\subseteq I$ and $B \not\subseteq I$.
Under the condition that $A \not\subseteq I$ and $B \not\subseteq I$, we will show the following:

(i) If $v \in C_i$ is an $A$-vertex, then all the vertices of $\overrightarrow{C}[a_i, v]$ are $A$-vertices. If $N(a_i) \cap C_{i-1} \neq \phi$, then $a_ib_{i-1} \in E(G)$.

(ii) If $v \in C_i$ is a $B$-vertex, then all the vertices of $\overrightarrow{C}[v, b_i]$ are $B$-vertices. If $N(b_i) \cap C_{i+1} \neq \phi$, then $a_{i+1}b_i \in E(G)$.

We will only show the proof of (i) as the proof of (ii) is similar.

Suppose the contrary, and assume that $y$ is the last vertex in $\overrightarrow{C}\left[a_i^+, v^+\right]$ that is not adjacent to $a_i$. Clearly, $y \notin \left\{a_i^+, v^+\right\}$, and $y$ is an $A$-vertex. By Lemma 25, $A \cup \{y^+\}$ is an independent set, and so $A \cup \{y^+, x_0\}$ is a maximum independent set containing $A$, contradicting the hypothesis.

Now suppose to the contrary that $N(a_i) \cap C_{i-1} \neq \phi$ and $a_i b_{i-1} \notin E(G)$. Let $y$ be the last vertex of $\overrightarrow{C}_{i-1}$ which is not adjacent to $a_i$. It follows from Lemma 26(a) that $y^+ a_j \notin E(G)$ for $j \neq i$. Therefore $A \cup \{y^+, x_0\}$ is a maximum independent set containing $A$, contradicting the hypothesis. This concludes the proof of (i).

Now consider the independent set $A$. By Lemma 4 we can find an ordering $F(A) = (a_{j_1}, \dots, a_{j_r})$ of the vertices of $A$ and a sequence $Y(A) \subseteq X$ such that

$$[a_{j_i}, y_i] \to a_{j_{i+1}} \quad \text{for each } i \text{ with } 1 \leq i \leq r-1.$$

Without loss of generality we may assume that

$$\{x_1\} = X \backslash Y(A).$$

We will first show the following:

(iii) $a_{i+1}b_i \notin E(G)$ for each $i$ with $1 \leq i \leq r-1$.

(iv) $y \in C_r$, where $[a_1, y] \to x_0$, and so $|C_r| \geq 2$.

(v) $a_1 b_{r-1} \notin E(G)$.

(iii) Suppose firstly that $r \geq 4$. Since $\{a_{j_t} | t \neq i, i+1\} \subseteq N(y_i)$,

$$|N(y_i) \cap A| \geq r - 2 \geq 2.$$

Therefore $|N(x_i) \cap A| \geq 2$ for each $i$ with $2 \leq i \leq r$, and so $x_{i+1}a_j \in E(G)$ for each $i$ with $1 \leq i \leq r-1$ and some $j \neq i+1$. Now $a_{i+1}b_i \notin E(G)$, for otherwise

$$x_{i+1}a_j \overrightarrow{C} b_i a_{i+1} \overrightarrow{C} x_j x_0 x_{i+1}$$

is a cycle longer than $C$.

Now consider $r = 3$ and assume without loss of generality that $Y(A) = (x_2, x_3)$. Suppose the contrary and let

$$p = \left|\{a_2b_1, a_3b_2, a_1b_3\} \cap E(G)\right|.$$

We first show that if $p \geq 2$, then $p = 3$. (This is similar to the proof used in Theorem 35).

Assume without loss of generality that $\{a_1b_3, a_2b_1\} \subseteq E(G)$ and $a_3b_2 \notin E(G)$. Then there exists some $x \in X \backslash \{x_3\}$ such that

$$[a_3, x] \rightarrow b_2 \quad \text{or} \quad [b_2, x] \rightarrow a_3.$$

The former case is impossible since $x_1$ does not dominate $a_2$ and $x_2$ does not dominate $a_1$ by Lemma 28. Similarly, $[b_2, x] \rightarrow a_3$ can be shown to be impossible and so $a_3b_2 \in E(G)$ and thus $p = 3$.

Now consider the vertices $a_3$ and $b_2$.

Suppose $a_3b_2 \in E(G)$. Then $\{a_1x_3, a_2x_3\} \cap E(G) = \phi$, for otherwise a cycle longer than $C$ results. Since $Y(A) = (x_2, x_3)$, $x_3a_{j_1} \in E(G)$ and so $a_{j_1} = a_3$. Also, $x_2a_{j_2} \notin E(G)$ and so $a_{j_2} = a_1$. Therefore $F(A) = (a_3, a_1, a_2)$, *i.e.* $[a_3, x_2] \rightarrow a_1$ and $[a_1, x_3] \rightarrow a_2$.

Consider $[a_1, x_3] \rightarrow a_2$ and note that $a_1b_3 \in E(G)$ since $x_3b_3 \notin E(G)$ by Lemma 28. So $p \geq 2$ and thus $p = 3$, *i.e.* $\{a_2b_1, a_3b_2, a_1b_3\} \subseteq E(G)$.

Now consider $[a_3, x_2] \rightarrow a_1$. Since $b_3x_2 \notin E(G)$ by Lemma 28, $a_3b_3 \in E(G)$. Hence $C_3 \backslash \{a_3, b_3\} \subseteq N(a_3) \cap N(b_3)$ by (i) and (ii). It follows from Lemma 25 that $(C_3 \backslash \{a_3, b_3\}) \cap N(a_1) = \phi$ and from Lemma 28 that $(C_3 \backslash \{a_3, b_3\}) \cap N(x_3) = \phi$. Thus $[a_1, x_3] \rightarrow a_2$ implies that $C_3 = \{a_3, b_3\}$. But this contradicts Lemma 27 because $a_3b_2 \in E(G)$ and $b_3a_1 (= a_3^+a_1) \in E(G)$.

Now suppose that $a_3b_2 \notin E(G)$. Since we have assumed (to the contrary) that $a_{i+1}b_i \in E(G)$ for some $i$ with $1 \leq i \leq r-1$, we may assume without loss of generality that $a_2b_1 \in E(G)$. By Lemma 28, it follows that

$$\{x_2a_1, x_2b_3, x_2a_3, x_2b_2\} \cap E(G) = \phi.$$

Since $[a_{j_1}, x_2] \rightarrow a_{j_2}$ and $[a_{j_2}, x_3] \rightarrow a_{j_3}$, it follows that $\{a_{j_1}, a_{j_2}\} = \{a_1, a_3\}$ and that $a_2 = a_{j_3}$. Suppose that $a_3 = a_{j_1}$. Then $[a_3, x_2] \rightarrow a_1$, which is impossible since $\{a_3, x_2\}$ does not dominate $b_2$. So $a_1 = a_{j_1}$ and $F(A) = (a_1, a_3, a_2)$. Thus $[a_1, x_2] \rightarrow a_3$ and since $x_2b_3 \notin E(G)$ by Lemma 2.26, $a_1b_3 \in E(G)$. Thus $p \geq 2$ and so $p = 3$, contradicting $a_3b_2 \notin E(G)$. This concludes the proof of (iii).

(iv) By Lemma 33, $[a_1, y] \rightarrow x_0$ for some $y \notin X$. Suppose to the contrary that $y \in C_k$ for some $k$ with $1 \leq k \leq r-1$. Then $a_{k+1} \neq a_1$, and so $ya_{k+1} \in E(G)$. By (ii) $a_{k+1}b_k \in E(G)$, contradicting (iii) and thus (iv) holds.

(v) Suppose to the contrary that $a_1b_{r-1} \in E(G)$. Then by Lemma 34(b),

$$\{b_1, b_2, \ldots, b_{r-2}\} \cap N(a_r) = \phi$$

and by (iii), $a_r b_{r-1} \notin E(G)$. It follows from 17 that $a_r b_r \in E(G)$, and (i) and (ii) imply that $C_r \backslash \{a_r, b_r\} \subseteq N(a_r) \cap N(b_r)$. Now, by (iv), $y \in C_r$ where $[a_1, y] \to x_0$. If $y \in C_r \backslash \{b_r\}$, then by Lemma 25 $a_2$ cannot be dominated. So $y = b_r$, *i.e.* $[a_1, b_r] \to x_0$. Therefore $b_r a_{r-1} \in E(G)$ and $a_1 b_{r-1} \in E(G)$, contradicting Lemma 2.32.

We can now complete the proof of Lemma 37.4.

Again by Lemma 33 there is a vertex $y \notin X$ such that $[a_1, y] \to x_0$. It follows from (v) that $y b_{r-1} \in E(G)$. Since $y \in C_r$ by (iv), it follows from (ii) that $a_r b_{r-1} \in E(G)$, contradicting (iii). □

Thus there exists a maximum independent set of the form

$$A \cup \{x_0, b\} \quad \text{for some } b \in B$$

or

$$B \cup \{x_0, a\} \quad \text{for some } a \in A.$$

It follows, therefore, that $A \neq B$. So we may assume without loss of generality that $b_1 \notin A$ and that $A \cup \{b_1, x_0\}$ is a maximum independent set. Let

$$A_1 = A \cup \{b_1\}.$$

Then there is an ordering $F(A_1)$ of the vertices of $A_1$ and a sequence $Y(A_1)$ such that Lemma 3 is satisfied. By Lemma 4, $Y(A_1) \subseteq N(x_0) = X$, and in fact $Y(A_1) = X$ since

$$|Y(A_1)| = |A_1| - 1 = r = |X|.$$

It now follows from Lemma 5 that for every $a \in A_1$,

$$|N(a) \cap X| = |N(a) \cap Y(A_1)| \geq r - 1 \geq 2 \tag{20}$$

and for every $x \in X$,

$$|N(x) \cap A_1| \geq r - 1 \geq 2. \tag{21}$$

We now formulate and prove the second lemma, which we will use to complete the proof of this case.

**Lemma 37.5** *$\{a_1\} \cup B$ is independent.*

**Proof.** Suppose firstly that $x_1 b_1 \in E(G)$. Then by Lemma 26(a), $B \backslash \{b_1\} \cap N(a_1) = \phi$. Also, $a_1 b_1 \notin E(G)$ since $A \cup \{b_1\}$ is independent. Therefore $\{a_1\} \cup B$ is independent.

Now suppose $x_1 b_1 \notin E(G)$. Since $b_1 \in A_1$, it follows from 20 that $X \backslash \{x_1\} \subseteq N(b_1)$. For each $i$ with $2 \leq i \leq r-1$, since $b_1 x_{i+1} \in E(G)$, it follows that $a_1 b_i \notin E(G)$, for otherwise the cycle

$$x_{i+1} b_1 \overleftarrow{C} a_1 b_i \overleftarrow{C} x_2 x_0 x_1 \overleftarrow{C} x_{i+1}$$

results, which is longer than $C$. Since $x_1 b_1 \notin E(G)$ it follows from 21 and Lemma 28 that $a_1 b_r \notin E(G)$. Furthermore, $a_1 b_1 \notin E(G)$ since $A_1$ is independent. Thus $B \cup \{a_1\}$ is independent. □

Let $B_1 = B \cup \{a_1\}$. By Lemma 37.5 there is an ordering $F(B_1)$ of the vertices of $B_1$ and a sequence $Y(B_1)$ such that Lemma 3 is satisfied. As for $A_1$, $Y(B_1) = X$ and for every $b \in B_1$,

$$|N(b) \cap X| \geq r-1,$$

while for every $x \in X$,

$$|N(x) \cap B_1| \geq r-1 \geq 2.$$

It follows from Lemma 33 that there is a vertex $y \notin X$ such that $[b_1, y] \rightarrow x_0$. Since $A_1$ and $B$ are independent,

$$A \cup B \backslash \{b_1\} \subseteq N(y) \tag{22}$$

and $y \notin A \cup B$.

We will now show the following:

(vi) $y \in C_1 \backslash \{a_1, b_1\}$

(vii) $y \notin \{b_1^-, a_1^+\}$ and thus $|C_1| \geq 5$.

(vi) Suppose the contrary and suppose firstly that $y \in C_r$. It follows from 22 that $\{yb_2, ya_1\} \subseteq E(G)$. Thus $\{y^+a_1, y^+b_1\} \cap E(G) = \phi$ by Lemma 27, and $(A \backslash \{a_1\}) \cap N(y^+) = \phi$ by Lemma 26(a). Thus $A_1 \cup \{y^+\}$ is a maximal independent set excluding $x_0$, a contradiction. Now suppose that $y \in C_k$ for some $k$ with $2 \leq k \leq r-1$. By 22, $yb_r \in E(G)$ and so it follows from Lemma 26(a) that $y^+b_1 \notin E(G)$. Since $yb_{k+1} \in E(G)$, Lemma 27 implies that $y^+a_{k+1} \notin E(G)$. Further, since $ya_{k+1} \in E(G)$, it follows from Lemma 26 (1) that $(A \backslash \{a_{k+1}\}) \cap N(y^+) = \phi$. Again $A_1 \cup \{y^+\}$ is a maximal independent set excluding $x_0$, and so (vi) holds.

(vii) Suppose to the contrary that $y = b_1^-$. By 22, $yb_r \in E(G)$ and so the cycle

$$yb_r \overleftarrow{C} x_2 x_0 x_1 \overrightarrow{C} y$$

results, which is as long as $C$ but excludes $b_1$, contradicting 16.

Now suppose that $y = a_1^+$. By 22, $ya_r \in E(G)$ and so the cycle

$$ya_r \overrightarrow{C} x_1 x_0 x_r \overleftarrow{C} y$$

results, which is as long as $C$ but excludes $a_1$, again contradicting 16.

Thus $y \in C_1 \setminus \{a_1, a_1^+, b_1^-, b_1\}$ and so $|C_1| \geq 5$. Therefore (vii) holds.

Let $A_2 = A \cup \{y^+\}$ and $B_2 = B \cup \{y^-\}$. It follows from Lemma 26(a) and the inclusion $A \cup B \subseteq N(y)$ that $A_2$ and $B_2$ are independent. Thus we can obtain an ordering $F(A_2)$ of the vertices in $A_2$ and a sequence $Y(A_2)$, and an ordering $F(B_2)$ of the vertices in $B_2$ and a sequence $Y(B_2)$, such that Lemma 3 is satisfied in both instances, and where $Y(A_2) = Y(B_2) = X$.

We will show the following:

(viii) $A \not\subseteq N(x)$, for any $x \in X$.

(ix) $B \not\subseteq N(x)$, for any $x \in X$.

(x) $y^+y^- \notin E(G)$.

(viii) Suppose the contrary. Then there exists $\tilde{x} \in X$ such that $A \subseteq N(\tilde{x})$. Let $\tilde{x} = y_t(A_1)$ for some $t$ with $1 \leq t \leq r$. Then

$$[f_t(A_1), \tilde{x}] \rightarrow f_{t+1}(A_1). \tag{23}$$

Since $A \subseteq N(\tilde{x})$, $f_{t+1}(A_1) = b_1$ and hence $f_t(A_1) \in A$.

Now consider $Y(A_2)$ and let $\tilde{x} = y_s(A_2)$ for some $s$ with $1 \leq s \leq r$. Then

$$[f_s(A_2), \tilde{x}] \rightarrow f_{s+1}(A_2).$$

Since $A \subseteq N(\tilde{x})$, $f_{s+1}(A_2) = y^+$ and thus $\tilde{x}y^+ \notin E(G)$. Since $A_2$ is independent, $\{f_t(A_1), \tilde{x}\}$ cannot dominate $y^+$, contradicting 23. Thus (viii) holds.

(ix) Using a similar argument to the one we used in (viii), we can show that $B \not\subseteq N(x)$ for any $x \in X$.

(x) Suppose to the contrary that $y^+y^- \in E(G)$. Then

$$ya_1 \overrightarrow{C} y^- y^+ \overrightarrow{C} x_2 x_0 x_1 \overleftarrow{C} a_2 y$$

is a cycle longer than $C$, a contradiction. Thus (x) holds.

We now complete the proof of Theorem 37. By (x), $y^+y^- \notin E(G)$. Therefore there exists a vertex $z \in V(G) \setminus \{y^+, y^-\}$ such that

$$[y^+, z] \rightarrow y^- \quad \text{or} \quad [y^-, z] \rightarrow y^+.$$

In either case $z \in X$ to dominate $x_0$. Suppose $[y^+, z] \to y^-$. Then $A \subseteq N(z)$ since $A_2$ is independent, contradicting (viii). Suppose $[y^-, z] \to y^+$. Then $B \subseteq N(z)$ since $B_2$ is independent, contradicting (ix).

This completes the proof. ■

Combining Theorem 35, Corollary 36 and Theorem 37, we have the following:

**Theorem 38** (*Wojcicka's Theorem*) *If $G$ is a connected, 3-$\gamma$-critical graph with $\delta(G) \geq 2$, then $G$ is hamiltonian.*

Wojcicka's Theorem can be restated (using Corollary 2.8) as:

*Any 2-connected, 3-$\gamma$-critical graph is hamiltonian.*

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