Independent Chess Pieces on Euclidean Boards

Jens-P. Bode and Heiko Harborth

Diskrete Mathematik
Technische Universität Braunschweig
D-38023 Braunschweig
Germany
email: h.harborth@tu-bs.de

ABSTRACT. Corresponding to chessboards we introduce game boards with triangles or hexagons as cells and chess-like pieces for these boards. The independence number β is determined for many of these pieces.

Dedicated to Ernie Cockayne on his sixtieth birthday.

1. Introduction

As Euclidean boards B_n we consider the classical square boards $B_n^{\square}$ (chessboards) and in addition hexagon and triangle boards, $B_n^{\hexagon}$ and $B_n^{\triangle}$, which are connected parts of the three Euclidean tessellations of the plane in such a way that B_1 is one cell, B_2 consists of all cells surrounding one vertex, and B_n for $n \geq 3$ consists of B_{n-2} together with all cells of the tessellation having at least one point in common with B_{n-2} (see Figure 1).

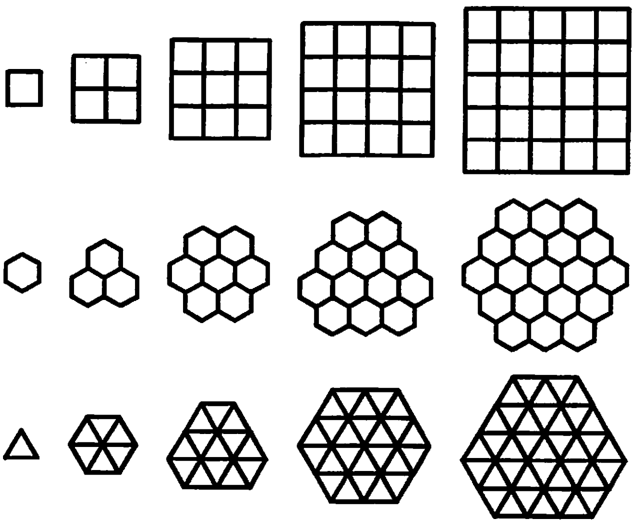

Figure 1. Boards $B_n^{\square}$, $B_n^{\hexagon}$, and $B_n^{\triangle}$ for $1 \leq n \leq 5$.

For chess pieces on $n \times n$-chessboards $B_n^{\square}$ several parameters are discussed frequently (see [6] for literature), for example, the independence number β_n which determines the maximum number of pairwise nonattacking chess pieces of one type on $B_n^{\square}$.

The possibilities of one move on $B_n^{\square}$ are represented in Figure 2 for the pieces grid (GD), king (KG), knight (KT), rook (RK), bishop (BP), and queen (QN). Here we define chess-like pieces for hexagon and triangle boards (see Figures 3 and 4).

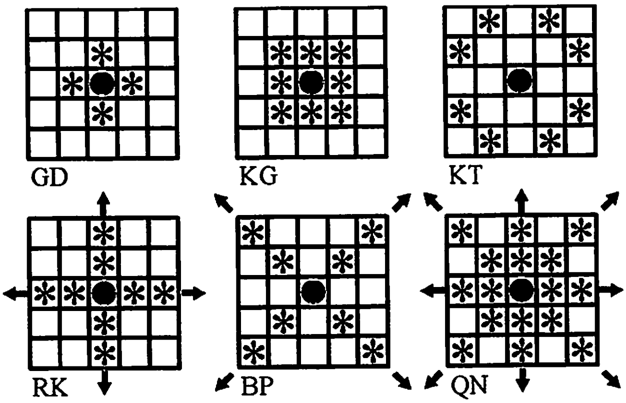

Figure 2. Moves on $B_n^{\square}$.

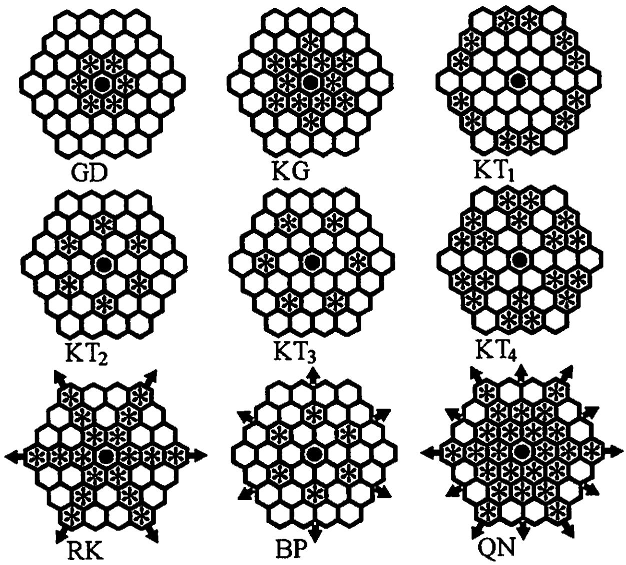

Figure 3. Moves on $B_n^{\bigcirc}$.

A grid attacks all edge-to-edge neighboring cells. A king has four possibilities for $B_n^{\triangle}$. In addition to the grid cells, KG_1 attacks the neighboring

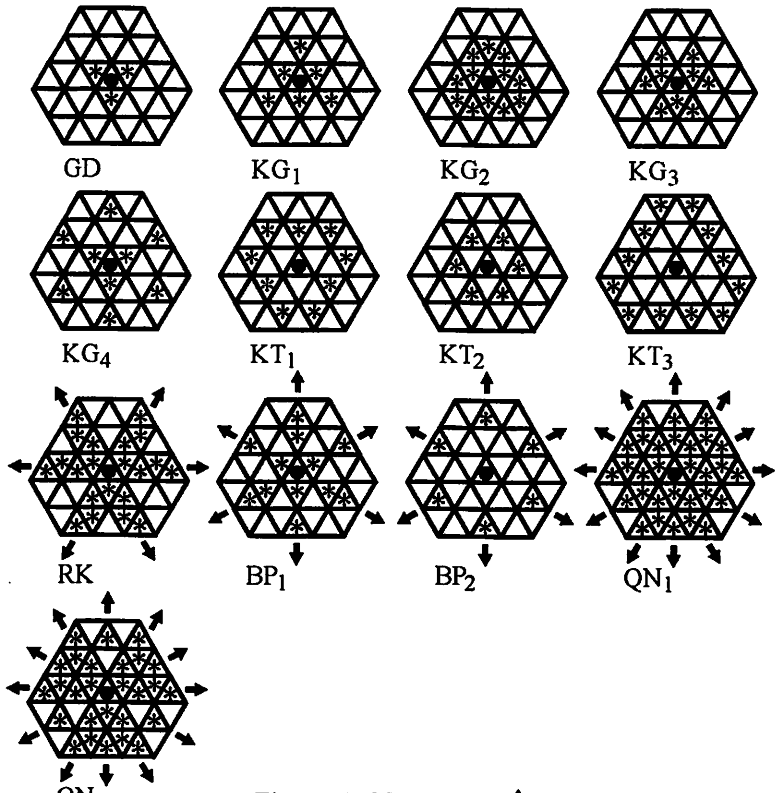

Figure 4. Moves on $B_n^{\triangle}$.

cells on the diagonals, KG_2 attacks all neighbors of the start cell, KG_3 attacks the edge-to-edge neighbors of the grid cells, and KG_4 attacks the two translated neighboring cells in the diagonal directions. A knight moves edge-to-edge two consecutive cells in one direction and attacks a left and a right neighbor for KT_1 and KT_2 on $B_n^{\bigcirc}$ and for KT_1 on $B_n^{\triangle}$. For KT_1 and KT_3 on $B_n^{\bigcirc}$ and for KT_2 on $B_n^{\triangle}$ the moves are to edge-to-edge neighbors of the neighboring cells on the diagonals. On $B_n^{\bigcirc}$ for KT_4 we have the union of the moves of KT_1 and KT_2. On $B_n^{\triangle}$ the moves of KT_3 go to the left and right edge-to-edge neighbors of translated neighboring triangles on the diagonals. All rooks, bishops, and queens move on straight lines of edge-to-edge cells where for $B_n^{\triangle}$ two bishops and two queens are distinguished, that are, for the complete diagonals and for the diagonals of translated cells only.

In this paper we list known results of the independence number β_n (mainly on $B_n^{\square}$), and we prove new results of β_n for many pieces of $B_n^{\bigcirc}$ and $B_n^{\triangle}$. For those values obtained by computer we have used the following method. We partition the cells of B_n into small parts of pairwise dependent cells. The number v of parts is an upper bound for β_n. We try to find v as small as possible using backtracking. If we do not obtain $v = \beta_n$ then we start another backtracking program. For one part after another we check all possibilities either to place one chess piece on a cell of that part observing the additional dependencies, or to leave that part free of a chess piece. Then the maximum number of chess pieces in all these possibilities determines β_n.

2. Square boards

The values of β_n are collected in Table 1. See [6] for references.

$\square$ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
GD	1	2	5	8	13	18	25	32	41	50	61	72	85	98	113
	$\beta_n = \lceil n^2/2 \rceil$														
KG	1	1	4	4	9	9	16	16	25	25	36	36	49	49	64
	$\beta_n = \lceil n/2 \rceil^2$														
KT	1	4	5	8	13	18	25	32	41	50	61	72	85	98	113
	$\beta_n = \lceil n^2/2 \rceil$ if $n \neq 2$														
RK	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	$\beta_n = n$														
BP	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28
	$\beta_n = 2n - 2$ if $n \neq 1$														
QN	1	1	2	4	5	6	7	8	9	10	11	12	13	14	15
	$\beta_n = n$ if $n \neq 2, 3$														

Table 1. Independence numbers β_n for chess pieces on square boards $B_n^{\square}$.

3. Hexagon boards

In case of hexagon boards we so far present terms of β_n for GD, KG, KT_1, KT_2, and RK (see Table 2). For all remaining pieces of Figure 3 we have determined by computer the first values of β_n as listed in Table 2.

3.1 Grid

To prove $\beta_n \geq \lceil n^2/4 \rceil$ we use the well-known unique 3-coloring of the hexagons such that two hexagons with a common edge have different colors. Any grid GD on a hexagon of one color attacks only hexagons of different colors so that GDs on all hexagons of one color are pairwise independent. Since $c(n) = \lceil 3n^2/4 \rceil$ is the number of all cells of the hexagon board B_n we obtain $\beta_n \geq \lceil c(n)/3 \rceil = \lceil n^2/4 \rceil$.

Inductive proofs of $\beta_n \leq \lceil n^2/4 \rceil$ for even and odd n are indicated by Figures 5 and 6, respectively. As upper bound of β_n for B_n we use the sum of the values of β for all parts of a tessellation of B_n.

⬡$_n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
GD	1	1	3	4	7	9	13	16	21	25	31	36	43	49	57
	$\beta_n = \lceil n^2/4 \rceil$														
KG	1	1	2	3	7	7	10	12	19	19	24	27	37	37	44
	$\beta_n = \begin{cases} \lceil 3n^2/16 \rceil & \text{if } n \equiv 0(2) \\ (3n^2+6n+7)/16 & \text{if } n \equiv 1(4) \\ (3n^2+2n-1)/16 & \text{if } n \equiv 3(4) \end{cases}$														
KT$_1$	1	3	7	7	7	10	15	17	22	25	31	39	49	49	57
	$\beta_n = \lceil n^2/4 \rceil$ if $n \geq 14$														
KT$_2$	1	3	3	6	9	12	15	21	21	30	36	42	47	57	57
	$\beta_n = \begin{cases} (3n^2+4n+5)/12 & \text{if } n \equiv 1(6) \\ (3n^2+6n+12)/12 & \text{if } n \equiv 2(6) \\ (3n^2+9)/12 & \text{if } n \equiv 3(6) \\ \lfloor (n+1)^2/4 \rfloor & \text{otherwise} \end{cases}$														
KT$_3$	1	3	4	4	9	12	15	16	22	27	31	36	43	51	58
KT$_4$	1	3	3	4	7	9	12	15	19	22	25	28	37	40	46
RK	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15
	$\beta_n = 2\lceil n/2 \rceil - 1$														
BP	1	3	3	6	7	9	9	15	15	15	19	21	21	24	27
QN	1	1	1	3	3	4	7	7	7	9	9	11	12	13	15

Table 2. Independence numbers β_n for chess pieces on hexagon boards $B_n^{\bigcirc}$.

For even n at first $\beta_2 = 1$ is trivial. For $n = 4$, the five parts of Figure 5 imply $\beta_4 \leq 5$. However, two GDs in the last row force two GDs in the second row and then the upmost B_2 cannot contain an independent GD so that $\beta_4 \leq 4$. Using the general tessellations of Figure 5 we obtain the desired upper bounds by induction. To see that the tessellations of Figure 5 do exist we note that the number of hexagons of each side is increased by 3 if B_n, B_{n-2}, and B_{n-4} are enlarged to B_{n+6}, B_{n+4}, and B_{n+2} respectively. Then 3×2- and 3×4-parallelograms partitioned into 2 and 4 boards B_2 are inserted at the arrows in Figure 5 to obtain a desired tessellation of B_{n+6}.

For odd n the tessellations of Figure 6 are used to prove $\beta_n \leq \lceil n^2/4 \rceil$ by induction. The number of hexagons of each side is increased by 2 if B_n is enlarged to B_{n+4}. Now the insertion of 2×3-parallelograms consisting of 2 boards B_2 at the arrows in Figure 6 determine a desired tessellation of B_{n+4}.

3.2 King

The independence number turns out to be one fourth of the number of cells of B_n for n even and by a linear term larger for n odd. If $n \equiv 1(\text{mod } 4)$ then $\beta_n = \beta_{n+1}$ for the asserted values (see Table 2). Thus it suffices to prove the lower bound for $n \not\equiv 2(\text{mod } 4)$ and the upper bound for $n \not\equiv 1(\text{mod } 4)$

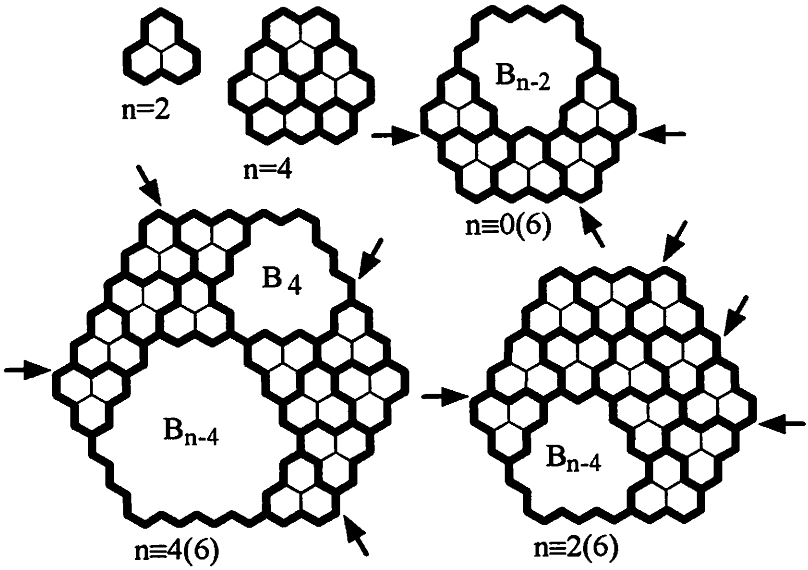

Figure 5. Upper bounds for $\beta_n^{\bigcirc}(GD)$ for even n.

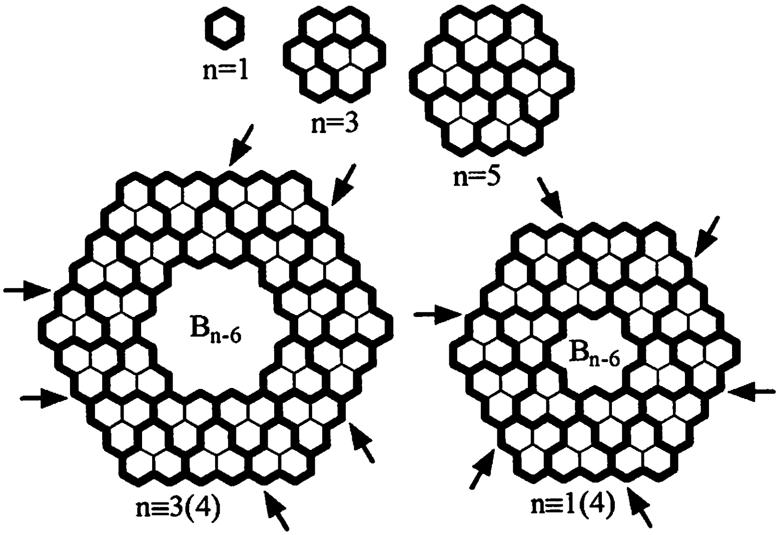

Figure 6. Upper bounds for $\beta_n^{\bigcirc}(GD)$ for odd n.

since $\beta_n \leq \beta_{n+1}$ in general.

The lower bounds follow by counting the kings of the three patterns indicated in Figure 7. The numbers of parts of the three tessellations in Figure 7 determine the upper bounds. The parts have four or three cells such that only one independent KG is possible. The tessellations grow modulo 4 by two rings of hexagons which are tessellated by insertion of one 2×2-rhomb at each side.

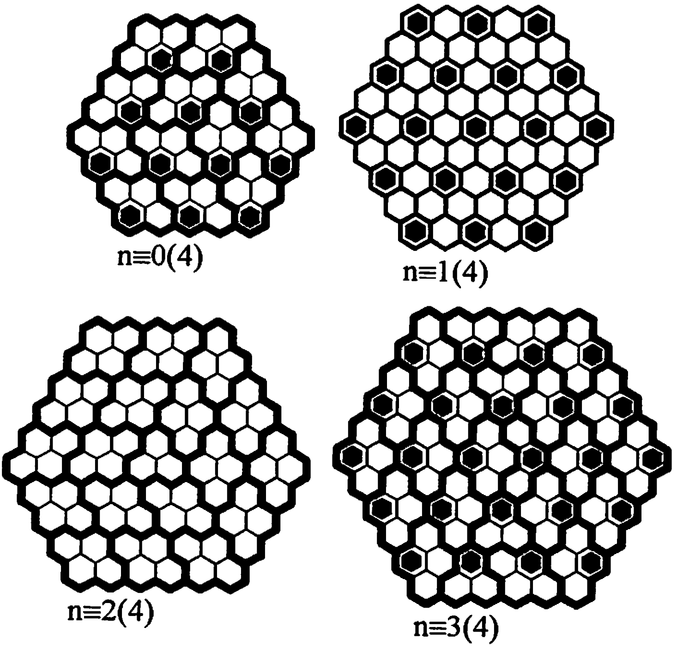

Figure 7. Upper and lower bounds for β_n°(KG).

3.3 Knight 1

One third rounded above of the number $c(n)$ of hexagons of B_n can be occupied by knights KT_1 in a maximum independent set ($n \geq 14$). The proofs are given in [4].

3.4 Knight 2

The sizes of the maximum independent sets differ modulo 6 by a linear factor. The independence number is close to one third of the number $c(n)$ of hexagons of B_n.

For $n \equiv 2 (\mathrm{mod}\ 6)$ the asserted values (see Table 2) fulfill $\beta_n = \beta_{n+1}$ so that it suffices to prove the lower bound for $n \not\equiv 3 (\mathrm{mod}\ 6)$ and the upper bound for $n \not\equiv 2 (\mathrm{mod}\ 6)$ since $\beta_n \leq \beta_{n+1}$ in general.

The lower bounds for $\beta_n(\mathrm{KT}_2)$ follow from the patterns in Figure 8 where the boards B_n grow modulo 6 by three rings of hexagons.

For the upper bounds we use partitions of B_n into triples and some pairs or singles of pairwise dependent hexagons as indicated in Figure 9 where the marked sixtuples denote two triples. The numbers of these tuples give

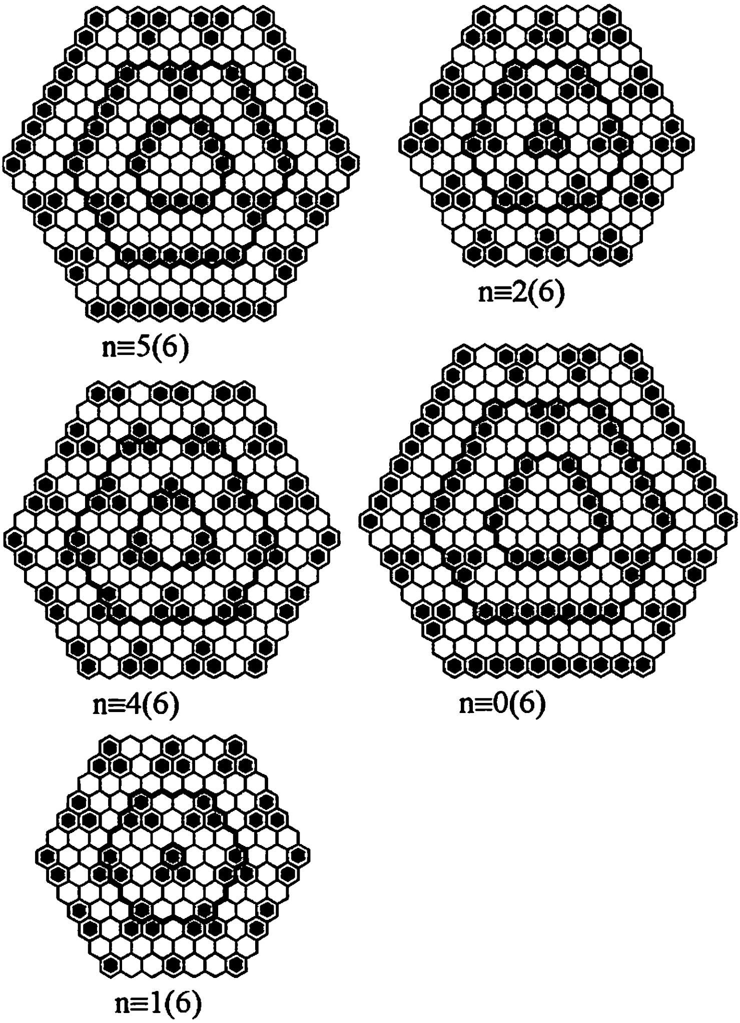

Figure 8. Lower bounds for $\beta_n^{\circ}(\mathrm{KT}_2)$.

the upper bounds (see Table 2) for $n \equiv 0$, 3, 4, and 5(mod 6) since every tuple contains at most one KT_2. For both, the general existence of the partitions of Figure 9 and the enumeration of the numbers of tuples, we remark that modulo 6 there are three additional rings of hexagons. These rings may be interpreted in such a way that at all six sides a block of nine hexagons is inserted. These blocks consist of one of the sixtuples together with one triple which uses the hexagon in the center of the sixtuple.

For $n \equiv 1$(mod 6) we have to observe that in general the unique 3-coloring mentioned in 3.1 has the property that KT_2 attacks only hexagons

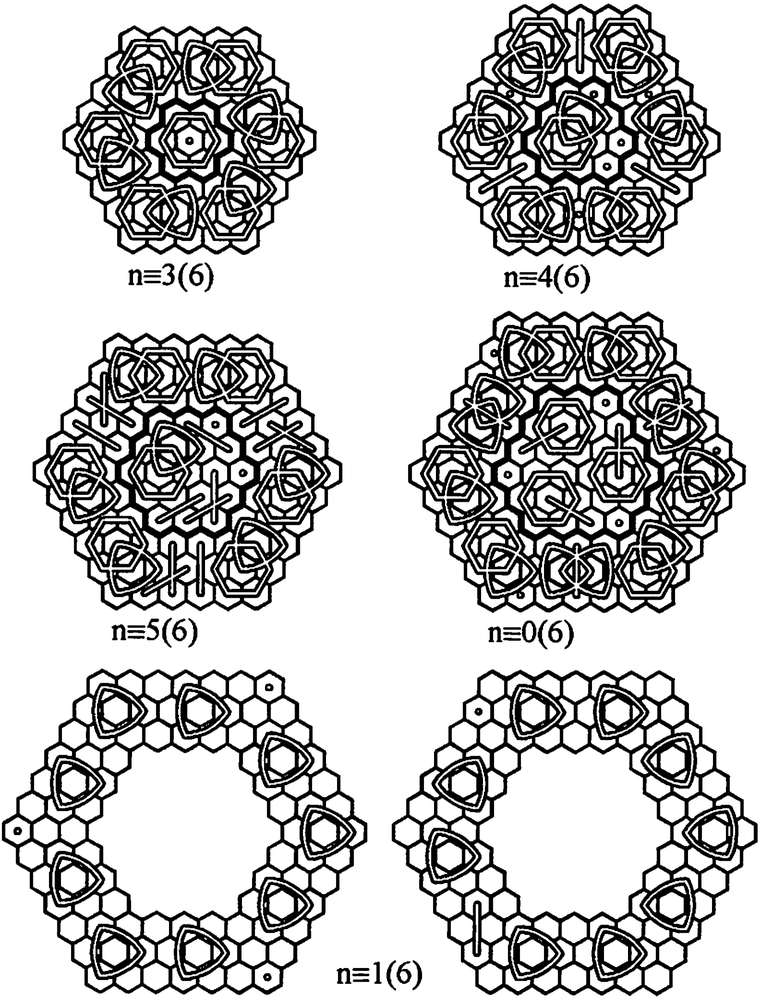

Figure 9. Upper bounds for $\beta_n^{\bigcirc}(\mathrm{KT}_2)$.

within one color class. Thus we can discuss the independence for the 3 color classes separately. For $n \equiv 1 (\mathrm{mod}\ 6)$ the left case in Figure 9 occurs once and the right case twice. The enumeration of the tuples in the right case is by one tuple too large. However, if the one single, the one pair, and all triples each are occupied by one KT_2 then at least one pair of KT_2s is dependent and we can subtract one.

3.5 Knights 3 and 4

General results for KT_3 and KT_4 will be discussed in [1].

3.6 Rook

To prove $\beta_n = 2\lceil n/2 \rceil - 1$ it suffices to prove $\beta_n \geq n$ for $n \equiv 1 (\text{mod } 2)$ and $\beta_n \leq n-1$ for $n \equiv 0 (\text{mod } 2)$ since $\beta_n \leq \beta_{n+1}$ in general.

For the lower bound it is possible to place the rooks in two parallel rows as shown in Figure 10.

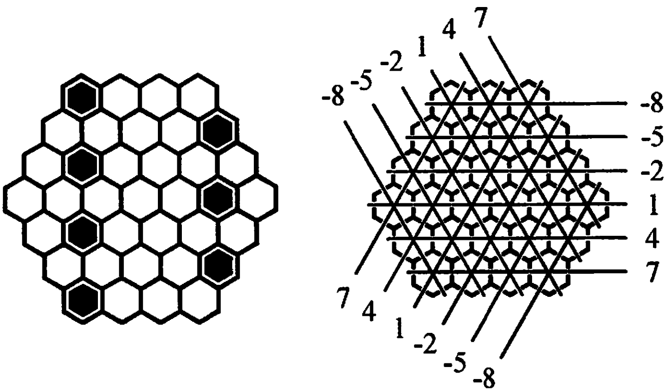

Figure 10. Rook independence $\beta_n^{\bigcirc}(\text{RK})$.

For the upper bound we consider three classes of weighted parallel lines of hexagons which give their weight to every hexagon of that line. To obtain zero for the sum of all three weights for each hexagon we start with a weight 1 for each of those three lines which intersect two of the three hexagons around the center vertex of B_n. Then we continue in such a way that $1+3i$ for i with $-n/2 \leq i \leq (n-2)/2$ are the weights for the consecutive parallels of the three classes. Assuming that n independent RKs are possible on B_n, we note that on the one hand the sum of all weights of lines meeting an RK vanishes and equals the sum of the weights of all lines on the other hand. However, the sum S of $1+3i$ for $-n/2 \leq i \leq (n-2)/2$ is $S = -n/2$ and thus $3S \neq 0$. This contradiction proves $\beta_n \leq n-1$.

3.7 Bishop

Results for BP will be discussed in [2]. The independence numbers vary from $2n-5$ to $2n-1$.

3.8 Queen

From rooks we know $\beta_n(\text{QN}) \leq \beta_n(\text{RK}) = 2\lceil n/2 \rceil - 1$. We do not know whether this bound is attained infinitely often.

4. Triangle boards

For triangle boards the independence numbers β_n will be presented for GD, KG_1, KG_3, KT_1, KT_3, RK, and BP_1 (see Table 3). The other values for $n \leq 15$ in Table 3 are determined by computer.

$\triangle n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
GD	1	3	7	12	19	27	37	48	61	75	91	108	127	147	169
	$\beta_n = \lceil 3n^2/4 \rceil$														
KG_1	1	3	7	12	19	27	37	48	61	75	91	108	127	147	169
	$\beta_n = \lceil 3n^2/4 \rceil$														
KG_2	1	1	3	6	8	12	15	19	25	30	36	42	49	55	
KG_3	1	2	4	6	10	15	19	24	31	39	46	54	64	75	85
	$\beta_n = \begin{cases} \lceil 3n^2/8 \rceil + 1 & \text{if } n \equiv 2(4) \text{ and } n \neq 2 \\ \lceil 3n^2/8 \rceil & \text{otherwise} \end{cases}$														
KG_4	1	3	5	9	14	18	26	34	41	52	64				
KT_1	1	3	7	12	19	27	37	48	61	75	91	108	127	147	169
	$\beta_n = \lceil 3n^2/4 \rceil$														
KT_2	1	2	6	8	13	18	25	32	41	50	61	72	85	98	113
KT_3	1	6	7	12	19	27	37	48	61	75	91	108	127	147	169
	$\beta_n = \lceil 3n^2/4 \rceil$ if $n \neq 2$														
RK	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	$\beta_n = n$														
BP_1	1	3	5	6	9	9	13	15	17	18	21	21	25	26	29
	$\beta_n = \begin{cases} 2n-1 & \text{if } n \equiv 1(2) \text{ or } n \equiv 2(6) \text{ and } n \neq 14 \\ 2n-2 & \text{if } n \equiv 4(6) \text{ or } n = 14 \\ 2n-3 & \text{if } n \equiv 0(6) \end{cases}$														
BP_2	1	6	6	12	16	18	18	30	30	30	40	42	45	48	54
QN_1	1	1	2	3	4	6	7	8	8	10	10	12	12	14	15
QN_2	1	2	2	4	4	6	7	8	8	10	10	12	12	14	15

Table 3. Independence numbers β_n for chess pieces on triangle boards $B_n^{\triangle}$.

4.1 Grid and king 1

The triangles of B_n are colorable by two colors such that triangles with a common side are of different color. Each of the pieces GD, KG_1, KT_1, and KT_3 move to triangles of the other color only. Thus the largest number of triangles of one color determines a lower bound of β_n. Since there are $c(n) = \lfloor 3n^2/2 \rfloor$ triangles in B_n we obtain $\beta_n \geq \lceil c(n)/2 \rceil = \lceil 3n^2/4 \rceil$ for all four pieces.

Figure 11 suggests partitions of B_n into 1×1-rhombs of two triangles and an additional triangle for n odd. Since each of the $\lceil c(n)/2 \rceil$ parts contains at most one piece the upper bound is proved.

4.2 King 2

This piece KG_2 will be discussed in [3]. Here β_n exceeds $c(n)/6$ by a linear factor.

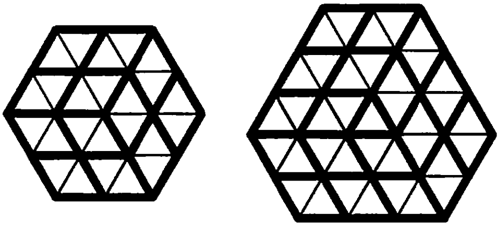

Figure 11. Upper bounds for $\beta_n^{\triangle}(\mathrm{GD})$ and $\beta_n^{\triangle}(\mathrm{KG}_1)$.

4.3 King 3

We use that 4-coloring of the triangles for the lower bound where any

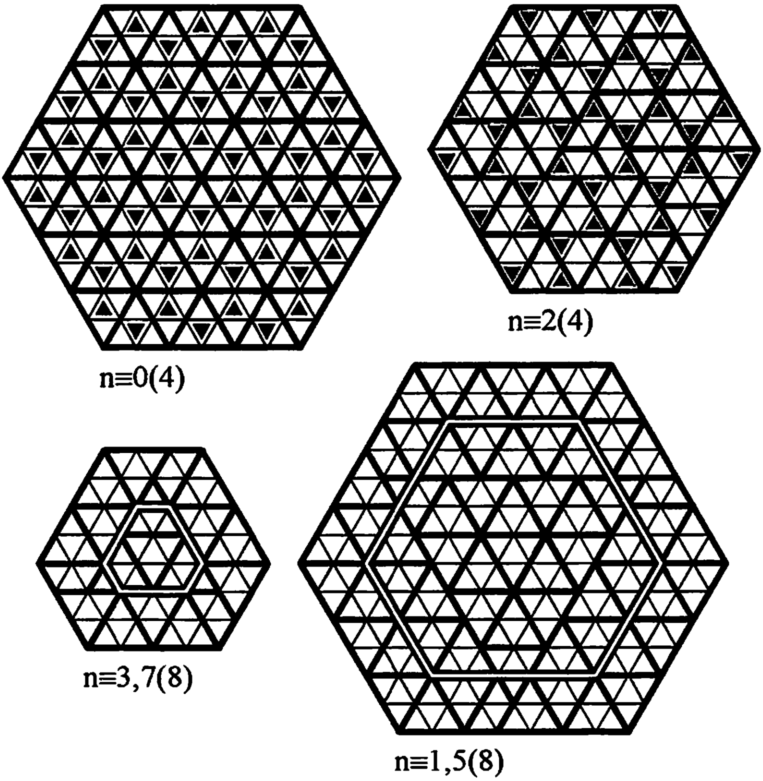

Figure 12. King 3 independence.

triangle and its vertex-to-vertex opposite triangles are in one color class (see Figure 12 for $n \equiv 0 (\mathrm{mod}\ 4)$). This implies the lower bound $\beta_n \geq \lceil c(n)/4 \rceil = \lceil 3n^2/8 \rceil$. For $n \equiv 2(\mathrm{mod}\ 4)$ we use that coloring which does not touch the center point. This coloring covers $\lceil 3n^2/8 \rceil - 2$ triangles. Then we move the KG_3s at every second side of B_n by two triangles in one direction. It follows that we are able to place three additional KG_3s at every second vertex point of B_n (see Figure 12). Thus for $n \equiv 2 \pmod 4$ we obtain $\beta_n \geq \lceil 3n^2/8 \rceil - 2 + 3$.

The upper bounds follow from the numbers of parts in the partitions of B_n indicated in Figure 12. For $n \equiv 0 \pmod 4$ it is possible to use only triangular sets of 4 triangles as parts. For $n \equiv 2 \pmod 4$ we may start with 3 parts of 3 triangles around the center vertex of B_n. Then we fill up B_n with triangular sets of 4 triangles and only 3 parts of 3 triangles remain at every second vertex point of B_n.

If $n \equiv 1$ or $3 \pmod 8$ then there are three sides of B_n having an odd number of sides of triangles. In the middle of each of these sides we place one part consisting of 3 triangles. Then B_n can be filled up by triangular parts consisting of 4 triangles (see inner B_ns in Figure 12 with $n = 3$ and $n = 9$). If $n \equiv 5$ or $7 \pmod 8$ then partitions of two rings of triangles can be added to the just mentioned cases as shown in Figure 12 with $n = 7$ and $n = 13$. These partitions consist of 3 singles neighboring the triples of the inner B_{n-4} and 3 triples (neighbors of the singles), and all remaining parts are triangular parts of 4 triangles. Thus again there are only 3 parts with 3 triangles. The numbers of all parts in these partitions prove the asserted values as upper bounds.

4.4 King 4

In [3] we will handle KG_4.

4.5 Knight 1

The lower bound was proved in 4.1.

For the upper bound we note that KT_1 on a ring of triangles attacks the third next triangles in both directions. Thus six consecutive triangles on a ring can be partitioned into three pairs of dependent triangles (see Figure 13). Since the number of triangles on a ring of B_n is a multiple

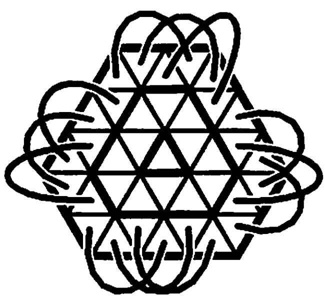

Figure 13. Upper bound for $\beta_n^{\triangle}(KT_1)$.

of 6, the triangles of B_n are partitioned into $\lceil c(n)/2 \rceil$ parts of dependent pairs of triangles and one triangle for n odd. This proves the upper bound $\lceil c(n)/2 \rceil$.

4.6 Knight 2

In [1] one third of $c(n)$ rounded above will be determined as β_n for KT_2.

4.7 Knight 3

In 4.1 the lower bound is proved. The numbers β_1 and β_2 are trivial. The upper bounds are obtained by induction modulo 6. The three outer rings of B_n are then partitioned into blocks of 6 triangles which then are partitioned into 3 pairs of dependent triangles (see the first board in Figure 14). Besides B_1 the five boards in Figure 14 (the first board with $n = 0$)

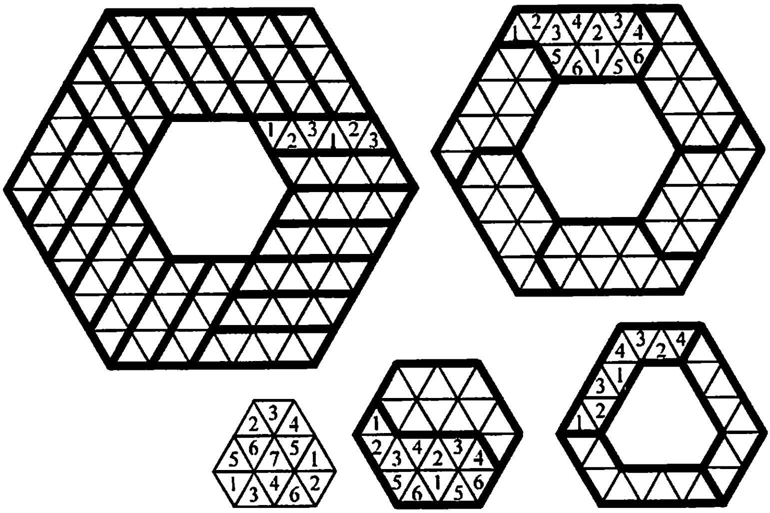

Figure 14. Upper bounds for $\beta_n^{\Delta}(KT_3)$.

serve as induction bases since partitions into dependent pairs of triangles and one single triangle for n odd are possible as indicated in the Figure.

4.8 Rook and bishop 1

The terms of β_n for RK and BP_1 (see Table 3) are proved in [5].

4.9 Bishop 2

The independence of BP_2 will be discussed in [2].

4.10 Queens

For QN_1 and QN_2 it holds $\beta_n(QN_1) \leq \beta_n(QN_2) \leq \beta_n(RK) = n$. We so far only know the values for small n given in Table 3 and [5]. It remains open whether $\beta_n = n$ can be attained infinitely often.

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