On the minimum influence of a maximal packing

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Dedicated to Professor E. J. Cockayne, seminal author of many topics in graph theory, on the occasion of his sixtieth birthday.

Abstract. A vertex set $S \subseteq V(G)$ is a perfect code or efficient dominating set for a graph G if each vertex of G is dominated by S exactly once. Not every graph has an efficient dominating set, and the efficient domination number $F(G)$ is the maximum number of vertices one can dominate given that no vertex is dominated more than once. That is, $F(G)$ is the maximum influence of a packing $S \subseteq V(G)$. In this paper we begin the study of $LF(G)$, the lower efficient domination number of G, which is the minimum number of vertices dominated by a maximal packing. We show that the decision problem associated with deciding if $LF(G) \leq K$ is an NP-complete problem. The principle result is a characterization of trees T where $LF(T) = F(T)$.

1 Introduction

For a graph $G = (V, E)$, a vertex $v \in V(G)$ is considered to dominate itself and each vertex in its open neighborhood, $N(v) = \{w \in V(G) : vw \in E(G)\}$. That is, v dominates each vertex in its closed neighborhood, $N[v] = \{v\} \cup N(v)$. For $S \subseteq V(G)$, we have $N[S] = \cup_{s \in S} N[s]$, and S is a dominating set if $N[S] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set, and the *upper domination number* $\Gamma(G)$ (see Cockayne and Hedetniemi [4] and Slater [13]) is the maximum number of vertices in a minimal dominating set. Similarly, the *independence number* $\beta(G)$ is the maximum number of vertices in an independent set in $V(G)$, and the *lower independence number* $i(G)$ is the minimum number of vertices in a maximal independent set. These and many other such pairs of parameters are considered in Haynes, Hedetniemi, and Slater [10, 11]. In particular, Cockayne, Hedetniemi, and Miller [5] consider irredundance and upper irredundance numbers $ir(G)$ and $IR(G)$, and we have the following famous inequality chain.

Theorem 1 (Cockayne, et al. [5]) *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

As described in [10], Plummer [12] introduced the now well-studied class of well-covered graphs, graphs G with $i(G) = \beta(G)$. Finbow, Hartnell, and Nowakowski [6] began the study of graphs with $\gamma(G) = \Gamma(G)$.

We can consider how many times a vertex w is dominated by a set S, that is, $|N[w] \cap S|$. Note that a vertex v of degree $deg(v)$ dominates $1 + deg(v) = |N[v]|$ vertices. As in Grinstead and Slater [8], the *influence* of vertex set S is $I(S) = \sum_{s \in S}(1 + deg(s)) = \sum_{w \in V(G)} |N[w] \cap S|$, which is the amount of domination done by S. For example, the *redundance* of G is $R(G) = \min\{I(S) : N[S] = V(G)\}$, the minimum amount of domination done by a dominating set. If we require that $|N[w] \cap S| \leq 1$ for all $w \in V(G)$, then for any two vertices u and v in S, we must have the distance $d(u, v) \geq 3$, that is, S is a *packing*. The maximum order of a packing is denoted $\rho(G)$. Note that S is a packing if and only if $I(S) = |N[S]|$. As in Biggs [3], a packing S with $N[S] = V(G)$ (equivalently, with $I(S) = |V(G)|$) is a *perfect code*, a set that dominates every vertex exactly once. As in Bange, Barkauskas, and Slater [1, 2], not every graph has a perfect code, so the *efficient domination number* of G is $F(G) = \max\{I(S) : S \text{ is a packing}\}$, the maximum number of vertices one can dominate given that no vertex is dominated more than once. When $F(G) = |V(G)|$, a perfect code is also called an efficient dominating set. In a manner similar to defining Γ and i, in [14] several "upper" and "lower" parameters were defined. In particular, the lower efficient domination number $LF(G) = \min\{I(S) : S \text{ is a maximal packing }\}$ is the minimum number of vertices dominated by a maximal packing, that is, the minimum influence of a maximal packing. For the tree T in Figure 1, $F(T) = I(\{v_1, v_4\}) = I(\{v_3\}) = 4$ and $LF(T) = I(\{v_2\}) = 3$.

Thus, $LF(G)$ and $F(G)$ are the minimum and maximum influence of maximal packings, and $R(G)$ and $UR(G)$ are the minimum and maximum influence of minimal dominating sets. One of the parameters considered in Grinstead and Slater [9] is $RI(G)$, where $RI(G)$ and $URI(G)$ are the minimum and maximum influence of independent dominating sets. A referee of this paper has observed that the influence of maximal irredundant sets might also be of interest. We let $I^-_{irr}(G)$ and $I^+_{irr}(G)$ denote the minimum and maximum influence, respectively, of maximal irredundant sets. Using similar notation, one could write $I^-_{dom}(G) = R(G)$, $I^+_{dom}(G) = UR(G)$, $I^-_{ind}(G) = RI(G)$, and $I^+_{ind}(G) = URI(G)$, and we have the following proposition.

Proposition 2 *For any graph G,*

$$I^-_{irr}(G) \le I^-_{dom}(G) \le I^-_{ind}(G) \le I^+_{ind}(G) \le I^+_{dom}(G) \le I^+_{irr}(G)$$

As noted in Grinstead and Slater [8], $|V(G)| \le R(G) = I^-_{dom}(G)$, but it can be easily verified that $|V(G)|$ and $I^-_{irr}(G)$ are incomparable.

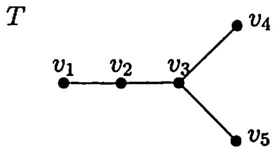

Figure 1: $LF(T) = 3$ and $F(T) = 4$.

In this paper we begin the study of the parameter LF. In the next section we present some examples and show that deciding, for an arbitrary graph G and positive integer K, if $LF(G) \le K$ is an NP-complete problem. In Section 3, trees T with $LF(T) = F(T)$ are characterized.

2 Examples and Complexity

Note that for the star $K_{1,n-1}$ we have $LF(K_{1,n-1}) = 2$ and $F(K_{1,n-1}) = n$. In general, if the packing number $\rho(G) = 1$, and $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum vertex degrees, then $LF(G) = 1 + \delta(G)$ and $F(G) = 1+\Delta(G)$. More generally, we have the following observation, where $L\rho(G)$ is the minimum cardinality of a maximal packing.

Proposition 3 *For any graph G,* $(1 + \delta(G)) \cdot L\rho(G) \le LF(G) \le F(G) \le (1+\Delta(G)) \cdot \rho(G)$. *If G is regular of degree r, then* $LF(G) = (1+r) \cdot L\rho(G)$ *and* $F(G) = (1 + r) \cdot \rho(G)$.

Proposition 4 *If a connected graph G with* $|V(G)| \ge 3$ *has* $\delta(G) = 1$, *then* $LF(G) < |V(G)|$.

Proof. Assume $deg(v) = 1$, and let w be a vertex with $d(v,w) = 2$. Simply note that any maximal packing S with $w \in S$ has $v \notin N[S]$ and $I(S) \le |V(G)| - 1$. □

Corollary 5 *Any tree T with* $|V(T)| \ge 3$ *has* $LF(T) < |V(T)|$.

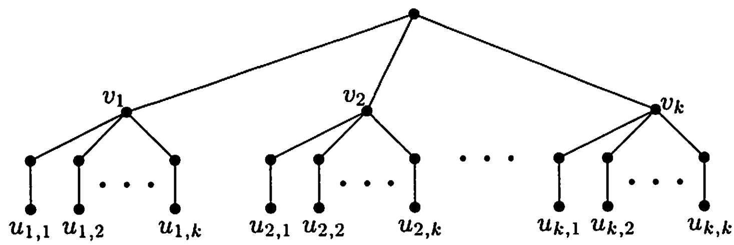

Figure 2: Tree T_n on $n = 2k^2 + k + 1$ vertices.

For the tree T_n on $n = 2k^2 + k + 1$ vertices in Figure 2, if $S = \{v_1, u_{2,1}, ..., u_{2,k}, u_{3,1}, ..., u_{3,k}, ..., u_{k,1}, ..., u_{k,k}\}$, then S is a maximal packing with $I(S) = 2k^2 - k + 2$. It can, in fact, be seen that $LF(T_{2k^2+k+1}) = 2k^2 - k + 2$. Note that $\lim_{k\to\infty} LF(T_{2k^2+k+1})/|V(T_{2k^2+k+1})| = \lim_{k\to\infty}(2k^2 - k + 2)/(2k^2 + k + 1) = 1$.

As a final example, if M is a perfect matching in $K_{t,t}$, then $LF(K_{t,t} - M) = F(K_{t,t} - M) = 2t = |V(K_{t,t})|$.

In order to show that the problem of deciding whether an arbitrary graph has a maximal packing with influence no greater that some positive integer K is an NP-complete problem, we present a transformation from the following restricted $3SAT$ problem which is known to be NP-complete [7].

3-SATISFIABILITY(3SAT)

INSTANCE: Set $U = \{u_1, u_2, ..., u_N\}$ of variables and collection C of clauses such that each clause $c \in C$ has $|c| = 3$, and for each $u_i \in U$ at most five clauses in C contain u_i or $\overline{u_i}$.

QUESTION: Is there a satisfying truth assignment for C?

LOWER EFFICIENT DOMINATION NUMBER(LED)

INSTANCE: Graph $G = (V, E)$ and positive integer $H \leq |V|$.

QUESTION: Does G have a maximal packing S with $I(S) \leq H$? (That is, is $LF(G) \leq H$?)

Theorem 6 *Problem LED is NP-complete.*

Proof. Since one can easily verify whether a vertex set $S \subseteq V(G)$ is a maximal packing and whether $I(S) \leq K$, it is clear that $LED \in NP$. To see that LED is NP-complete, we reduce 3SAT to it.

Given an instance of 3SAT, with N variables and M clauses, construct the graph G illustrated in Figure 3 in which each clause vertex c_j is connected by paths of length two to the three vertices corresponding to the literals contained in c_j. Because at most five clauses contain u_i or $\overline{u_i}$, for $1 \le i \le N$, we have $M \le 5N/3$, and there are $3M \le 5N$ vertices of degree two on the paths of length two. We let $H = 2 + 8N$. Each $N(v_i)$ contains u_i,$\overline{u_i}$, and H vertices that form a complete subgraph K_H, and each c_j has degree $3 + H$. (For example, in Figure 3 we have $c_1 = \{u_1, \overline{u_2}, u_3\}$, $c_2 = \{\overline{u_1}, \overline{u_2}, \overline{u_3}\}$,...) Graph G has $8N^2 + 13N + 4 + 4M \le 8N^2 + 59N/3 + 4$ vertices, and it can be constructed in polynomial time.

Note that if S is a maximal packing, then each $S \cap N[v_i] \neq \emptyset$ and $S \cap N[w] \neq \emptyset$. Consequently, if S is a maximal packing either with $x \notin S$ or $S \cap \{u_i, \overline{u_i}\} = \emptyset$ for at least one value of i with $1 \le i \le N$, then the influence $I(S) > H$. On the other hand, if $|S| = N + 1$ with $x \in S$ and $S \cap \{u_i, \overline{u_i}\} \neq \emptyset$ for $1 \le i \le N$, then $I(S) \le 2 + 3N + M \le 2 + 8N = H$. Hence, there is a maximal packing S with $I(S) \le H$ if and only if there is a maximal packing S with $|S| = N + 1$ which contains one element from each $\{u_i, \overline{u_i}\}$ and vertex x. It follows that the instance of 3SAT has a satisfying truth assignment if and only if the corresponding graph G has a maximal packing S with $I(S) \le H$. □

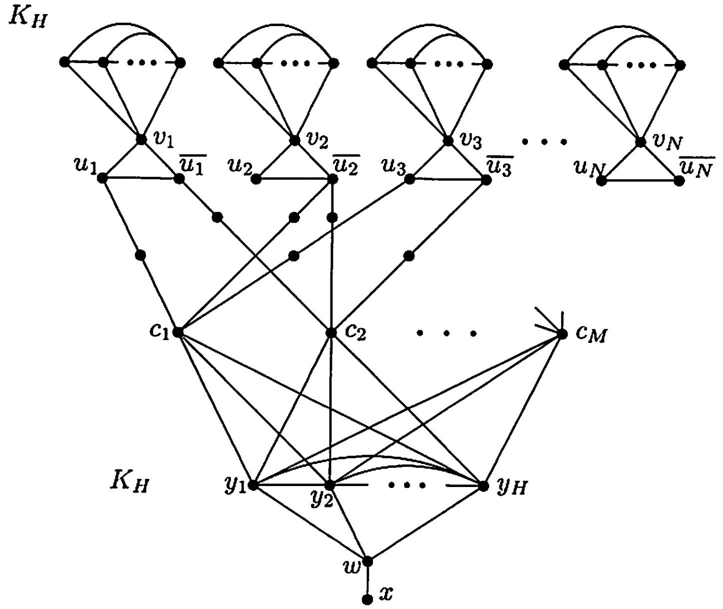

Figure 3: Graph G.

3 Characterization of Trees T where $LF(T) = F(T)$

We shall make use of the following notation. Any vertex adjacent to an endvertex v is called the *support vertex* of v or just a *support vertex*. For any subset $S \subseteq V(G)$, recall that $I(S) = \sum_{s \in S}(1+deg(s))$ denotes the *influence of* S, that is, the total amount of domination done by S. Therefore given a graph G, $LF(G) = F(G)$ if and only if for all $S \subseteq V(G)$ and $S' \subseteq V(G)$ such that S and S' are maximal packings we have $I(S) = I(S')$.

In order to characterize the trees in which $LF(T) = F(T)$, we define the family $\mathcal{T}$ where a tree $T \in \mathcal{T}$ if and only if the following two properties hold:

(1) If $x \in V(T)$ such that x is a support vertex and j is the number of vertices adjacent to x that are also support vertices, then $deg(x) = 2j + 1$.

(2) If $x \in V(T)$ such that x is not a support vertex and j is the number of vertices adjacent to x that are support vertices, then $deg(x) = 2j - 1$.

See Figure 4 for an example of a tree in $\mathcal{T}$. Note, for example, that u is a support vertex adjacent to three other support vertices and $deg(u) = 2 \cdot 3 + 1 = 7$ and v is a non-support vertex adjacent to two support vertices and $deg(v) = 2 \cdot 2 - 1 = 3$.

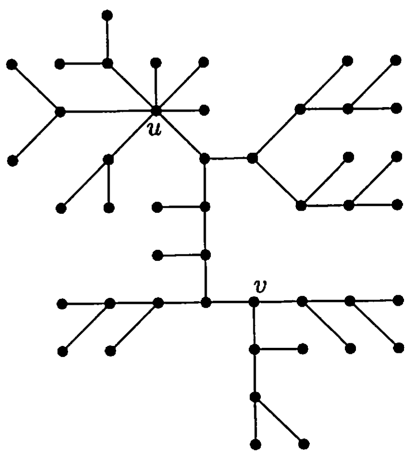

Figure 4: Tree $T \in \mathcal{T}$.

We will prove that $T \in \mathcal{T}$ if and only if $LF(T) = F(T)$.

Lemma 7 *Suppose that T is a tree where $LF(T) = F(T)$ and $diam(T) \geq 4$. If x is a support vertex and is adjacent to j other support vertices, then $deg(x) = 2j + 1$.*

Proof. Let T be rooted at x. Let w be an endvertex adjacent to x. For $1 \leq i \leq t$, let $u_i \in N(x)$ where $deg(u_i) \geq 2$ and u_i is not a support vertex. Let u_{i,r_i} be the children of u_i where $1 \leq r_i \leq deg(u_i) - 1$ and let u'_{i,r_i} be a child of u_{i,r_i}. For $1 \leq h \leq j$, let $v_h \in N(x)$ where $deg(v_h) \geq 2$ and v_h is a support vertex to some endvertex s_h. If v_h has any grandchildren, let v_{h,q_h} be the children of v_h such that $deg(v_{h,q_h}) \geq 2$ and let v'_{h,q_h} be a child of v_{h,q_h}. Let $LF(T) = F(T) = K$.
Case 1. Let $t \geq 1$ and $j = 0$, that is, suppose that no child of x is a support vertex. Choose any maximal packing P such that $x \in P$ and such that for all u_{i,r_i}, P contains a child of u_{i,r_i}, namely u'_{i,r_i}. In other words, choose P so that P contains x and is as tight around x as possible. Define $P' = P - \{x\}$. Since P is a maximal packing, the influence $I(P) = I(x) + I(P') = K$. However $P - \{x\} \cup \{w\}$ must also be a maximal packing, thus $I(w) + I(P') = K$. This implies that $I(x) = I(w) = 2$ and so $deg(x) = 1$, a contradiction. Thus there must exist at least one vertex adjacent to x that is a support vertex.

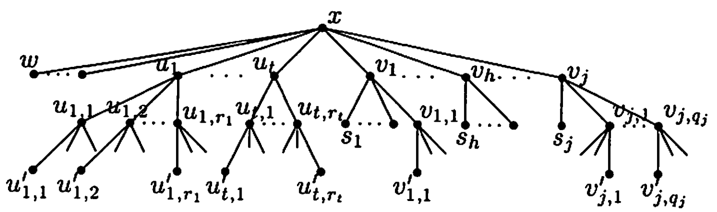

Figure 5: Tree T rooted at x where x is a support vertex.

Case 2. Let $j \geq 1$. Choose P to be a maximal packing that contains x and, for all u_{i,r_i}, P contains a child of u_{i,r_i}, namely u'_{i,r_i}. Also for all v_h such that v_h has a grandchild, then for all v_{h,q_h}, P contains a child of v_{h,q_h}, namely v'_{h,q_h}. Again, this is a maximal packing chosen in such a way that P contains x and is as tight around x as possible. Define $P' = P - \{x\}$. Since P is a maximal packing, $I(x) + I(P') = K$. However $P' \cup \{w\} \cup \{s_1, s_2, ..., s_j\}$ is also a maximal packing. Thus $K = I(P') + I(w) + I(s_1) + ... + I(s_j)$. Hence $I(x) = I(w) + I(s_1) + ... + I(s_j) = 2 + 2j$. Therefore $deg(x) = 2j + 1$. □

For ease of notation, define $V_{u,v}$ to be the set of all vertices in the tree that are closer to u than to v.

Lemma 8 *Suppose that T is a tree where $LF(T) = F(T)$ and $diam(T) \geq 4$. If x is not a support vertex and is adjacent to j support vertices, then $deg(x) = 2j - 1$.*

Proof. Let T be rooted at x. Observe that if x is an endvertex, then x is adjacent to one support vertex and $deg(x) = 2j - 1$ holds. Therefore, assume that $deg(x) \geq 2$.

For $1 \leq i \leq t$, let $u_i \in N(x)$ where $deg(u_i) \geq 2$ and u_i is not a support vertex. Let u_{i,r_i} be the children of u_i where $1 \leq r_i \leq deg(u_i) - 1$, and let u'_{i,r_i} be a child of u_{i,r_i}. For $1 \leq h \leq j$, let $v_h \in N(x)$ where $deg(v_h) \geq 2$ and v_h is a support vertex to some endvertex s_h. If v_h has any grandchildren, let v_{h,q_h} be the children of v_h such that $deg(v_{h,q_h}) \geq 2$, and let v'_{h,q_h} be a child of v_{h,q_h}. Thus $deg(x) = t + j = k \geq 2$. Let $LF(T) = F(T) = K$.

Case 1. Let $t = k$ and $j = 0$, that is, assume that all children of x are not support vertices. There exists a maximal packing P such that $x \in P$ and for all u_{i,r_i}, we have $u'_{i,r_i} \in P$. Let $P_{u'_i} = P \cap V_{u_i,x}$. Hence $P = \{x\} \cup \bigcup_{i=1}^{t} P_{u'_i}$, and, since P is a maximal packing, $I(x) + \sum_{i=1}^{t} I(P_{u'_i}) = K$. However, there exists a maximal packing P' of T such that for each u_i exactly one child of u_i is in P', namely $u_{i,1}$, and for each other child u_{i,r_i}, $r_i \neq 1$, of u_i, we have $u'_{i,r_i} \in P'$. Let $P_{u_i} = P \cap V_{u_i,x}$. Thus $P' = \bigcup_{i=1}^{t} P_{u_i}$ and $\sum_{i=1}^{t} I(P_{u_i}) = K$. Finally, for any i, $P' - P_{u_i} \cup P_{u'_i}$ is maximal. Therefore $I(P_{u'_i}) + \sum_{l \neq i} I(P_{u_l}) = K$. Hence, for all $1 \leq i \leq k$, $I(P_{u'_i}) = I(P_{u_i})$ and so $\sum_{i=1}^{t} I(P_{u'_i}) = K$, a contradiction.

Case 2. Let $t = 0$ and $j = k$, that is, assume that all children of x are support vertices. Let P be a maximal packing such that $x \in P$. Consider $P - \{x\}$, $I(P) = I(P - \{x\}) + I(x) = K$. But $P - \{x\} \cup \bigcup_{h=1}^{j} \{s_h\}$ is a maximal packing. Therefore $I(P - \{x\}) + \sum_{h=1}^{j} I(s_h) = K$ and so $I(x) = \sum_{h=1}^{j} I(s_h) = 2k$. However since $deg(x) = k$, $I(x) = k + 1$. Whence $k = j = 1$ and x is an endvertex adjacent to one support vertex. Thus $deg(x) = 2j - 1$

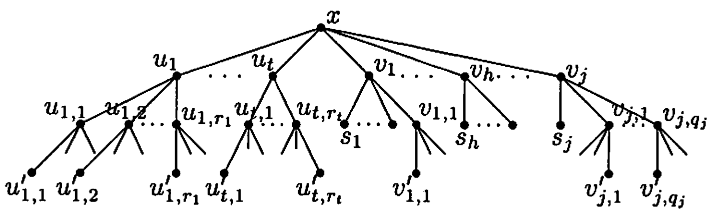

Figure 6: Tree T rooted at x where x is not a support vertex.

holds.

Case 3. Let $t \geq 1$ and $j \geq 1$. Choose P to be a maximal packing that contains x and, for all u_{i,r_i}, P contains a child of u_{i,r_i}, namely u'_{i,r_i}, and, for all v_h such that v_h has a grandchild, then for all v_{h,q_h}, P contains a child of v_{h,q_h}, namely v'_{h,q_h}. Again, we have selected P to contain x and be as tight around x as possible. Let $P_{u'_i} = P \cap V_{u_i,x}$, and let $P_{v'_h} = P \cap V_{v_h,x}$, noting that $P_{v'_h}$ could be empty for some h. So $P = \{x\} \cup \bigcup_{i=1}^{t} P_{u'_i} \cup \bigcup_{h=1}^{j} P_{v'_h}$, thus $I(x) + \sum_{i=1}^{t} I(P_{u'_i}) + \sum_{h=1}^{j} I(P_{v'_h}) = K$.

Consider the maximal packing $P^* = P - \{x\} \cup \bigcup_{h=1}^{j} \{s_h\}$. Thus $\sum_{h=1}^{j} I(s_h) + \sum_{i=1}^{t} I(P_{u'_i}) + \sum_{h=1}^{j} I(P_{v'_h}) = K$. Moreover, $I(x) = \sum_{h=1}^{j} I(s_h) = 2j$. Therefore $deg(x) = 2j - 1$. □

Theorem 9 *For a tree T, $LF(T) = F(T)$ if and only if $T \in \mathcal{T}$.*

Proof. Suppose that the $diam(T) = 1$. Then $T = K_2$, $LF(T) = F(T) = 2$ and $T \in \mathcal{T}$. Suppose that the $diam(T) = 2$. Consider a longest path in T, call it D, labeling an endvertex of the path x and the support vertex of x as y. Hence, $deg(x) = 1$ and $deg(y) \geq 2$. Notice that $\{x\}$ is a maximal packing in T and $\{y\}$ is a maximal packing in T. Thus if $LF(T) = F(T)$ then $deg(x) = deg(y)$, which is a contradiction. Therefore, there does not exist a tree of diameter two such that $LF(T) = F(T)$, nor does there exist a tree of diameter two such that $T \in \mathcal{T}$. Assume that $diam(T) = 3$. Consider (a longest path) D in T, and label the vertices of this longest path, x, y, z, u, where x and u are endvertices and y and z

are their support vertices, respectively. Notice that $\{x, u\}$, $\{y\}$ and $\{z\}$ are all maximal packings in T. Therefore, if $LF(T) = F(T)$ then $deg(y) = deg(z) = 3$. Hence, T is the double star $S_{2,2}$. It can be easily verified that $S_{2,2} \in \mathcal{T}$ and that there does not exist another tree T' of diameter three such that $T' \in \mathcal{T}$. Therefore, if $diam(T) \leq 3$, $LF(T) = F(T)$ if and only if $T \in \mathcal{T}$.

Let $diam(T) \geq 4$, then by Lemma 7 and Lemma 8, $LF(T) = F(T)$ implies that $T \in \mathcal{T}$. We will proceed to show that if $T \in \mathcal{T}$ then $LF(T) = F(T)$ by induction on the order of T. We know that the double star $S_{2,2}$ is an element of $\mathcal{T}$ and that $LF(S_{2,2}) = F(S_{2,2}) = 4$. Indeed, K_2 and $S_{2,2}$ are easily seen to be the only trees of order at most six that are in $\mathcal{T}$. Let $T \in \mathcal{T}$ and assume that for all trees $T^* \in \mathcal{T}$ where $|T^*| < |T|$ then $LF(T^*) = F(T^*)$. Take a longest path D and root T at one of the endvertices of D. Consider the opposite endvertex of D. This vertex must have a support vertex in D adjacent to it, call this support vertex x. Let y be the parent of x and z be the parent of y. Since x is a support vertex and $deg(x) \geq 2$, x must be adjacent to at least one other support vertex. If this support vertex is a child of x, then the endpoint adjacent to x would not be on a longest path, a contradiction. Thus, y must also be a support vertex, and $deg(x) = 3$ where x has two children, x' and x'', both of which are endvertices.

Now, $deg(y) = 2j + 1$ where j is the number of support vertices adjacent to y, and each child of y is either an endvertex or is a support vertex u_i that is similar to x. Each u_i has two children that are endvertices, u_i' and u_i''. Because $deg(y) = 2j + 1$, we can associate with each such u_i an endvertex of T adjacent to y, call it y_i, and associate with x an endvertex adjacent to y, call it y_x.

Case 1. Suppose there exists a support vertex $u_i \neq x$ that is a child of y. Then consider $T' = T - \{u_i, u_i', u_i'', y_i\}$. Note that $T' \in \mathcal{T}$ since y is adjacent to $j-1$ support vertices in T' and $deg_{T'}(y) = 2(j-1)+1 = 2j-1$. Consider any maximal packing P in T. Define $P' = P \cap T'$. By the induction hypothesis, all maximal packings of T' have the same influence, call it K.

Subcase a. Suppose that $y \in P$. Then $y \in P'$ and P' is a maximal packing in T'. Thus $I(P') = K$. But then $P = P'$ is a maximal packing in T and $I(P) = K + 2$, since $deg_T(y) = deg_{T'}(y) + 2$.

Subcase b. Suppose that P contains a neighbor of y.

i) Suppose that $u_i \notin P$ and $y_i \notin P$. Then P' is maximal in T'. Thus $I(P') = K$ and $P = P' \cup \{u_i'\}$ or $P = P' \cup \{u_i''\}$. Hence $I(P) = I(P') + I(u_i') = I(P') + I(u_i'') = K + 2$.

ii) Suppose that $y_i \in P$. Then $P^* = P' - \{y_i\} \cup \{y_x\}$ is a maximal packing in T', and $I(P^*) = I(P' \cup \{y_x\}) = K$. In T, $P = P' \cup \{y_i\} \cup \{u_i'\}$ or $P = P' \cup \{y_i\} \cup \{u_i''\}$. Thus $I(P) = K + 2$.

iii) Suppose that $u_i \in P$. Then P' is not maximal in T', but $P' \cup \{y_x\}$

is maximal in T'. Thus $I(P') = K - 2$. In T, $P = P' \cup \{u_i\}$ and $I(P) = I(P') + I(u_i) = (K-2) + 4 = K + 2$.

Hence, for all maximal packings P of T, $I(P) = K+2$, and so $LF(T) = F(T)$.

We can assume that y has exactly one child that is a support vertex, namely x. We consider two cases, based on whether or not the parent z of y is a support vertex.

Case 2. Let z be a support vertex, and call one of its endvertices z_y. Since y is adjacent to two support vertices, $deg(y) = 5$ and y is adjacent to three endvertices, call them $\{y_x, y_1, y_2\}$ where y_x is associated with x as stated above. Let $T' = T - \{x, x', x'', y_x\}$. By an argument parallel to Case 1, for all maximal packings P of T when z is a support vertex, $I(P)$ is a constant and $LF(T) = F(T)$.

Case 3. Suppose that z is not a support vertex. Then $deg(y) = 3$ and y has only one child that is an endvertex, namely y_x. If $deg(z) = 1$ then $T = S_{2,2}$ and $LF(T) = F(T) = 4$, so let $deg(z) \geq 3$. By the argument that x, y and z are on a longest path and that z is not a support vertex, all the children of z must be support vertices that are similar to y. Thus, z can be adjacent to only one non-support vertex, which must be the parent of z, call it w. Hence $deg(w) \geq 2$ and $deg(z) = 3$. Let the children of z be y and y^*, and let their respective descendants be labeled similarly to previous cases (see Figure 7). Consider the children of w. Since w is not a support vertex and is the parent of the non-support vertex z, w must be adjacent to at least two support vertices. Note that if the parent of w is not a support vertex, then $deg(w) \geq 5$. Therefore, at least one child of w is a support vertex, call it v. We know by the longest path assumption that no non-endvertex child of v can be a non-support vertex, and we know by our previous arguments, that if support vertex v is on a longest path then T has a form similar to Case 1 or Case 2, and so we are done. Therefore, v must have a form similar to y. Let $\overline{v}$ be the endvertex adjacent to v, and let s be the support vertex adjacent to v. Finally, label the endvertices of s by s' and s''. Let $T' = T - \{z, y, y_x, x, x', x'', y^*, y^*_x, x^*, x^{*\prime}, x^{*\prime\prime}, v, \overline{v}, s, s', s''\}$. Notice that w lost both a support vertex v and a non-support vertex z, that is, w is adjacent to $j-1$ support vertices and $deg(w) = 2(j-1)-1 = 2j-3$. Also, note that if the parent of w is not a support vertex then $deg(w) \geq 5$, and the parent of w did not become a support vertex. Hence, $T' \in \mathcal{T}$. Again let P be a maximal packing in T and define $P' = P \cap T'$. Let $LF(T') = F(T') = K$

Subcase a. Suppose that $w \in P$. Then P' is a maximal packing in T'. Therefore $I(P') = K$ in T'. In T, $I(P') = K + 2$, since $deg_T(w) = deg_{T'}(w) + 2$. Let T_z be the tree containing z and all the descendants of z. Let T_v be the tree containing v and all the descendants of v. It can easily be verified that any maximal packing in T_z, that does not contain z or

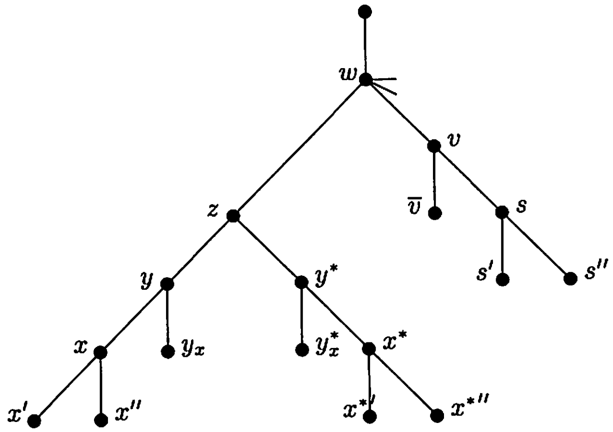

Figure 7: Tree T where z is not a support vertex.

the children of z, must have an influence of eight. Similarly, any maximal packing of T_v that does not contain v or the children of v, must have an influence of 2. Thus, for any P that contains w, $I(P) = (K + 2) + 8 + 2 = K + 12$.

Subcase b. Suppose that P contains a neighbor of w.

i) Suppose that $z \notin P$ and $v \notin P$. Then P' is maximal in T'. Hence, $I(P') = K$ in T' and $I(P') = K$ in T. Any maximal packing of T_z that does not contain z must have an influence of eight, and any maximal packing of T_v that does not contain v must have an influence of four. Thus, for all maximal packings P in T, $I(P) = K + 8 + 4 = K + 12$.

ii) Suppose that $z \in P$ and consider P'. Since $T' \in \mathcal{T}$, w must be adjacent to a support vertex different from v, call it $\overline{w}$. Let $\overline{w}'$ be an endvertex adjacent to $\overline{w}$. Also, $\overline{w}$ must be adjacent to at least one other support vertex. Now, P' must contain a vertex in $N(\overline{w}) - w$ or else $P \cup \{\overline{w}'\}$ is a packing, contradicting the maximality of P. Therefore $d(w, q) = 2$ for some $q \in P'$. By a similar argument, any non-support vertex adjacent to w must also be at most distance two from some element in P'. Therefore, P' is a maximal packing in T', and so $I(P') = K$ in T' and $I(P') = K$ in T. Any maximal packing of T_z that contains z must have an influence of eight in T, and any maximal packing of T_v that does not contain v must have an influence of four in T. Thus, $I(P) = K + 8 + 4 = K + 12$.

iii) Suppose that $v \in P$. By the same argument as above, P' is maximal in T'. Thus, $I(P') = K$ in T' and $I(P') = K$ in T. Any maximal packing of T_z that does not contain z must have an influence of eight in T, and the maximal packing $\{v\}$ of T_v has an influence of four in T. Thus, $I(P) = K + 8 + 4 = K + 12$.

Subcase c. Suppose that P does not contain any neighbors of w. Then P' must be maximal in T' and $I(P') = K$ in T' and in T. Also, for any maximal packing of T_z that does not contain z, the influence must be eight, and for any maximal packing of T_v that does not contain v, the influence must be four. Thus, $I(P) = K + 12$.

Therefore, for all maximal packings P of T, $I(P) = K + 12$, and so $LF(T) = F(T)$. Hence, we have proven that for a tree T, $LF(T) = F(T)$ if and only if $T \in \mathcal{T}$. □

4 Concluding Remarks

As noted, this paper begins the study of the parameter LF. The duals of F and R are the parameters W and P, closed neighborhood order domination and closed neighborhood order packing, respectively. The parameters UW, LP, and UR defined in [14] have yet to be studied.

Complexity questions concerning the parameter LF are of interest. For example, for which classes of graphs G (1) can $LF(G)$ be computed in polynomial/linear time, or (2) is the decision problem associated with determining $LF(G)$ NP-complete?

Characterizing other classes of graphs G with $LF(G) = F(G)$ would be interesting. Note that, from Proposition 3, if G is regular then $LF(G) = F(G)$ if and only if $L\rho(G) = \rho(G)$.

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