Upper Domination Parameters and Edge-Critical Graphs

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Abstract

For π one of the upper domination parameters β, Γ or IR, we investigate graphs for which π decreases (π-edge-critical graphs) and graphs for π increases (π^+-edge-critical graphs) whenever an edge is added. We find characterisations of β- and Γ-edge-critical graphs and show that a graph is IR-edge-critical if and only if it is Γ-edge-critical. We also exhibit a class of Γ^+-edge-critical graphs.

We dedicate this paper to Ernie
To wish him the happiest journey
Through the rest of his life –
May the theorems be rife
And the beer flow as fast as the Smirny[1]!

1 Introduction

Unless stated otherwise we follow the notation and terminology of [11]. Specifically, $N_G(v) = \{u \in V_G : uv \in E_G\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open* and *closed neighbourhoods*, respectively, of a vertex v of a graph $G = (V_G, E_G)$. The *closed neighbourhood of a set* $S \subseteq V_G$, denoted by $N_G[S]$, is the set $\cup_{s\in S} N_G[s]$. If $s \in S$, then the *private neighbourhood of* s relative to S, denoted by $pn_G(s, S)$, is the set $N_G[s] - N_G[S - \{s\}]$. The vertices of $pn_G(s, S)$ are called the *private neighbours* of s *relative to* S. We often refer to the vertices of $pn_G(s, S) - S$ as the *external private neighbours of s relative to S*. If $pn_G(s, S) \subseteq N[v]$, where $v \in V - S$, we

[1]A river in Russia

often say that v *annihilates* s (relative to S). If confusion is unlikely we omit the subscript G from the above notation.

The lower and upper irredundant, domination and independence numbers of the graph $G = (V, E)$ are denoted by $ir(G)$, $IR(G)$, $\gamma(G)$, $\Gamma(G)$, $i(G)$ and $\beta(G)$ respectively, where in the case of the independence number we shorten the $\beta_0(G)$ used in [11] as confusion with the edge independence number $\beta_1(G)$ is unlikely. The lower independence number is of course more generally known as the independent domination number. In this paper these six parameters are called the *domination parameters*; ir, γ and i are called the *lower domination parameters*, while β, Γ and IR are referred to as the *upper domination parameters.* By a *π-set* of G, where π is a domination parameter, we mean a vertex-set of G realising $\pi(G)$, *e.g.* a *β-set* of G is a maximal independent set X of G with $|X| = \beta(G)$.

For each of the six domination parameters π, we define the graph G to be

C1 *π-critical* if $\pi(G - v) < \pi(G)$ for all $v \in V_G$,
C2 *π^+-critical* if $\pi(G - v) > \pi(G)$ for all $v \in V_G$,
C3 *π-edge-critical* if $\pi(G + e) < \pi(G)$ for all $e \in E_{\overline{G}}$,
C4 *π^+-edge-critical* if $\pi(G + e) > \pi(G)$ for all $e \in E_{\overline{G}}$,
C5 *π-ER-critical* if $\pi(G - uv) > \pi(G)$ for all $uv \in E_G$,
C6 *π^--ER-critical* if $\pi(G - uv) < \pi(G)$ for all $uv \in E_G$.

The existence of critical graphs of types C1 – C6 for each of the domination parameters is discussed in Section 2.

Graphs that are γ-critical were first studied by Brigham, Chinn and Dutton in [2], where they showed that the only 2-γ-critical graphs are $\overline{nK_2}$, $n \geq 1$. They presented some properties of γ-critical graphs and a method of constructing them, and concluded their study with some open problems, which were answered by Fulman, Hanson and MacGillivray in [7]. Graphs that are i-critical were studied by Ao in [1], in which she obtained results analogous to those in [2] and [7].

The study of γ-edge-critical graphs was initiated by Sumner and Blitch in [14], where they showed that G is 2-γ-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars. They also obtained several properties of 3-γ-edge-critical graphs. Hamiltonian properties of 3-γ-edge-critical graphs were studied in [5, 6, 16, 18, 19] and surveyed in [12] and k-γ-edge-critical graphs with $k \geq 4$ were studied in [4] and [13]. For a recent survey on γ-edge-critical graphs we refer the reader to [15].

Graphs that are i-edge-critical were studied by Ao in [1], where she obtained results analogous to those in [14]. For example, G is 2-i-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars. Since $\gamma(G) = 2$ implies $ir(G) = 2$, it is evident that the same characterisation holds for ir-edge-critical graphs.

Walikar and Acharya [17] and Ao [1] characterised graphs that are γ-ER-critical and i-ER-critical, respectively. These two classes of graphs coincide, and consist of those graphs that are disjoint unions of stars. Clearly, disjoint unions of stars are also ir-ER-critical.

Graphs that are π-critical and π^+-critical, where π is an upper domination parameter, were studied in [9]. It was shown that $K_m \times K_n$ is Γ^+-critical if $m, n \geq 5$. Also, π-ER-critical ($\pi \in \{ir, \beta, \Gamma, IR\}$) and π^--ER-critical graphs were investigated in [10], where three classes of i^--ER-critical graphs were exhibited.

In this paper we concentrate on π-edge-critical and π^+-edge-critical graphs, where π is an upper domination parameter. We find characterisations of β- and Γ-edge-critical graphs and show that a graph is IR-edge-critical if and only if it is Γ-edge-critical. (Note that this characterisation of IR-edge-critical graphs was also obtained by Dunbar, Monroe and Whitehead [3], but with a completely different proof.) We show that $K_m \times K_n$ is Γ^+-edge-critical for $m, n \geq 5$, but whether there exist IR^+-edge-critical graphs remains an open problem. The work displayed here forms part of the doctoral thesis [8].

2 Existence results

For any domination parameter π, all edgeless graphs with more than one vertex are both π-critical and π-edge-critical. If π is an upper parameter, then all complete graphs with more than one vertex are π-ER-critical and if π is a lower parameter, then all stars $K_{1,n}$ ($n \geq 1$) are π-ER-critical. This establishes the existence of π-critical, π-edge-critical and π-ER-critical graphs for all domination parameters π.

Since $\beta(G - v) \leq \beta(G)$ and $IR(G - v) \leq IR(G)$ for all $v \in V_G$ (any IR-set of $G - v$ is irredundant in G), there do not exist any β^+- or IR^+-critical graphs. Similarly, since $\gamma(G + e) \leq \gamma(G)$ and $\beta(G + e) \leq \beta(G)$ for all $e \in E_{\overline{G}}$, there are no γ^+- or β^+-edge-critical graphs. Further, since $\gamma(G - e) \geq \gamma(G)$ and $\beta(G - e) \geq \beta(G)$ for all $e \in E_G$, there are no γ^-- or β^--ER-critical graphs.

The following proposition shows that there are no π^+-critical graphs if π is a lower parameter.

Proposition 1 [8, 9] *Let π be a lower domination parameter. For any graph G with more than one vertex, $\pi(G - v) \leq \pi(G)$ for at least one $v \in V_G$.*

The following proposition implies that there are no π^+-edge-critical graphs if $\pi \in \{ir, i\}$.

Proposition 2 [8, 9] *Let $\pi \in \{ir, i\}$. For any graph G which is not complete, $\pi(G+e) \leq \pi(G)$ for at least one $e \in E_{\overline{G}}$.*

There also are no π^--ER-critical graphs for $\pi \in \{\Gamma, IR\}$, as is stated next.

Proposition 3 [8, 9] *Let $\pi \in \{\Gamma, IR\}$. For any graph G with at least one edge, $\pi(G-e) \geq \pi(G)$ for at least one $e \in E_G$.*

We summarise the existence or non-existence of the six types of criticality for the six domination parameters in Table 1.

	ir	γ	i	β	Γ	IR
π-critical	yes	yes	yes	yes	yes	yes
π^+-critical	no	no	no	no	yes	no
π-edge-critical	yes	yes	yes	yes	yes	yes
π^+-edge-critical	no	no	no	no	yes	?
π-ER-critical	yes	yes	yes	yes	yes	yes
π^--ER-critical	?	no	yes	no	no	no

Table 1. Existence of critical graphs

3 Upper parameter edge-critical graphs

In this section we find characterisations of β- and Γ-edge-critical graphs and show that a graph is IR-critical if and only if it is Γ-edge-critical. We need the following definition.

Consider a graph G with n vertices. A partition $\{S, T\}$ of V_G is called a *one-to-one perfect matching*, abbreviated to $1-1$ *p.m.*, of G if each $s \in S$ is adjacent to exactly one $t \in T$, and each $t \in T$ is adjacent to exactly one $s \in S$. If G has a $1-1$ p.m. $\{S, T\}$, then clearly G has an even number of vertices. Furthermore, S is an irredundant dominating set of G (every vertex of S has a unique private neighbour in T); hence $\Gamma(G) \geq n/2$. Note that if G has a $1-1$ p.m. $\{S, T\}$, where $\langle S \rangle \cong \langle T \rangle$, then $G = \langle S \rangle \times K_2$.

For any graph G and integer $q \geq 0$, let $G+q$ denote the graph obtained by adding q universal vertices to V_G, *i.e.*, if $q \geq 1$, then $G + q \cong G + K_q$.

Lemma 4 *Suppose π is an upper parameter and $q \geq 0$. Then G is π-edge-critical if and only if $G+q$ is π-edge-critical.*

Proof. We make three observations.
O1. $E_{\overline{G+q}} = E_{\overline{G}}$.
O2. If $e \in E_{\overline{G}}$, then $(G+e)+q = (G+q)+e$.

O3. $\pi(G+q) = \pi(G)$.

O1 and O2 are obvious. We prove O3. Obviously, G and $G+q$ are either both complete or both non-complete. In the first case $\pi(G) = \pi(G+q) = 1$. In the second case, S is an independent (dominating, irredundant) set of G if and only if S is an independent (dominating, irredundant) set of $G+q$. It follows that if S is a π-set of G, then $\pi(G+q) \leq |S| = \pi(G)$ and if S is a π-set of $G+q$, then $\pi(G) \leq |S| \leq \pi(G+q)$; hence $\pi(G+q) = \pi(G)$.

It now follows from O1 and O3 that $\pi(G+e) < \pi(G)$ for all $e \in E_{\overline{G}}$ if and only if $\pi((G+e)+q) < \pi(G+q)$ for all $e \in E_{\overline{G+q}}$. Applying O2 completes the proof of the lemma. ■

The characterisation of β-edge-critical graphs now is a simple matter.

Proposition 5 *The graph G is β-edge-critical if and only if $G = \overline{K_p} + q$, where $p \geq 2$ and $q \geq 0$.*

Proof. Suppose G is β-edge-critical and consider a β-set S of G. Clearly $\langle S \rangle = \overline{K_p}$ for some integer $p \geq 2$. Let $q = |V_G - S|$. For each $u \in V_G - S$ and $v \in V_G - \{u\}$, $uv \in E_G$, for otherwise S is an independent set of $G+uv$, in which case $\beta(G) = |S| \leq \beta(G+uv)$. It follows that $G = \overline{K_p} + q$.

Conversely, for $p \geq 2$, $\overline{K_p}$ is clearly β-edge-critical. It now follows from Lemma 4 that $\overline{K_p} + q$ is β-edge-critical for any $p \geq 2$ and $q \geq 0$. ■

We next give the characterisation of Γ- and IR-edge-critical graphs. (Also see [3].)

Theorem 6 *The following statements are equivalent for any graph G:*

(a) *G is Γ-edge-critical.*

(b) *$G = H + q$, where* (i) *$H \cong \overline{K_p}$ with $p \geq 2$, or* (ii) *the non-isolated vertices of H induce a graph M with at least six vertices and a $1-1$ perfect matching $\{S, T\}$ such that $\langle S \rangle$ and $\langle T \rangle$ are complete graphs.*

(c) *G is IR-edge-critical.*

Proof. (a) $\Rightarrow$ (b): Consider a Γ-set X of G and let Z and S be the sets of isolated and non-isolated vertices of X, respectively. Let T be the set of external private neighbours of vertices in X and let W be the set of vertices of $V_G - X$ that are adjacent to more than one vertex of X. Clearly, $X = Z \cup S$ and $\{Z, S, T, W\}$ is a partition of V_G.

If $e \in E_{\overline{G}}$ and X is an irredundant set of $G+e$, then X is a minimal dominating set of $G+e$. Therefore $\Gamma(G) = |X| \leq \Gamma(G+e)$, which contradicts the Γ-edge-criticality of G. Hence

$$\text{for all } e \in E_{\overline{G}},\ X \text{ is not an irredundant set of } G+e. \qquad (1)$$

Furthermore, if $\beta(G) = \Gamma(G)$, then G is β-edge-critical and it follows from Proposition 5 that $G = H + q$, where H is a graph satisfying (i). Henceforth, assume that

$$\beta(G) < \Gamma(G). \tag{2}$$

Suppose $x \in X$ and $|pn(x, X)| > 1$. Let $u \in X - \{x\}$ and $v \in pn(x, X)$. Then X is an irredundant set of $G + uv$, which contradicts (1). Therefore $|pn(x, X)| = 1$ for all $x \in X$. It follows that Z is the set of isolated vertices of $H = \langle Z \cup S \cup T \rangle$ and that $\{S, T\}$ is a $1-1$ p.m. of $M = \langle S \cup T \rangle$.

Suppose $uv \in E_{\overline{G}}$ and $\{u, v\} \subseteq S$ or $\{u, v\} \subseteq T$. Then X is an irredundant set of $G + uv$, which contradicts (1). Therefore $\langle S \rangle$ and $\langle T \rangle$ are complete graphs.

Suppose $|S| = 2$ and let $s \in S$, $t \in T$ with s and t nonadjacent. Then $Z \cup \{s, t\}$ is an independent set of G. Therefore

$$\Gamma(G) = |X| = |Z \cup \{s, t\}| \leq \beta(G),$$

which contradicts (2). Consequently $|S| \geq 3$ and so M has at least six vertices.

Finally, suppose $uv \in E_{\overline{G}}$ with $u \in W$ and $v \in V_G - \{u\}$. Then X is an irredundant set of $G + uv$, which contradicts (1). Therefore u is a universal vertex of G. It thus follows that $G = H + q$ with H satisfying (ii).

(b) $\Rightarrow$ (c): In the case of (i) it is clear that H is IR-edge-critical. Hence by Lemma 4, $H + q$ is IR-edge-critical. Now consider condition (ii) and let Z be the set of isolated vertices of H. Since $S \cup Z$ and $T \cup Z$ are IR-sets of H,

$$IR(G) = |S| + |Z| = |T| + |Z|.$$

Consider any $e \in E_{\overline{H}}$ and let B be an IR-set of $H + e$.

Case 1: $B \cap S \neq \emptyset$ and $B \cap T \neq \emptyset$. Let $s \in B \cap S$ and $t \in B \cap T$. Since $\langle S \rangle$ and $\langle T \rangle$ are complete graphs, $B \cap S = \{s\}$ and $B \cap T = \{t\}$, for otherwise B is not irredundant in $H + e$. Therefore $B \subseteq \{s, t\} \cup Z$ and so

$$IR(H + e) = |B| \leq 2 + |Z| < |S| + |Z| = IR(H).$$

Case 2: $B \cap S = \emptyset$ or $B \cap T = \emptyset$. Assume without loss of generality that $B \cap S = \emptyset$. Then $B \subseteq T \cup Z$. Since $\langle T \rangle$ is complete, $T \cup Z$ is not irredundant in $H + e$. Therefore

$$IR(H + e) = |B| < |T| + |Z| = IR(H).$$

This proves that H is IR-critical and by Lemma 4, $H + q$ is IR-edge-critical.

(c) $\Rightarrow$ (a): Let S be an IR-set of G and let $s \in S$. If $r \in V_G - N_G[S]$, then $sr \in E_{\overline{G}}$ and S is irredundant in $G + sr$; hence $IR(G) = |S| \leq IR(G + sr)$, which contradicts the IR-edge-criticality of G. Therefore S

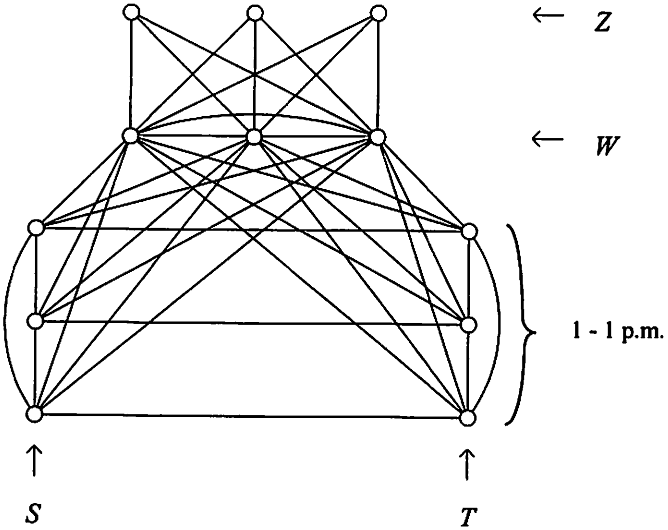

Figure 1: A Γ-edge-critical graph

is a dominating irredundant set of G and thus $\Gamma(G) = IR(G)$. It follows that for all $e \in E_{\overline{G}}$,

$$\Gamma(G+e) \leq IR(G+e) < IR(G) = \Gamma(G). \quad \blacksquare$$

Figure 1 shows the Γ-edge-critical graph $G = H + q$ with $|Z| = |S| = |T| = |W| = 3$.

4 Γ^+- and IR^+-edge-critical graphs

In this section we show that $K_m \times K_n$ is Γ^+-edge-critical if $m, n \geq 5$. Whether there exist IR^+-edge-critical graphs or not remains an open problem.

The *product* $G_1 \times G_2$ of two graphs G_1 and G_2 has vertex-set $V_{G_1} \times V_{G_2}$ and two vertices $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are adjacent in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2v_2 \in E_{G_2}$, or $u_2 = v_2$ and $u_1v_1 \in E_{G_1}$.

Let

$$V = \{v_{ij} : i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$$

and

$$E = \{\{v_{ij}, v_{kl}\} : v_{ij}, v_{kl} \in V,\ i = k \text{ and } j \neq l, \text{ or } j = l \text{ and } i \neq k\}$$

be the vertex and edge sets of the graph $K_m \times K_n$, respectively. Furthermore, let

$$X_i = \{v_{ik} : k = 1, 2, ..., n\}$$

for each $i = 1, 2, ..., m$ and

$$Y_j = \{v_{kj} : k = 1, 2, ..., m\}$$

for each $j = 1, 2, ..., n$. Note that $\langle X_i \rangle \cong K_n$ for each $i = 1, 2, ..., m$ and $\langle Y_j \rangle \cong K_m$ for each $j = 1, 2, ..., n$. We first determine the domination parameters of $K_m \times K_n$.

Theorem 7 *Let $G = K_m \times K_n$ for $n \geq m \geq 2$. Then*

$$ir(G) = \gamma(G) = i(G) = \beta(G) = m,$$

$$\Gamma(G) = n$$

and

$$IR(G) = \begin{cases} n & \text{if } m \leq 4 \\ m + n - 4 & \text{if } m \geq 4. \end{cases}$$

Proof. Consider any maximal independent set S of G. Since S is independent, $|X_i \cap S| \leq 1$ for all $i = 1, 2, ..., m$. Therefore

$$|S| = \sum_{i=1}^{m} |X_i \cap S| \leq m.$$

Since S is dominating, $X_i \cap S \neq \emptyset$ for all $i = 1, 2, ..., m$ or $Y_j \cap S \neq \emptyset$ for all $j = 1, 2, ..., n$. Therefore

$$|S| = \sum_{i=1}^{m} |X_i \cap S| \geq m.$$

or

$$|S| = \sum_{j=1}^{n} |Y_j \cap S| \geq n \geq m.$$

It follows that $|S| = m$ for every maximal independent set S of G ; hence

$$i(G) = \beta(G) = m.$$

Consider any minimal dominating set S of G. Again, since S is dominating, $X_i \cap S \neq \emptyset$ for all i or $Y_j \cap S \neq \emptyset$ for all j. If $X_i \cap S \neq \emptyset$ for all i, then choose $x_i \in X_i \cap S$ for each i. Since $\{x_1, x_2, ..., x_m\}$ is a dominating subset of the minimal dominating set S, it follows that $S = \{x_1, x_2, ..., x_m\}$; hence $|S| = m$. Similarly, if $Y_j \cap S \neq \emptyset$ for all j, then $|S| = n$. It follows that

$|S| \in \{m, n\}$ for every minimal dominating set S of G. Furthermore, Y_1 and X_1 are minimal dominating sets with cardinalities m and n, respectively; hence

$$\gamma(G) = m \quad \text{and} \quad \Gamma(G) = n.$$

To complete the proof, we show that $n \leq |S| \leq m + n - 4$ for any maximal irredundant set S of G that is not dominating, and that there exists one with cardinality $m+n-4$ if $m \geq 4$. To be more precise, we show that for each $c \in \{n, n+1, ..., m+n-4\}$ there exists a non-dominating maximal irredundant set with cardinality c.

Consider any maximal irredundant set S of G that is not dominating. Assume without loss of generality that $X_m \cap S = \emptyset$ and $Y_n \cap S = \emptyset$, *i.e.*, the vertex v_{mn} is not dominated by S.

If $|X_i \cap S| \leq 1$ for all i, let $T = S \cup \{v_{m1}\}$. Since $Y_n \cap S = \emptyset$ and $v_{m1} \notin Y_n$, we see that $v_{in} \in pn(v_{ij}, T)$ for every $v_{ij} \in T$. Therefore T is an irredundant superset of the maximal irredundant set S, which is impossible. Hence assume, without loss of generality, that $|X_{m-1} \cap S| > 1$ and, similarly, that $|Y_{n-1} \cap S| > 1$.

Let r be the number of sets X_i for which $|X_i \cap S| = 1$ and s the number of sets Y_j for which $|Y_j \cap S| = 1$. It follows that $r \leq m-2$ and $s \leq n-2$. Assume without loss of generality that

$$\begin{aligned} |X_i \cap S| &= 1 \quad \text{for all} \quad i = 1, 2, ..., r \\ |X_i \cap S| &\neq 1 \quad \text{for all} \quad i = r+1, ..., m \\ |Y_j \cap S| &= 1 \quad \text{for all} \quad j = 1, 2, ..., s \\ |Y_j \cap S| &\neq 1 \quad \text{for all} \quad j = s+1, ..., n. \end{aligned}$$

Since S is irredundant, $|X_i \cap S| = 1$ or $|Y_j \cap S| = 1$ for every $v_{ij} \in S$. Therefore

$$\bigcup_{i=r+1}^{m} (X_i \cap S) \subseteq \bigcup_{j=1}^{s} (Y_j \cap S).$$

These unions are disjoint, so

$$|S| - r = \sum_{i=r+1}^{m} |X_i \cap S| \leq \sum_{j=1}^{s} |Y_j \cap S| = s.$$

Hence

$$|S| \leq r + s \leq (m-2) + (n-2) = m + n - 4.$$

Furthermore, suppose $Y_k \cap S = \emptyset$ for $k \neq n$. Then v_{mk} is not dominated by S and therefore $v_{(m-1)k}$ annihilates some $v_{ij} \in S$. If $|Y_j \cap S| = 1$, then $v_{mj} \in pn(v_{ij}, S)$; thus v_{mj} is adjacent to $v_{(m-1)k}$ and so $j = k$, which is impossible since $Y_k \cap S = \emptyset$. Consequently, $|X_i \cap S| = 1$ and it follows

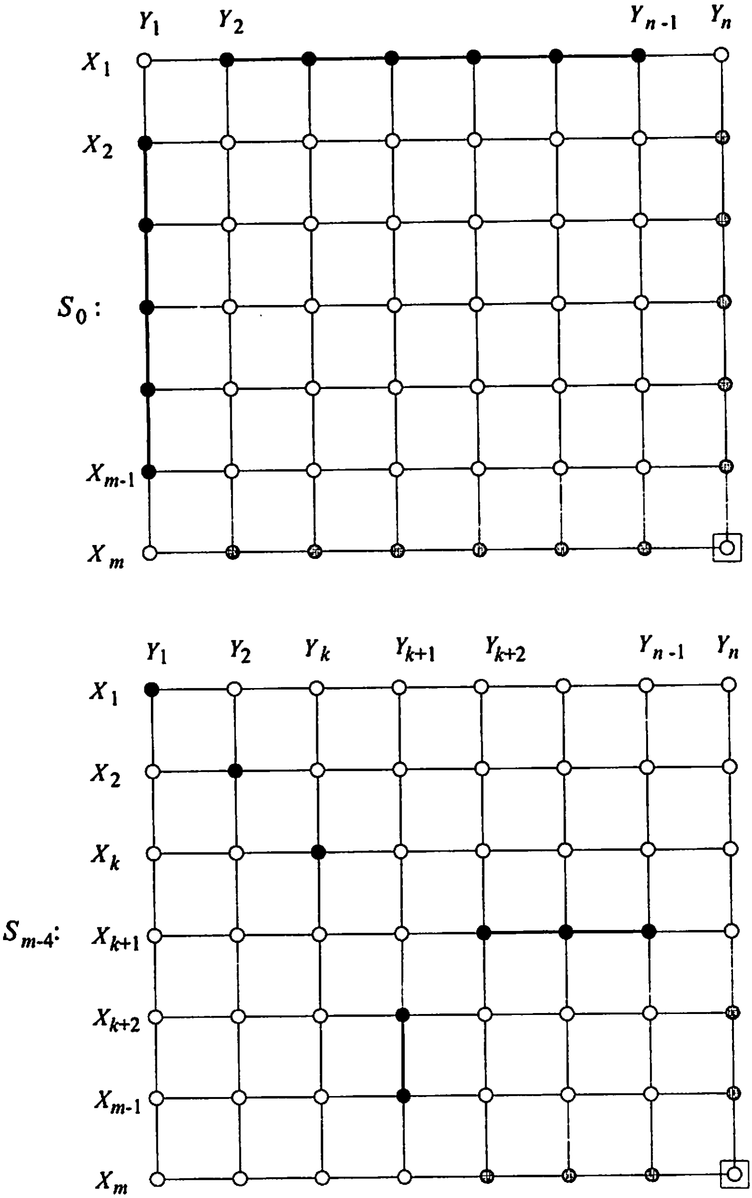

Figure 2: Irredundant sets (black vertices) of $K_m \times K_n$

that $v_{in} \in pn(v_{ij}, S)$. This implies that v_{in} is adjacent to $v_{(m-1)k}$; hence $i = m-1$, which is impossible since $|X_{m-1} \cap S| > 1$.

It now follows that $|Y_j \cap S| > 1$ for all $j = s+1, ..., n-1$ and therefore

$$\begin{aligned} |S| &= \sum_{j=1}^{n} |Y_j \cap S| \\ &= \sum_{j=1}^{s} |Y_j \cap S| + \sum_{j=s+1}^{n-1} |Y_j \cap S| + |Y_n \cap S| \\ &\geq s + 2(n-s-1) + 0 \\ &= 2n - s - 2 \\ &\geq n \quad \text{since} \quad s \leq n-2. \end{aligned}$$

For each $k \in \{0, 1, ..., m-4\}$, let

$$\begin{aligned} S_k &= \{v_{11}, v_{22}, ..., v_{kk}\} \cup \{v_{(k+1)(k+2)}, ..., v_{(k+1)(n-1)}\} \\ &\quad \cup \{v_{(k+2)(k+1)}, ..., v_{(m-1)(k+1)}\}. \end{aligned}$$

See Figure 2 (but note that not all the edges are shown) for S_0 and S_{m-4} and note that k is the number of isolated vertices of S_k. (Black dots denote the vertices of the set, grey dots the external private neighbours and white squares the vertices not dominated by the set.) For each $k \in \{0, 1, ..., m-4\}$, S_k is a maximal irredundant set which is not dominating and $|S_k| = m + n - 4 - k$. Therefore the non-dominating maximal irredundant sets have cardinalities $n, n+1, ..., m+n-4$. ■

Proposition 8 *For any $m, n \geq 5$, the graph $K_m \times K_n$ is Γ^+-edge-critical.*

Proof. Let $G = K_m \times K_n$ where $n \geq m \geq 5$ and consider any $uv \in E_{\overline{G}}$. Since $\overline{G}$ is edge-transitive, assume without loss of generality that $u = v_{12}$ and $v = v_{mn}$. Now S_0 in the proof of Theorem 7 (see Figure 2) is a minimal dominating set of $G + uv$. Therefore $\Gamma(G) = n < m + n - 4 = |S_0| \leq \Gamma(G + uv)$. ■

Although we have not found an IR^+-edge-critical graph, we show that for such a graph G, if it exists, the upper irredundance number increases by exactly one whenever an edge is added to G.

Proposition 9 *Suppose G is IR^+-edge-critical. For every $uv \in E_{\overline{G}}$ and every IR-set S of $G+uv$, (without loss of generality) $u \in S$, $pn_{G+uv}(u, S) = \{v\}$ and $S - \{u\}$ is an IR-set of G.*

Proof. S is not an irredundant set of G, for otherwise $IR(G+uv) = |S| \leq IR(G)$. Thus we may assume without losing generality that $u \in S$ and $pn_{G+uv}(u, S) = \{v\}$. Furthermore, $S - \{u\}$ is an irredundant set of G. Therefore

$$IR(G+uv) - 1 = |S| - 1 \leq IR(G) < IR(G+uv)$$

and hence $S - \{u\}$ is an IR-set of G. ∎

Corollary 10 *If G is IR^+-edge-critical, then $IR(G+uv) = IR(G) + 1$ for each $uv \in E_{\overline{G}}$.*

5 Open Problems

We conclude with a brief list of unsolved problems.

1. Are there ir-critical graphs which are not γ-critical, or γ-critical graphs which are not ir-critical?

2. Are there ir-edge-critical graphs which are not γ-edge-critical, or γ-edge-critical graphs which are not ir-edge-critical?

3. $K_m \times K_n$ with $m, n \geq 5$ are the only known Γ^+-critical graphs (see [9]) and also the only known Γ^+-edge-critical graphs. Are these the only Γ^+-critical or Γ^+-edge-critical graphs? (This would be very surprising.) Do these two types of criticality coincide? Do they imply vertex-transitivity? (This also seems unlikely.)

4. Determine properties of Γ^+-edge-critical graphs.

5. Are there any IR^+-edge-critical or ir^--ER-critical graphs?

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