Distance-k independent domination sequences

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is a k-packing if the distance between every pair of distinct vertices in S is at least $k+1$, and $\rho_k(G)$ is the maximum cardinality of a k-packing. A set $S \subseteq V$ is a distance-k dominating set if for each vertex $u \in V-S$, the distance $d(u, v) \leq k$ for some $v \in S$. Call a vertex set S a k-independent dominating set if it is both a k-packing and a distance-k dominating set, and let the k-independent domination number $i_k(G)$ be the minimum cardinality of a k-independent dominating set. We show that deciding if a graph G is not k-equipackable (that is, $i_k(G) < \rho_k(G)$) is an NP-complete problem, and we present a lower bound on $i_k(G)$. Our main result shows that the sequence $(i_1(G), i_2(G), i_3(G), \ldots)$ is surprisingly not monotone. In fact, the difference $i_{k+1}(G) - i_k(G)$ can be arbitrarily large.

Dedicated to Ernie Cockayne on the occasion of his 60th birthday

1 Introduction

The *closed k-neighborhood* of a vertex v in a graph $G = (V, E)$ is $N_k[v] = \{w \in V \mid d(v,w) \leq k\}$, where $d(v,w)$ denotes the distance between v and w in G. A vertex set S is said to be *distance-k dominating*, or just *k-dominating*, if $N_k[v] \cap S \neq \emptyset$ for every $v \in V$, and $\gamma_{\leq k}(G)$ is the minimum cardinality of a k-dominating set. These sets were first studied by Meir and Moon under the term of k-coverings [17]. The even more general $\mathcal{R}$-dominating sets were introduced in Slater [20], and k-dominating sets are considered, for example, in [2, 10, 11, 14]. Note that $\gamma_{\leq 1}(G)$ is simply the domination number $\gamma(G)$. For a comprehensive study of domination and distance domination, the reader is referred to [8, 9]. Clearly, every k-dominating set is also $(k+1)$-dominating, and we have the following.

$$\gamma(G) = \gamma_{\leq 1}(G) \geq \gamma_{\leq 2}(G) \geq \gamma_{\leq 3}(G) \geq \dots \tag{1}$$

Meir and Moon [17] call S a *k-packing* if $d(u,v) \geq k+1$ for each pair of distinct vertices u and v in S. We let $\rho_k(G)$ denote the *k-packing number* of G, that is, the maximum cardinality of a k-packing. Note that the independence number $\beta(G) = \rho_1(G)$ and the packing number $\rho(G) = \rho_2(G)$, and every $(k+1)$-packing is a k-packing. Thus for any graph G, we have the following.

$$\beta(G) = \rho_1(G) \geq \rho_2(G) \geq \rho_3(G) \geq \dots \tag{2}$$

The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set. Note that for the star $K_{1,k}$ of order $n = k+1$, we have $i(K_{1,k}) = 1$ and $\beta(K_{1,k}) = k$, but graphs G such as cycles C_4 and C_5, complete graphs K_n, and complete multipartite graphs $K_{t,t,\dots,t}$ have $i(G) = \beta(G)$. The complementary property for independence is covering, that is, a vertex set S is independent if and only if $V - S$ is a covering of E. Thus, $i(G) = \beta(G)$ if and only if all minimal covers have the same cardinality. Plummer [18] initiated the study of *well-covered graphs* G, those with $i(G) = \beta(G)$. For a survey on well-covered graphs, see [19].

In general, a k-packing is maximal if and only if it is k-dominating. Call a vertex set S a *k-independent dominating set*, as in [1, 4, 5, 6, 7, 12, 13, 16], if it is both a k-packing and a k-dominating set, and let the *k-independent domination number* $i_k(G)$ be the minimum cardinality of a k-independent dominating set. In particular, $i(G) = i_1(G)$. Thus, $i_k(G)$ and $\rho_k(G)$ are the minimum and maximum cardinalities, respectively, of any maximal k-

packing. Clearly, $i_k(G) \leq \rho_k(G)$. Also, any $i_k(G)$-set is k-dominating, so $\gamma_{\leq k}(G) \leq i_k(G)$.

Proposition 1 *For any graph G and positive integer k,*

$$\gamma_{\leq k}(G) \leq i_k(G) \leq \rho_k(G).$$

We next mention a well-known lower bound for the domination number of a graph.

Proposition 2 (Walikar, Acharya, and Sampathkumar [22]) *For any graph G,*

$$\lceil n/(1+\Delta(G)) \rceil \leq \gamma(G).$$

A stronger result is given in [21].

Proposition 3 (Slater [21]) *If G has degree sequence $(d_1, d_2, ..., d_n)$ with $d_i \geq d_{i+1}$, then $\gamma(G) \geq \min\{t \mid t + (d_1 + d_2 + \cdots + d_t) \geq n\}$.*

The *open k-neighborhood* of a vertex $v \in V$, denoted $N_k(v)$, is the set $N_k(v) = \{u \mid u \neq v \text{ and } d(u,v) \leq k\}$. Then the *$k$-degree* $deg_k(v) = |N_k(v)|$, and the *maximum k-degree* $\Delta_k(G) = max\{deg_k(v) \mid v \in V\}$. Note that $deg_k(v)$ equals the degree of v in the kth power G^k of G. Assume that $V = \{v_1, v_2, ..., v_n\}$, and let $d_i^k = deg_k(v_i)$.

Fricke, Hedetniemi, and Henning [5] extended Proposition 2 to the following bound for k-domination.

Proposition 4 *For any graph G with $\Delta_k(G) \geq 2k \geq 2$,*

$$\left\lceil \frac{n}{\frac{k+1}{k}\Delta_k(G) - 1} \right\rceil \leq i_k(G).$$

This result follows by simply observing that

$$i_k(G) \geq \gamma_{\leq k}(G) \geq \frac{n}{\Delta_k(G)+1} \geq \frac{n}{\frac{k+1}{k}\Delta_k(G) - 1}$$

for $\Delta_k(G) \geq 2k$.

Further using Proposition 3 along with the facts that $\gamma_{\leq k}(G) = \gamma(G^k)$ and $i_k(G) = i(G^k)$, we have the following.

Proposition 5 *If G has k-degree sequence $(d_1^k, d_2^k, ..., d_n^k)$ with $d_i^k \geq d_{i+1}^k$, then*

$$i_k(G) \geq \gamma_{\leq k}(G) \geq min\{t \mid t + (d_1^k + d_2^k + ...d_t^k) \geq n\} \geq \frac{n}{\Delta_k(G)+1}.$$

Writing $(a_1, a_2, ...) \leq (b_1, b_2, ...)$ if every $a_i \leq b_i$, and using (1), (2), and Proposition 1, we have the following.

Proposition 6 *For any graph G,*

$$\begin{aligned}&(\gamma_{\leq 1}(G), \gamma_{\leq 2}(G), \gamma_{\leq 3}(G), ...)\\ &\leq (i_1(G), i_2(G), i_3(G), ...)\\ &\leq (\rho_1(G), \rho_2(G), \rho_3(G), ...),\end{aligned}$$

and the first and third sequences are nonincreasing.

Moreover, it is known that $\rho_{2k}(G) \leq \gamma_{\leq k}$ for any graph G and any $k \geq 1$ (the proof given for trees in [17] is valid for any graph). This implies $i_{2k} \leq \rho_{2k} \leq \gamma_{\leq k} \leq i_k$ for all $k \geq 1$. This could lead us to think that the sequence i_k is nonincreasing as is the case for the sequences ρ_k and $\gamma_{\leq k}$. Surprisingly, as shown in Section 3, the sequence $(i_1(G), i_2(G), i_3(G), ...)$ is not necessarily nonincreasing. That is, $i_k(G) = i(G^k)$, but adding edges to a graph can actually increase the value of the parameter i, so it is possible to have $i(G^k) < i(G^{k+1})$. In fact, we show that the difference $i_{k+1}(G) - i_k(G)$ can be arbitrarily large. First, we present complexity results, in Section 2.

2 Complexity

In terms of computing $i_k(G)$, there is little hope of finding an efficient algorithm to determine i_k for arbitrary graphs. In fact, Irving [15] has shown that approximating $i_1(G)$ within a factor of t is NP-hard.

Theorem 7 (Irving [15]) *Unless $P = NP$, there cannot exist a polynomial time approximation algorithm A satisfying $A(G) \leq t \cdot i(G)$ for an arbitrary fixed constant $t > 1$.*

Based on some additions to the construction presented by Irving for 1-independent domination, McRae and Hedetniemi [16] extended this result to k-independent domination.

Theorem 8 (McRae and Hedetniemi [16]) *Unless $P = NP$, there cannot exist a polynomial approximation algorithm A satisfying $A(G) < t \cdot i_k(G)$ for an arbitrary fixed constant $t > 1$.*

Hartnell and Whitehead [7] call a graph G *k-equipackable* if all maximal k-packings of G have the same order, that is, $i_k(G) = \rho_k(G)$. The 1-equipackable graphs are precisely the well-covered graphs of Plummer [18].

Theorem 9 (Chvátal and Slater [3]) *Deciding if a graph G is not well-covered (that is, $i(G) < \beta(G)$) is an NP-complete problem.*

We extend Theorem 9 to show that deciding if a graph G is not k-equipackable (that is, $i_k(G) < \rho_k(G)$) is an NP-complete problem. We demonstrate reductions from the known NP-complete problem 3-Satisfiability.

Problem: 3-Satisfiability (3SAT).

INSTANCE: Set $\mathcal{U} = \{u_1, u_2, ..., u_N\}$ of literals and collection $\mathcal{C} = \{c_1, c_2, ..., c_M\}$ of clauses where each c_i is a 3-element subset of $\{u_1, u_2, ..., u_N, \overline{u}_1, \overline{u}_2, ..., \overline{u}_N\}$.

QUESTION: Does there exist a satisfying truth-assignment for $\mathcal{C}$?

Clearly, we can assume no clause contains both a literal and its complement, because such a clause is satisfied by every truth assignment.

Problem: Not k-equipackable (NkE).

INSTANCE: Graph $G = (V, E)$.

QUESTION: Is $i_k(G) < \rho_k(G)$?

Theorem 10 *For each positive integer k, the problem NkE is NP-complete.*

Proof. As noted, the not-well-covered problem ($k = 1$) is NP-complete [3]. Clearly, NkE $\in$ NP because we can test two vertex sets S_1 and S_2 in polynomial time to see if each is a maximal k-packing and $|S_1| \neq |S_2|$.

First, assume k is even, $k = 2j \geq 2$. Given an instance of 3SAT, construct graph G as follows. For each u_i with $1 \leq i \leq N$, construct a path $P_i = u_i, u_i^2, u_i^3, ..., u_i^k, \overline{u}_i$ on $k+1$ vertices. Construct a complete graph K_M on $L_1 = \{c_1, c_2, ..., c_M\}$. Each literal u_i or $\overline{u}_i$ is therefore identified with a

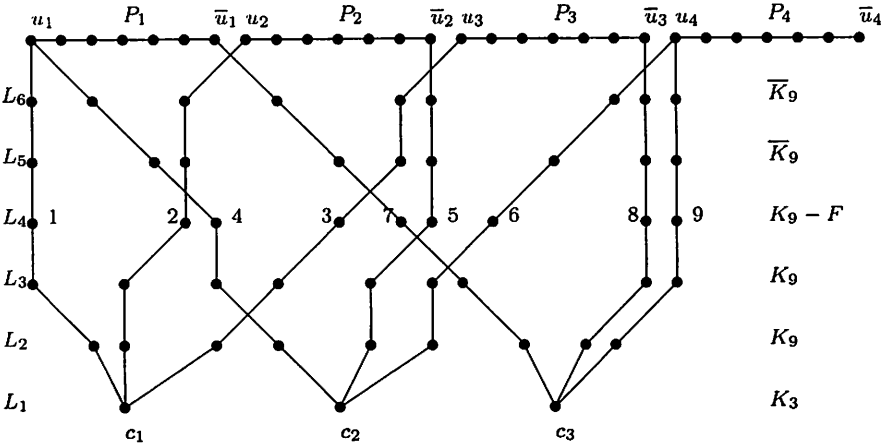

Figure 1: Graph G with $k = 6$, $F = \{\{1,7\},\{4,7\},\{2,5\},\{3,8\}\}$, and $C = \{\{u_1,u_2,u_3\},\{u_1,\overline{u}_2,u_4\},\{\overline{u}_1,\overline{u}_3,u_4\}\}$.

vertex of G, as is each clause c_j. For each clause c_j connect vertex c_j to vertex u_i (respectively, $\overline{u}_i$) by a path of length k if and only if $u_i \in c_j$ (respectively, $\overline{u}_i \in c_j$). For $2 \le h \le k$, let L_h consist of the vertices w on a c_j-to-u_i or c_j-to-$\overline{u}_i$ path whose distance from the clause vertex is $h-1$, that is, $d(w,c_j) = h-1$. Note that for $2 \le h \le k$ we have $|L_h| = 3M$, each $c_j \in L_1$ is adjacent to three vertices in L_2, for $2 \le h \le k-1$ there is a perfect matching between L_h and L_{h+1}, and the number of vertices in L_k adjacent to a u_i or $\overline{u}_i$ vertex is the number of clauses containing u_i or $\overline{u}_i$, respectively. We next define each induced subgraph on L_h, $\langle L_h\rangle$ for $1 \le h \le k$. Let $\langle L_h\rangle$ be complete for $1 \le h \le j$, and for $j+2 \le h \le k = 2j$ no two vertices of L_h are adjacent (that is, $\langle L_h\rangle = \overline{K}_{3M}$ for $j+2 \le h \le k$). Finally, two vertices v and w in L_{j+1} are adjacent unless one of them is at distance j from some u_i and the other is at distance j from $\overline{u}_i$. See Figure 1 for $k=6$, $\mathcal{U} = \{u_1,u_2,u_3,u_4\}$, and $C = \{\{u_1,u_2,u_3\},\{u_1,\overline{u}_2,u_4\},\{\overline{u}_1,\overline{u}_3,u_4\}\}$.

Let $V_1 = L_2 \cup L_3 ... \cup L_k$. Any two vertices v and w in V_1 satisfy $d(v,w) \le k$, and $d(v,w) = k$ if and only if either $vu_i \in E$ and $w\overline{u}_i \in E$ or else $wu_i \in E$ and $v\overline{u}_i \in E$ for some i. Any two clause vertices are adjacent, and $w \in V_1$ implies $d(c_j,w) \le k$. Thus considering $V_1 \cup L_1, V(P_1), V(P_2), ..., V(P_N)$, we have $\rho_k(G) \le N+1$. Also, $v \in V(P_i)$ and $w \in V(P_j)$ with $i \ne j$ implies $d(v,w) \ge k+1$. Relabelling, if necessary, we can assume $c_1 = \{u_1,u_2,u_3\}$. Then $\{c_1,\overline{u}_1,\overline{u}_2,\overline{u}_3,u_4,u_5,...,u_N\}$ is a k-packing. Hence, $\rho_k(G) = N+1$.

Assume there is a satisfying truth assignment $t : \mathcal{U} \rightarrow \{\text{true, false}\}$. Let $S = \{u_i \in V \mid t(u_i) = \text{true}\} \cup \{\overline{u}_i \in V \mid t(u_i) = \text{false}\}$. Note that each vertex c_j is within distance k of at least one vertex in S. Without loss of generality, assume $u_1 \in S$ and $u_2 \in S$. If $w \in V_1 - N(\overline{u}_1)$, then $d(u_1, w) \leq k$, and if $w \in V_1 \cap N(\overline{u}_1)$, then $d(u_2, w) = k$. Hence, S is a maximal k-packing, and $i_k(G) < \rho_k(G)$.

Assume there does not exist a satisfying truth assignment $t : \mathcal{U} \rightarrow \{\text{true,false}\}$, and let $S \subseteq V(G)$ be a maximal k-packing. Let $S_1 = S \cap (V_1 \cup L_1)$ and $S_2 = S - S_1$. Note that $|S_2 \cap V(P_i)| \leq 1$. Because there is no satisfying truth assignment, at least one clause vertex c_j has $N_k[c_j] \cap S_2 = \emptyset$. It follows that $S_1 \neq \emptyset$. Since any two vertices in $V_1 \cup L_1$ are at distance at most k, $|S_1| = 1$. Let $S_1 = \{w\}$. It remains solely to observe that each $V(P_i)$ has at least one vertex $x \in V(P_i)$ with $d(x, w) \geq k + 1$. Hence, $|S| = N + 1 = \rho_k(G)$.

Thus, $i_k(G) < \rho_k(G)$ if and only if there is a satisfying truth assignment, completing the proof when k is even.

For $k = 2j + 1 \geq 3$ we construct a similar graph G with paths P_i of length k connecting vertices u_i and $\overline{u}_i$, $1 \leq i \leq N$. Each clause vertex $c_j \in L_1$ is again connected by a path of length k to each of the three vertices corresponding to literals in clause c_j. For this case where k is odd, we make $\langle L_1 \rangle = K_M$, $\langle L_h \rangle = K_{3M}$ for $2 \leq h \leq j+1$, and $\langle L_h \rangle = \overline{K}_{3M}$ for $j + 2 \leq h \leq k$. This time, therefore, every $\langle L_h \rangle$ is either complete or the empty graph, but between L_{j+1} and L_{j+2} we have more than just a matching. Specifically, for each set of three paths connecting a clause vertex to its literal vertices between L_{j+1} and L_{j+2} we add six more edges to form a $K_{3,3}$.

Again, $i_k(G) < \rho_k(G)$ if and only if there is a satisfying truth assignment, completing the proof. □

3 The Sequence $(i_1(G), i_2(G), i_3(G), \ldots)$

As stated in the introduction, $(i_1(G), i_2(G), i_3(G), \ldots)$ is not monotone for all graphs G. Since $i_{2k}(G) \leq i_k(G)$ for any k, the inequality $i_2(G) \leq i_1(G)$ is always satisfied. However, the graph G in Figure 2 shows that the sequence is not necessarily nonincreasing for $k \geq 2$ (here $\{u, v, w\}$ is an $i_2(G)$-set and $\{x, y, z, w\}$ is an $i_3(G)$-set, that is, $i_2(G) = 3 < i_3(G) = 4$). Also, for the graph in Figure 3, $\{u, v\}$ is an $i_3(G)$-set and $\{x, y, z\}$ is an $i_4(G)$-set. Next we show infinite families where $i_{k+1} > i_k$ for $k \geq 4$, and such that, in fact,

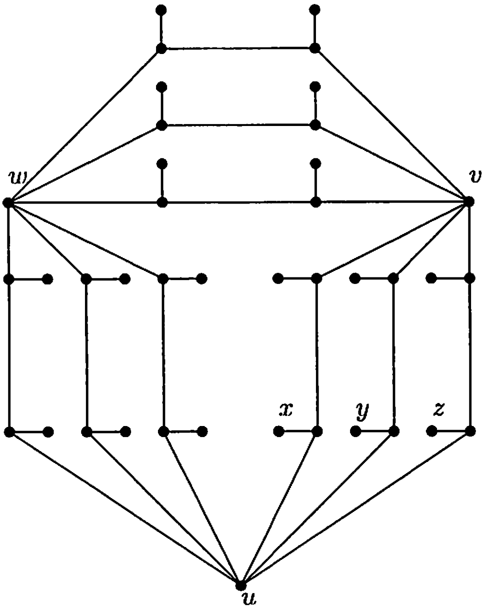

Figure 2: Graph G with $i_2(G) = 3 < i_3(G) = 4$.

the difference $i_{k+1} - i_k$ can be made arbitrarily large.

Theorem 11 *For any integers $k \geq 4$, $r \geq 2$, and l arbitrarily large, there exist graphs G such that $i_k(G) = r$ and $i_{k+1}(G) \geq l$.*

Proof. Consider the graph $G(r, k, l)$ constructed as follows. For $r \geq 3$, begin with a complete graph K_r with vertex set $X = \{x_1, x_2, ..., x_r\}$. Subdivide each edge $x_i x_j$ with k vertices denoted, from x_i to x_j,

$$x_{ij}(1), x_{ij}(2), ..., x_{ij}(k/2), x_{ji}(k/2), ..., x_{ji}(2), x_{ji}(1)$$

if k is even, and

$$x_{ij}(1), x_{ij}(2), ..., x_{ij}((k-1)/2), x_{ij}((k+1)/2) = x_{ji}((k+1)/2), x_{ji}((k-1)/2),$$
$$..., x_{ji}(2), x_{ji}(1)$$

if k is odd. Replace each of the $2\binom{r}{2}$ vertices $x_{ij}(1)$ by an independent set X_{ij} of l vertices, and attach l disjoint paths of length $k-1$, respectively, at the l vertices of X_{ij}. Replace the edge $x_i x_{ij}(1)$ (respectively, $x_{ij}(2)x_{ij}(1)$)

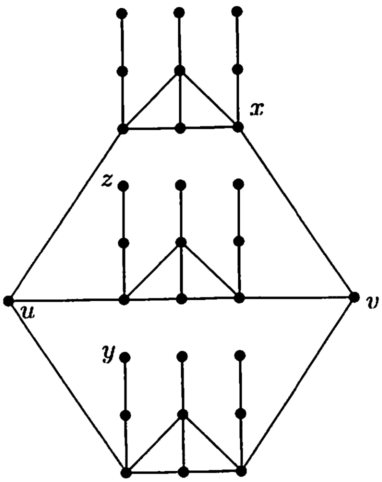

Figure 3: Graph G with $i_3(G) = 2 < i_4(G) = 3$.

by all the edges between x_i and X_{ij} (respectively, between $x_{ij}(2)$ and X_{ij}). For $r = 2$, the graph $G(2,k,l)$ is obtained from $G(3,k,l)$ by identifying x_2 and x_3 into one vertex x_2 and deleting $X_{23}, x_{23}(2), ..., x_{32}(2), X_{32}$ and the $2l$ attached paths. For examples of this construction, see the graph $G(4,5,2)$ in Figure 4 and the graph $G(2,5,2)$ in Figure 5.

Let

B_{ij} be the set of the $(k-1)l$ vertices of the l paths attached at the vertices of X_{ij}, minus X_{ij};

$A_i = \{x_i\} \cup \bigcup_{1 \le j \ne i \le r} X_{ij}$ for $1 \le i \le r$;

$C_{ij} = A_i \cup \{x_{ij}(2), x_{ij}(3)\}$ with $x_{ij}(3) = x_{ji}(2)$ in the case $k = 4$; and

$D_{ij} = X_{ij} \cup \{x_i, x_{ij}(2)\}$.

Recall that $x_{ij}(1) \ne x_{ji}(1)$, $B_{ij} \ne B_{ji}$, and so on.

Claim 1 *Every maximal $(k+1)$-packing S such that $S \cap B_{ij} \ne \emptyset$ for some B_{ij} contains at least l vertices.*

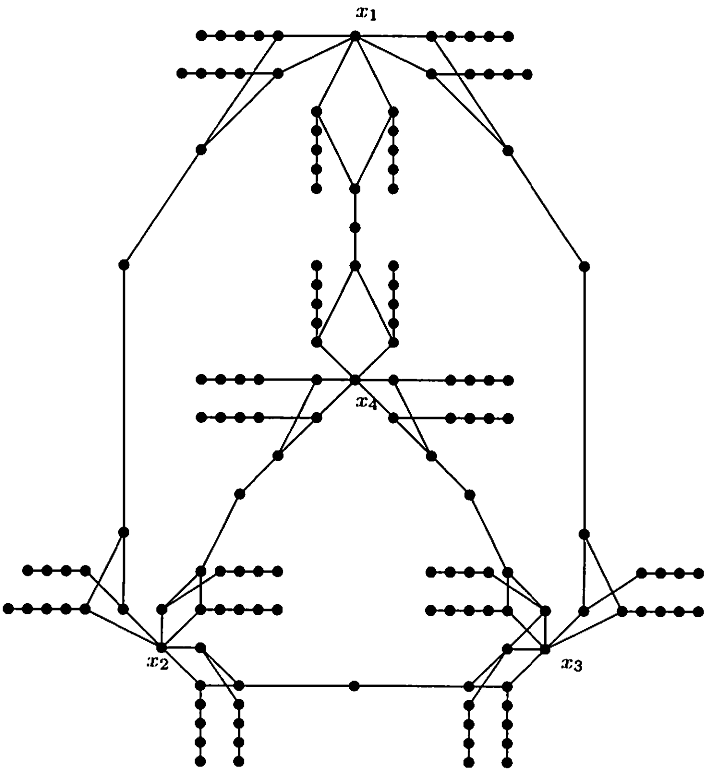

Figure 4: Graph $G(4, 5, 2)$.

Proof. If $x \in S \cap B_{ij}$, then every vertex y of $S - B_{ij}$ is at distance at least 4 from B_{ij}. Therefore, $\{x, y\}$ cannot $(k+1)$-dominate the leaves of the paths of B_{ij} not containing x. Hence, S has at least one vertex on each of these l paths.

Remark. The same result holds for a maximal k-packing, the only difference is that y is at distance at least 3 from B_{ij}. □

Let $G = G(r, k, l)$.

Claim 2 *$i_k(G) = r$ and X is the unique i_k-set of G.*

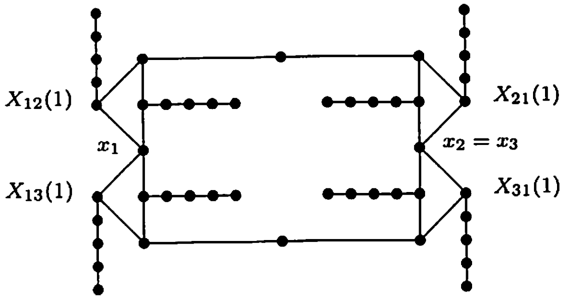

Figure 5: Graph $G(2, 5, 2)$.

Proof. The vertices x_i are pairwise at distance $k+1$, and hence they form a k-packing of G. Moreover, they k-dominate G. Hence, X is a maximal k-packing, and $i_k(G) \le |X| = r$.

Conversely, consider an i_k-set S. Then $|S| \le r < l$ and by the remark of Claim 1, $S \cap B_{ij} = \emptyset$ for all $i \neq j$. To k-dominate the leaves of B_{ij}, S contains at least one vertex in the set D_{ij} (and exactly one since diameter$(\langle D_{ij} \rangle) = 2 < k$). Since $k \ge 4$, $D_{ij} \cap D_{i'j'} = \emptyset$ for every $i' \neq i$. Moreover, $D_{ij} \cap D_{ij'} = \{x_i\}$ for all $j \neq j'$. Hence, $|S| \ge r$, and thus $i_k(G) = r$, and the only possibility to have $|S| = r$ with $|S \cap D_{ij}| \neq 0$ for every D_{ij} is to let $S = X$. □

Claim 3 $i_{k+1}(G) \ge l$.

Proof. Suppose there exists a maximal $(k+1)$-packing S with cardinality less than l. By Claim 1, $S \cap B_{ij} = \emptyset$ for all B_{ij}. To $(k+1)$-dominate the leaves of B_{ij}, S contains at least one vertex in each set C_{ij} (exactly one since diameter$(\langle C_{ij} \rangle) = 4$). If, say $S \cap C_{12} = \{x_{12}(2)\}$, then $S \cap C_{13} = \emptyset$ since $x_{12}(2)$ is at distance at most $5 \le k+1$ from any vertex of C_{13}. If, say, $S \cap C_{12} = \{x_{12}(3)\}$, then $S \cap C_{13} = \emptyset$ if $k \ge 5$; and if $k = 4$, then $S \cap C_{23} = \emptyset$ since then, $x_{13}(3)$ is at distance at most $5 = k+1$ from any vertex of C_{23} (when $r = 2$, consider C_{31} instead of C_{23}). In both cases, we get a contradiction with $S \cap C_{ij} \neq \emptyset$ for all C_{ij}. Hence, $S \cap C_{ij} = S \cap A_i$ for all $i \neq j$ and thus, $|S \cap A_i| = 1$ for all i. If S contains a vertex y of, say, X_{12}, then $S \cap A_2 = \emptyset$ since y is at distance at most $k+1$ from any vertex of A_2, a contradiction. Hence, $S \cap A_i = \{x_i\}$ for all i, again a contradiction since the $x_i's$ are pairwise at distance $k+1$.

Therefore, every maximal $(k+1)$-packing has at least l elements. □

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