Fall Colorings of Graphs

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Abstract

We introduce a new class of colorings of graphs and define and study two new graph coloring parameters. A *coloring* of a graph $G = (V, E)$ is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertices of G into independent sets V_i, or *color classes*. A vertex $v \in V_i$ is called *colorful* if it is adjacent to at least one vertex in every color class V_j, $i \neq j$. A *fall coloring* is a coloring in which every vertex is colorful. If a graph G has a fall coloring, we define the *fall chromatic number* (*fall achromatic number*) of G, denoted $\chi_f(G)$, $(\psi_f(G))$ to equal the minimum (maximum) order of a fall coloring of G, respectively. In this paper we relate fall colorings to other colorings of graphs and to independent dominating sets in graphs.

Dedicated to Ernie Cockayne: 60 and going strong

Keywords: colorings, chromatic number, achromatic number, idomatic number, Grundy number.

1 Introduction

Unless stated otherwise, we follow the notation and terminology in [9]. A *k-coloring* of a graph $G = (V, E)$ with vertex set V and edge set E, is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V(G)$ into independent sets

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V_i, each of which is called a *color class*. The minimum integer k for which a graph G has a k-coloring is called the *chromatic number* of G and is denoted $\chi(G)$.

A coloring is called *complete* if for every $1 \leq i < j \leq k$, there is a vertex $u \in V_i$ and a vertex $v \in V_j$ such that u is adjacent to v. The maximum order of a complete coloring of a graph G is called the *achromatic number* and is denoted $\psi(G)$. First defined and studied by Harary and Hedetniemi [5] in 1970, the achromatic number provides an upper bound for the chromatic number of a graph, i.e. for any graph G, $\chi(G) \leq \psi(G)$. Notice that every coloring of a graph G with $\chi(G)$ colors must be a complete coloring.

A *Grundy k-coloring* of a graph G is a k-coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ such that for each color class V_i, $1 < i \leq k$, every vertex $v \in V_i$ is adjacent to at least one vertex in color class V_j for every $j < i$. The *Grundy number* $\Gamma(G)$ is the maximum integer k for which G has a Grundy k-coloring. First introduced by Christen and Selkow [1] in 1979, and later in [6], the Grundy number can be seen to satisfy the following inequality for every graph G: $\chi(G) \leq \Gamma(G) \leq \psi(G)$. Notice further that every Grundy coloring of a graph G is a complete coloring. One can also observe that the minimum integer k for which a graph G has a Grundy k-coloring is always equal to the chromatic number, $\chi(G)$, but there exist graphs G for which $\Gamma(G) < \psi(G)$.

Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a coloring of a graph G and let $v \in V_i$. We say that vertex v is a *Grundy* vertex if it is adjacent to at least one vertex in every set V_j, $1 \leq j < i$. A coloring Π is called a *partial Grundy coloring* if every color class V_i contains at least one Grundy vertex, for every $2 \leq i \leq k$. The *partial Grundy number* of a graph G, denoted $\partial\Gamma(G)$, is the maximum order of a partial Grundy coloring of G. Note that since every Grundy coloring of a graph is also a partial Grundy coloring, it follows that: $\Gamma(G) \leq \partial\Gamma(G)$.

The next type of coloring which we will consider was introduced in 1997 by Irving and Manlove [7]. We define this type of coloring using terminology that is newly defined here. We say that a vertex $v \in V_i$ in a coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ is *colorful* if it is adjacent to at least one vertex in each color class V_j, $i \neq j$. A color class V_i is called *colorful* if it contains at least one colorful vertex, and a coloring Π is called *colorful* if every color class is colorful. The maximum order of a colorful coloring of a graph G is called the *b-chromatic number*, and is denoted $\phi(G)$. It is easy to see from the definition, that every colorful coloring is a complete coloring. Therefore, for every graph G, $\chi(G) \leq \phi(G) \leq \psi(G)$. One can observe that the minimum order of a colorful coloring of a graph G is always equal to the chromatic number $\chi(G)$. It is also worth noting, as observed in [7], that no inequality relation holds between $\phi(G)$ and $\Gamma(G)$.

A set $S \subseteq V$ is called a *dominating set* if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in a graph G is called the *domination number* of G, and is denoted $\gamma(G)$. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. The minimum cardinality of an independent dominating set in a graph is called the *independent domination number* of G and is denoted $i(G)$.

2 Fall Colorings of Graphs

We say that a colorful k-coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ is a *fall coloring* if every vertex $v \in V$ is colorful. The minimum integer k for which a graph G has a fall k-coloring is called the *fall chromatic number* and is denoted $\chi_f(G)$. The maximum integer k for which a graph G has a fall k-coloring is called the *fall achromatic number*, $\psi_f(G)$. Notice that it follows immediately from the definition of $\psi_f(G)$ that for any graph G, $\psi_f(G) \leq \delta(G) + 1$, where $\delta(G)$ equals the minimum degree of a vertex in G.

It is easy to see that every bipartite graph without isolated vertices has a fall 2-coloring. In fact, every 2-coloring of a bipartite graph G without isolated vertices is a fall 2-coloring, since every vertex colored 1 is adjacent to at least one vertex colored 2, and every vertex colored 2 is adjacent to at least one vertex colored 1. Notice in this case that each color class is both an independent set and a dominating set.

Theorem 1 *The cycle C_m has a fall 3-coloring if and only if $m = 0 \bmod 3$.*

Proof. It is trivial to see that the cycle C_{3m} has a fall 3-coloring; simply color the vertices, in consecutive order, $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$. Assume therefore that $m = 3k + 1$ or $3k + 2$. If C_m has a fall 3-coloring, then each color class must be an independent dominating set. But it is well-known that $\gamma(C_m) = i(C_m) = \lceil m/3 \rceil$, and since $m = 3k + 1$ or $3k + 2$, then $3\lceil m/3 \rceil > m$. Thus, at least one color class cannot be a dominating set, i.e. neither C_{3k+1} nor C_{3k+2} has fall 3-coloring. □

Not every graph has a fall k-coloring, for any k. For example, the 5−cycle C_5 has chromatic number 3 but does not have a fall k-coloring for any k. If a graph G does not have a fall k-coloring, for any k, we say that $\chi_f(G) = 0$. Later in this paper, we will consider the algorithmic complexity of answering the following question:

> FALL COLORING
> INSTANCE: Graph $G = (V, E)$.
> QUESTION: Does G have a fall k-coloring for any positive integer k?

We must first point out that there is a very close connection between fall colorings of a graph G and the existence of disjoint independent dominating sets in G. The *domatic number* $d(G)$ is the maximum order of a partition of $V(G)$ into dominating sets. The domatic number was introduced in 1977 by Cockayne and Hedetniemi [3], who observed from a well-known theorem of Ore [8], that since the complement $V - S$ of every minimal dominating set S in a graph G without isolated vertices is a dominating set, it must be the case that for graphs without isolated vertices, $d(G) \geq 2$.

A partition of the vertex set $V(G)$ into independent dominating sets is called an *idomatic partition* of G, also introduced in [3]. The maximum order of an idomatic partition of G is called the *idomatic number* $id(G)$. If a graph G has no idomatic partition, then we say that $id(G) = 0$. Notice that any 2-coloring of a connected bipartite graph into color classes V_1 and V_2 is, in fact, a partition of G into two disjoint independent dominating sets. Therefore, every connected bipartite graph G has $id(G) \geq 2$.

Proposition 1 *For every graph G, $\psi_f(G) = id(G)$.*

Proof. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a fall coloring of a graph $G = (V, E)$ with $k = \psi_f(G)$ colors. Assume that $k > 0$. Since Π is a coloring, we know that every color class V_i is an independent set. We must show that V_i is also a dominating set, i.e. for every vertex $v \in V - V_i$ there exists a vertex $u \in V_i$ such that u is adjacent to v. But since every vertex in a fall coloring is colorful, v must be adjacent to at least one vertex in each color class, and in particular v must be adjacent to at least one vertex in V_i. Therefore, V_i is a dominating set, Π is an idomatic partition, and $\psi_f(G) \leq id(G)$.

Conversely, let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ into $k = id(G)$ independent dominating sets. We must show that Π is a fall k-coloring, i.e. we must show that every vertex $v \in V(G)$ is colorful. Let $v \in V_i$ and let V_j, $j \neq i$, be an arbitrary set of Π. Since V_j is an independent dominating set of G, there must be at least one vertex in V_j which is adjacent to v. Therefore, v is colorful and Π is a fall k-coloring. Therefore, $id(G) \leq \psi_f(G)$. □

In 1976 Cockayne and Hedetniemi [2] first studied disjoint independent dominating sets in graphs. They defined a parameter $b(G)$ to equal the maximum number of disjoint independent dominating sets which can be found in a graph G. Further, they called a graph G *indominable* if the vertex set $V(G)$ can be partitioned (completely) into disjoint independent dominating sets. In the terminology of the current paper, we would say that a graph has a fall coloring if and only if it is indominable.

In [2] the following classes of graphs are shown to have fall colorings:

1. complete graphs K_n have (only) fall n-colorings;

2. connected bipartite graphs, including all $m \times n$ grid graphs, trees, and n-cubes, have fall 2-colorings;

3. complete k-partite graphs have (only) fall k-colorings;

4. cycles of the form C_{2n} have fall 2-colorings and cycles of the form C_{3n} have fall 3-colorings;

5. uniquely k-colorable graphs, including maximal outerplanar graphs, have fall k-colorings;

6. k-trees have fall $(k+1)$-colorings;

7. graphs of the form $K_{m,m}-$ 1-factor have (only) fall 2- and m-colorings;

8. domatic-critical graphs, i.e. graphs for which $d(G) = k$, but for which $d(G-e) < k$ for every edge $e \in E$, have fall k-colorings;

9. regular graphs for which $d(G) = k = \delta(G) + 1$ have fall k-colorings;

10. the complements of graphs G of order n, which have no triangles and have a 1-factor, have fall $n/2$-colorings;

11. the complement of the Petersen graph has a fall 5-coloring, although the Petersen graph has no fall k-coloring for any k;

12. the join $G + H$ of two graphs, each of which has a fall coloring, say of orders k and l, has a fall $(k+l)$-coloring.

3 The Fall Chromatic and Achromatic Numbers

In this section we assume that every graph considered has a fall coloring, and therefore $\chi_f(G)$ and $\psi_f(G)$ are well-defined. The inequality chain below follows immediately from the definitions of these types of colorings, and the next five observations:

1. every fall coloring is a colorful coloring;

2. every fall coloring is a Grundy coloring;

3. every colorful coloring is a complete coloring;

4. every Grundy coloring is a partial Grundy coloring;

5. every partial Grundy coloring is a complete coloring.

Proposition 2 *If a graph G has a fall coloring, then*

$$\chi(G) \leq \chi_f(G) \leq \psi_f(G) \leq \left\{ \begin{array}{c} \phi(G) \\ \Gamma(G) \end{array} \right\} \leq \partial\Gamma(G) \leq \psi(G)$$

Given this new inequality chain, it is appropriate to consider, for each consecutive pair of parameters, whether there exist classes of graphs for which these parameters are different, or for which the difference in value between these two parameters can be arbitrarily large.

For arbitrary graphs G and H, we define the *Cartesian product* of G and H to be the graph $G\Box H$ with vertices $\{(u,v)|u \in G, v \in H\}$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G\Box H$ if and only if one of the following is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

1. $\chi(G) \leq \chi_f(G)$. The graph $G = C_4\Box C_5$ is a graph for which $\chi(G) = 3$, yet $\chi_f(G) = 4$. Furthermore, $\chi(C_5\Box C_5) = 3$, while $\chi_f(C_5\Box C_5) = 5$. To date, we have not been able to construct a graph G for which $\chi_f(G) - \chi(G) \geq 3$.

2. $\chi_f(G) \leq \psi_f(G)$. It can be inferred from an observation in [2] that the difference between $\chi_f(G)$ and $\psi_f(G)$ can be arbitrarily large. Consider only the graph $H = K_{n,n}$ − 1-factor. For this graph $\chi_f(H) = 2$ while $\psi_f(H) = n$.

3. $\psi_f(G) \leq \Gamma(G)$. Note that for any graph G, $\psi_f(G) \leq \delta(G) + 1$, since every vertex must be colorful. Therefore, for any tree T, $\psi_f(T) = 2$. However, it is well-known that for any positive integer k, there exists a tree $T(k)$ for which $\Gamma(T(k)) = k$. Such a tree $T(k)$ can be constructed as follows: (i) let $T(1)$ be the tree on a single vertex, (ii) let $T(k)$ be the tree constructed from $T(k-1)$ by adding a pendant vertex to every vertex of $T(k-1)$. The tree $T(k)$ is known as the *binomial* tree B_k; it has 2^{k-1} vertices and satisfies: $\Gamma(T(k)) = k$.

4. $\psi_f(G) \leq \phi(G)$. Let P_n be a path with vertices labeled $v_1, v_2, \ldots, v_n$. To each vertex v_i, $1 \leq i \leq n$, add $n-2$ pendant vertices. It can be seen that the graph H so constructed satisfies: $\psi_f(H) = 2$, while $\phi(H) = n$.

5. $\Gamma(G) \leq \partial\Gamma(G)$. Let P_n be a path with vertices labeled $v_1, v_2, \ldots, v_n$. To each vertex v_i, $3 \leq i \leq n$, add $i-2$ pendant vertices. It can be seen that the graph H so constructed satisfies: $\Gamma(H) = 4$, while $\partial\Gamma(H) = n$.

6. $\phi(G) \leq \partial\Gamma(G)$. It is easy to see that if a graph G has $\phi(G) = k+1$, then it must have at least $k+1$ vertices of degree at least k. The tree H in item 5 above can be seen to satisfy

$$\phi(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

whereas $\partial\Gamma(G) = n$. Thus, the difference between $\phi(G)$ and $\partial\Gamma(G)$ can be arbitrarily large.

7. $\partial\Gamma(G) \leq \psi(G)$. It is easy to see that for any graph G, $\partial\Gamma(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. Therefore, for a path P_n, $\partial\Gamma(P_n) \leq 3$. However, it is well-known that for any positive integer k, $\psi(P_n) \geq k$, for n sufficiently large. In particular, if k is odd and $n = k(k-1)/2$, then $\psi(P_n) = k$. Thus, the difference between $\partial\Gamma(G)$ and $\psi(G)$ can be arbitrarily large.

4 Fall Colorings of Cartesian Products

In this section we study fall colorings of the families of Cartesian products: $P_m \square P_n$, $C_m \square P_n$ and $C_m \square C_n$. Since these graphs G satisfy: $2 \leq \delta(G) \leq 4$ and $\Delta(G) \leq 4$, they can only have fall colorings of orders 2, 3, 4 and 5.

Theorem 2 *For any positive integers m and n, the Cartesian product $P_m \square P_n$ has a fall 2-coloring, but does not have a fall k-coloring for any integer $k \geq 3$.*

Proof. Let G be the Cartesian product for $P_m \square P_n$ and label the vertices of G as $a_{1,1}, a_{1,2}, \ldots, a_{1,n}, a_{2,1}, a_{2,2}, \ldots, a_{2,n}, \ldots, a_{m,1}, a_{m,2}, \ldots, a_{m,n}$, where $a_{i,j} = (v_i, v_j)$.

Since $P_n \square P_m$ is bipartite, it clearly has a fall 2-coloring. Further, since $\delta(G) = 2$, it has no fall k-coloring for $k > 3$. Suppose a fall 3-coloring exists. Without loss of generality, we assume that the vertices $a_{1,1}, a_{1,2}$, and $a_{2,1}$ are assigned colors 1, 2, and 3, respectively. Then $a_{2,2}$ must have color 1 in order to have a proper coloring. Next, $a_{3,1}$ must have color 2, in order for $a_{2,1}$ to be a colorful vertex. Again $a_{3,2}$ must be colored 3 to retain a proper coloring, and then $a_{4,1}$ must be colored 1 in order for $a_{3,1}$ to be a colorful vertex. Continuing in this way, $a_{m-1,2} \neq a_{m,1}$ and so $a_{m,2}$ must be assigned the same color as $a_{m-1,1}$ to maintain a proper coloring. But this means that $a_{m,1}$ cannot be a colorful vertex. □

Proposition 3 *The graph $C_m \Box P_n$ has a fall 2-coloring if and only if $m \geq 4$ is even.*

Proof. Clearly no odd cycle can be 2-colored. Thus, if $C_m \Box P_n$ has a fall 2-coloring, then m must be even. But every graph $C_m \Box P_n$, for m even, is a connected, bipartite graph, and hence has a (unique) fall 2-coloring. □

Proposition 4 *The graph $C_{3m} \Box P_n$ has a fall 3-coloring for all $n \geq 1$.*

Proof. By Theorem 1 we know that C_{3m} has a fall 3-coloring. Figure 1 shows that a fall 3-coloring of C_{3m} can easily be extended to a fall 3-coloring of $C_{3m} \Box P_n$. □

1	2	3	1	...
2	3	1	2	...
3	1	2	3	...
1	2	3	1	...
2	3	1	2	...
3	1	2	3	...
⋮	⋮	⋮	⋮	

Figure 1: Fall 3-coloring of $C_{3m} \Box P_n$

Proposition 5 *The graphs $C_{3k+1} \Box P_n$ and $C_{3k+2} \Box P_n$ do not have a fall 3-coloring for any value of $k \geq 1$.*

Proof. Let us assume that the graph $C_{3k+1} \Box P_n$ consists of n columns of $3k+1$ vertices, each column defining a cycle C_{3k+1}. In every 3-coloring of the cycle C_{3k+1} there must exist two vertices which are colored the same and have a common neighbor (i.e. are distance two apart). This gives rise to the situation in Figure 2.

1	2	3	1	...
2	3	1	2	...
1	2	3	1	...
⋮	⋮	⋮	⋮	

Figure 2: No fall 3-coloring of $C_{3k+1} \Box P_n$

Without loss of generality, we can assume that vertices in the first column labeled v_{11} and v_{13} are colored the same, say color 1. We can further assume that vertex v_{12} is colored 2. But this implies that vertex v_{22} must be colored 3, else vertex v_{12} is not colorful.

Now in the second column, it must be the case that vertices v_{21} and v_{23} are both colored 2. This means that vertex v_{22} is not (yet) colorful, since it is not adjacent with a vertex colored 1. If there is a third column, then vertex v_{32} must be colored 1, else vertex v_{22} is not colorful. But this implies that vertices v_{31} and v_{33} must both be colored 3, and this means that vertex v_{32} is not (yet) colorful, since it is not adjacent with a vertex colored 2.

This in turn means that if there were a fourth column, then vertex v_{42} would have to be colored 2 so that vertex v_{32} can be colorful. It follows from this argument that by continuing to color $C_{3k+1} \square P_n$ in this way, vertex v_{n2} will not be colorful. Therefore, $C_{3k+1} \square P_n$ does not have a fall 3-coloring.

The same argument applies to the graphs $C_{3k+2} \square P_n$, since every 3-coloring of the cycle C_{3k+2} must result in two vertices having the same color which are distance two apart. This means that any attempt to produce a fall 3-coloring will be forced to consider the same cases as above. □

Proposition 6 *The graph $C_m \square C_n$ has a fall 2-coloring if and only if both $m \geq 4$ and $n \geq 4$ are even.*

Proof. Clearly $C_m \square C_n$ has a fall 2-coloring if and only if it is bipartite, and $C_m \square C_n$ is bipartite if and only if both m and n are even, for $m, n \geq 4$. □

Theorem 3 *The graph $C_m \square C_n$ has a fall 3-coloring if and only if either $m = 0 \bmod 3$ or $n = 0 \bmod 3$.*

Proof. We first show that the graph $C_{3m} \square C_n$ has a fall 3-coloring for all $n \geq 1$. This result can be proved by a simple additions to Figure 1 which is used to show that $C_{3m} \square P_n$ has a fall 3-coloring, for all $n \geq 1$. We need to point out that in the graph $C_{3m} \square C_n$ there are edges connecting each vertex $v_{i,1}$ in the first column to the vertex $v_{i,n}$ in the last column.

In order to show that $C_{3m} \square C_n$ has fall 3-colorings for every value of $n \geq 1$, we will consider three cases: $n = 0 \bmod 3$, $n = 1 \bmod 3$ and $n = 2 \bmod 3$.

If $n = 0 \bmod 3$, i.e. $n = 3k$ for some positive integer k, we repeat the first three columns of Figure 1 k times.

If $n = 1 \bmod 3$, i.e. $n = 3k + 1$ for some positive integer k, we repeat the first three columns of Figure 1 k times and then add a copy of column 2.

Finally, if $n = 2 \bmod 3$, we repeat the first three columns of Figure 1 k times and then add a copy of column 2 and column 3.

We next show that the graphs $C_{3m+1} \square C_n$ and $C_{3m+2} \square C_n$ have a fall 3-coloring if and only if $n = 0 \bmod 3$. Figure 3 illustrates a fall 3-coloring of $C_7 \square C_6$.

1	2	3	1	2	3
2	3	1	2	3	1
1	2	3	1	2	3
2	3	1	2	3	1
1	2	3	1	2	3
2	3	1	2	3	1
3	1	2	3	1	2

Figure 3: Fall 3-coloring of $C_7 \square C_6$

Clearly this pattern generalizes to $C_{3m+1} \square C_{3n}$. Let the first row of Figure 3 be called A, the second row called B, and the last row called C. A fall 3-coloring of $C_{3m+1} \square C_{3n}$ is obtained by an alternating sequence of A's and B's of length $3m$ followed by one C.

A fall 3-coloring of $C_{3m+2} \square C_{3n}$ is obtained by an alternating sequence of A's and B's of length $3m+1$ followed by one C.

The fact that these graphs do not have fall 3-colorings if $n \neq 0 \bmod 3$ is a consequence of the observation that in the first column there will have to be two vertices at distance two apart which have the same color. For example, the first and third vertex in the first column in Figure 3. In any possible fall 3-coloring of such a graph the colors in the first three rows are uniquely forced (up to isomorphism). Therefore, terminating the coloring at any column other than those which are multiples of three will fail to produce a fall 3-coloring. □

Proposition 7 *The graphs $C_{4m} \square P_{2n}$ and $C_{4m} \square C_{2n}$ have a fall 4-coloring for all $m \geq 1$ and $n \geq 2$.*

Proof. Figure 4 shows that a fall 4-coloring of $C_4 \square P_6$ or $C_4 \square C_6$ can easily be extended to a fall 4-coloring of $C_{4m} \square P_{2n}$ or $C_{4m} \square C_{2n}$. □

1	3	1	3	1	3	...
2	4	2	4	2	4	...
3	1	3	1	3	1	...
4	2	4	2	4	2	...
⋮	⋮	⋮	⋮	⋮	⋮	

Figure 4: Fall 4-coloring of $C_{4m} \square C_{2n}$

Proposition 8 *The graph $C_m \square C_n$ has a fall 5-coloring if and only if $m = 0 \bmod 5$ and $n = 0 \bmod 5$.*

Proof. Figure 5 shows that a fall 5-coloring of $C_5 \Box C_5$ can easily be extended to a fall 5-coloring of $C_{5m} \Box C_{5n}$ by repeating this 5×5 pattern in each direction.

$$\begin{array}{cccccc} 1 & 3 & 5 & 2 & 4 & \dots \\ 2 & 4 & 1 & 3 & 5 & \dots \\ 3 & 5 & 2 & 4 & 1 & \dots \\ 4 & 1 & 3 & 5 & 2 & \dots \\ 5 & 2 & 4 & 1 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Figure 5: Fall 5-coloring of $C_{5m} \Box C_{5n}$

Suppose, conversely, that a graph $G = C_m \Box C_n$ has a fall 5-coloring. Since G is a 4-regular graph (i.e. every vertex is adjacent to exactly four vertices), in any fall 5-coloring of G, every vertex must be adjacent to exactly one vertex of each color, other than its own color. Equivalently, no vertex can be adjacent to two vertices having the same color.

But since each of the five color classes is an independent dominating set, each color class is an *efficient dominating set*, i.e. for any two vertices of the same color, say $u, v \in V_i$, $N[u] \cap N[v] = \emptyset$, and furthermore, $\bigcup_{v_i \in V_i} N[v_i] = V$. This implies that $|V| = mn = 0 \bmod 5$, since $|N[v_i]| = 5$, for every vertex $v_i \in V(G)$. It follows that $mn = 0 \bmod 5$ if and only if either $m = 0 \bmod 5$ or $n = 0 \bmod 5$. In fact, we can show that if G has a fall 5-coloring, then both $m = 0 \bmod 5$ and $n = 0 \bmod 5$. The proof of this involves a case analysis very similar to that used in the proof of Theorem 3 and is omitted. □

We conclude this section with the following result about fall colorings of Cartesian products.

Theorem 4 *If a graph G has a fall s-coloring and a graph H has a fall r-coloring, for $s \geq r$, then the Cartesian product $G \Box H$ has a fall s-coloring.*

Proof. Let $\{V_1, V_2, \dots, V_s\}$ be a fall s-coloring of G and let $\{U_1, U_2, \dots, U_r\}$ be a fall r-coloring of H. Consider subscripts (on the first coordinate) modulo s and define the subsets $W_1, W_2, \dots, W_s$ of the vertex set of graph $G \Box H$ as follows:

$$W_j = \cup_{i=1}^{r} \left(V_{j+(i-1)} \times U_i \right).$$

The partition $\{W_1, W_2, \dots, W_s\}$ is a fall s-coloring of the graph $G \Box H$.
□

5 Fall Colorings of Categorical Products

For arbitrary graphs G and H, we define the *categorical product* of G and H to be the graph $G \times H$ with vertices $\{(u,v)|u \in G, v \in H\}$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G \times H$ if and only if u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. In this section we show that the categorical product can be used to specify graphs having fall k-colorings for every integer k in an arbitrarily specified set $S = \{n_1, n_2, \ldots, n_m\}$ of positive integers.

Theorem 5 *If $r \geq 2$ and $s \geq 2$ are distinct positive integers, then the categorical product $K_r \times K_s$ has a fall r-coloring and a fall s-coloring. Furthermore, if n is a positive integer different from both r and s, then $K_r \times K_s$ does not have a fall n-coloring.*

Proof. Let $V(K_r) = \{a_1, a_2, \ldots, a_r\}$ and let $V(K_s) = \{b_1, b_2, \ldots, b_s\}$. Note that the vertices $x = (a_p, b_q)$ and $y = (a_m, b_n)$ are adjacent if and only if $p \neq m$ and $q \neq n$. For $1 \leq i \leq r$, let $A_i = \{(a_i, b_j)|1 \leq j \leq s\}$, and for $1 \leq j \leq s$, let $B_j = \{(a_i, b_j)|1 \leq i \leq r\}$.

Assume M is an independent dominating set of $K_r \times K_s$. If $|M \cap A_i| \geq 2$ for some i, then since A_i is independent it follows that $M = A_i$. Similarly, if $|M \cap B_j| \geq 2$ for some j, then $M = B_j$. But if (a_p, b_q) and (a_m, b_n) belong to M, then since M is independent, it follows that $p = m$ or $q = n$. Therefore, either $M = A_i$ for some i or $M = B_j$ for some j. That is, $K_r \times K_s$ has a fall r-coloring and a fall s-coloring and no others. □

It is interesting to note that Theorem 5 does not generalize to categorical products of three or more complete graphs. For example, the categorical product $K_2 \times K_3 \times K_4$ has fall 2−, 3− and 4-colorings, but also has a fall 6-coloring, as follows. Label the vertices of $K_2 \times K_3 \times K_4$ lexicographically: $1 = (1,1,1), 2 = (1,1,2), 3 = (1,1,3), 4 = (1,1,4), 5 = (1,2,1)$, etc. A fall 6-coloring is given by the following partition: $\{1,6,14,17\}$, $\{5,10,18,21\}$, $\{2,9,13,22\}$, $\{3,8,16,19\}$, $\{7,12,20,23\}$, $\{4,11,15,24\}$.

Therefore, given an arbitrary set S of positive integers, an analogous proof shows that the categorical product of complete graphs K_a for every integer $a \in S$, has fall a-colorings, for every integer $a \in S$. However, this categorical product may have other (perhaps unwanted) fall k-colorings.

Notice also that such categorical products have rather large order. For example, if you seek a graph having fall k-colorings for every positive integer k, $2 \leq k \leq m$, then the categorical product $G = K_2 \times K_3 \times \ldots \times K_m$ will have fall colorings for all of these values of k. However, this graph will have $m!$ vertices, and will be regular of degree $(m-1)!$

There is a another method of constructing graphs having fall k-colorings for consecutive values of k, which does not require an exponential number of vertices. Consider the following example. Let $G = K_{3,3,3,3,3,3}$, the complete 6-partite graph, each component of which has three vertices. We assume that the vertices of G are partitioned naturally into six independent sets, $V_1, V_2, V_3, V_4, V_5, V_6$, each of size three. The subgraph induced by any two of these sets defines a complete bipartite graph $K_{3,3}$.

Now from the subgraph $K_{3,3}$ defined by the pair V_1, V_2 remove a maximum matching, i.e. three independent edges. Do the same for the subgraphs defined by V_3 and V_4 and by V_5 and V_6. Let $H_{3,3,3,3,3,3} = H_{6(3)}$ be the resulting graph.

Proposition 9 *The graph $H_{6(3)}$ has fall 6−, 7−, 8− and 9-colorings. Furthermore, $H_{6(3)}$ has no fall k-colorings for any other values of k.*

Proof. It is easy to see that the natural partition $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ defines a fall 6-coloring. Let the vertices of $V_1 = \{v_{11}, v_{12}, v_{13}\}$, and $V_2 = \{v_{21}, v_{22}, v_{23}\}$. We assume that edges $(v_{11}, v_{21}), (v_{12}, v_{22})$ and (v_{13}, v_{23}) have been removed between V_1 and V_2.

A fall 7-coloring is defined by the partition:
$\{\{v_{11}, v_{21}\}, \{v_{12}, v_{22}\}, \{v_{13}, v_{23}\}, V_3, V_4, V_5, V_6\}$.

Similarly, a fall 8-coloring is defined by the partition:
$\{\{v_{11}, v_{21}\}, \{v_{12}, v_{22}\}, \{v_{13}, v_{23}\}, \{v_{31}, v_{41}\}, \{v_{32}, v_{42}\}, \{v_{33}, v_{43}\}, V_5, V_6\}$.

A fall 9-coloring can be obtained from the fall 8-coloring above by splitting sets V_5 and V_6 into sets $\{v_{51}, v_{61}\}, \{v_{52}, v_{62}\}$, and $\{v_{53}, v_{63}\}$.

It is easy to see that the graph $H_{6(3)}$ does not have any fall k-colorings for any other values of k. This follows immediately from the observation that the only independent dominating sets in $H_{6(3)}$ are of the form V_i or $\{v_{(2i-1)j}, v_{(2i)j}\}$. □

If, for example, we want five consecutive values of fall colorings, then we start with the graph $K_{8(3)}$. We then pairwise remove matchings between all consecutive sets of vertices V_{2i-1} and V_{2i}. The resulting graph $H_{8(3)}$ has fall k-colorings for $k = 8, 9, 10, 11, 12$.

In this way, if we seek a graph having fall k-colorings for m consecutive values of k, we construct the graph $H_{(2m-2)(3)}$ having $2m - 2$ vertex sets of size three. In this case, the number of vertices grows linearly with the value of m, rather than exponentially using categorical products.

6 Complexity Results

In this section we address the complexity issues connected with fall colorings. First we define three distinct, but related, decision problems.

K-FALL-COLORING (KFC)
INSTANCE: A graph $G = (V, E)$
QUESTION: Does G have a fall K-coloring?

FALL COLORING (FC)
INSTANCE: A graph $G = (V, E)$
QUESTION: Does G have a fall coloring?

FALL K-COLORING (FKC)
INSTANCE: A graph $G = (V, E)$, an integer K
QUESTION: Does G have a fall K-coloring?

Before addressing these three problems we will prove the following lemma.

Lemma 1 *If KFC is NP-complete for some integer K, then $(K+1)FC$ is NP-complete.*

Proof. For a graph G, let G^* denote the graph obtained from G by adding one new vertex, and adjoining it to all other vertices. It is easy to see that G has a fall K-coloring if and only if G^* has a fall $(K+1)$-coloring. Thus the map $G \to G^*$ is a transformation from KFC to $(K+1)$FC. □

Theorem 6 *Problem KFC is NP-complete for each $K \geq 3$.*

Proof. By Lemma 1 it is sufficient to show $3FC$ is NP-complete. To see that $3FC \in NP$, let $f : V \to \{1,2,3\}$ be any 3-coloring of the vertices of a graph. It can be verified in polynomial time whether f defines a fall 3-coloring. To show that $3FC$ is an NP-complete problem, we will establish a polynomial transformation from the NP-complete problem NOT-ALL-EQUAL-3SAT [4]. An instance I of NOT-ALL-EQUAL-3SAT consists of a set $X = \{X_1, X_2, \ldots, X_k\}$ of variables and a set $C = \{C_1, C_2, \ldots, C_j\}$ of clauses, each of which has three literals from the set X. A solution to an instance I of NOT-ALL-EQUAL-3SAT consists of an assignment of truth values to the variables in X such that each clause in C has at least one true literal and at least one false literal.

We transform each instance I of NOT-ALL-EQUAL-3SAT to an instance G_I of KFC as shown in Figure 6.

Initialize G_I with k disjoint copies of the graph K_3 and label the vertices of the ith copy as $\{y_i, x_i, \overline{x_i}\}$. Label the vertices of another copy of K_3 with $\{a, b, c\}$, and add edges by_i and cy_i for $i = 1, 2, \ldots, k$. Corresponding to each clause $C_i \in C$ add a single vertex c_i to the graph G_I. Finally, join each vertex c_i to the three vertices in clause C_i. Clearly the construction can be accomplished in polynomial time. All that remains is to show that I has a satisfying truth assignment if and only if G_I has a fall 3-coloring. Assume first that I has a satisfying truth assignment $f : X \to \{T, F\}$. Color the vertices $\{a, b, c\}$ in G_I with the colors $\{1, 2, 3\}$, respectively. Next color every vertex y_i and every vertex c_i with the color 1. Finally, for $i = 1, 2, \ldots, k$, if $f(X_i) = T$, color the associated vertex x_i with the color

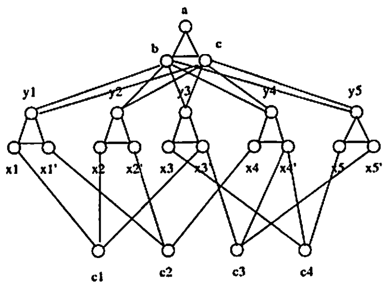

Figure 6: The graph G_I obtained from NOT-ALL-EQUAL-3SAT instance

2. Otherwise color $\overline{x_i}$ with the color 2. Color all remaining vertices 3. Clearly this coloring is a proper coloring, and every vertex in a triangle is a colorful vertex. The only vertices to check are the vertices c_i. But since f is a satisfying truth assignment, every clause C_i has at least one true literal, and its literals are not all equal, so it has at least one false literal. Thus every vertex c_i is colorful. Assume next that G_I has a fall 3-coloring. Then without loss of generality, we may assume the vertex a is colored 1. Since a is colorful, b and c must have the colors 2 and 3. This forces each y_i to have the color 1. Since each color class is an independent set, this means the vertices x_i and $\overline{x_i}$ must have colors 2 or 3. Thus each c_i must have color 1. We define a function $f : X \to \{T, F\}$ by saying $f(X_i) = T$ iff x_i has color 2, and otherwise $f(X_i) = F$. Since the coloring is a fall 3-coloring, each vertex c_i is adjacent to at least one vertex with color 2 and at least one vertex with color 3. Thus the function f defined here is a satisfying truth assignment for NOT-ALL-EQUAL-3SAT. □

Theorem 7 *Problems FC and FKC are NP-complete.*

Proof. In the mapping $I \to G_I$ used in Theorem 6, G_I has a vertex of degree 2. Thus G_I has a fall coloring if and only if it has a fall 3-coloring. Hence this mapping is also a transformation from NOT-ALL-EQUAL-3SAT to FC, and so FC is NP-complete. Finally, problem FKC is NP-complete since the mapping $I \to (G_I, 3)$ is a transformation from NOT-ALL-EQUAL-3-SAT into FKC. □

7 Summary and Open Problems

As the following list of open problems suggests, we have only scratched the surface of the subject of fall colorings of graphs. In addition to the questions below, it can be seen that fall colorings of graphs can be found in the study of latin squares, colorings of Queens graphs and Rooks graphs, and the study of block designs.

1. Can the difference between $\chi(G)$ and $\chi_f(G)$ be arbitrarily large?

2. Does there exist a graph G for which the following strict inequality chain holds:

$$\chi(G) < \chi_f(G) < \psi_f(G) < \phi(G) < \partial\Gamma(G) < \psi(G)?$$

3. What is the smallest graph with $\delta = k$ which does not have a fall coloring? For $\delta = 1$ the smallest graph with no fall coloring is the graph K_3 with a pendant vertex. For $\delta = 2$ the smallest graph with no fall coloring is C_5. For $\delta = 3$ the smallest graphs with no fall colorings are the wheel of order 6 and the graph which is the complement of the graph $K_3 \cup P_3$. Finally, it can be shown that the complement of the cycle C_7 is the unique smallest graph with $\delta = 4$ having no fall coloring.

4. Do colorful colorings interpolate? i.e. if a graph G has colorful colorings of orders i and k, does it also have colorful colorings of every order j, $i \leq j \leq k$? Recall that we have shown that fall colorings do not interpolate.

5. What fall colorings do the n-cubes have? We note that the 3-cube Q_3 has a fall 2-coloring and a fall 4-coloring, but does not have a fall 3-coloring.

6. Settle the NP-completeness of the decision problem associated with the parameter $\psi_{fs}(G)$.

7. Is $\psi_{fs}(G) \leq \phi(G)$?

8. Under what conditions does the categorical product of a set of complete graphs K_r, for every integer r in some specified set S, have a fall k-coloring for some integer k not in S?

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