On the number of edges in a graph with given domination-type parameters

Gayla S. Domke and Johannes H. Hattingh
Department of Mathematics and Statistics
Georgia State University
Atlanta, GA 30303, U.S.A.

Lisa R. Markus
Department of Mathematics
De Anza College
Cupertino, CA 95014, U.S.A.

Abstract

Let G be a graph. The weak domination number of G, $\gamma_w(G)$, is the minimum cardinality of a set D of vertices where every vertex $u \notin D$ is adjacent to a vertex $v \in D$, where $\deg(v) \leq \deg(u)$. The strong domination number of G, $\gamma_s(G)$, is the minimum cardinality of a set D of vertices where every vertex $u \notin D$ is adjacent to a vertex $v \in D$, where $\deg(v) \geq \deg(u)$. Similarly, the independent weak domination number, $i_w(G)$, and the independent strong domination number, $i_{st}(G)$, are defined with the additional requirement that the set D is independent. We find upper bounds on the number of edges of a graph in terms of the number of vertices and for each of these four domination parameters. We also characterize all graphs where equality is achieved in each of the four bounds.

Dedicated to Ernie Cockayne on the occasion of his 60th birthday.

1 Introduction

Let $G = (V, E)$ be a graph. For any vertex $v \in V$, the *open neighborhood* of v, denoted $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, denoted $N[v]$, is $N(v) \cup \{v\}$. The *degree* of a vertex v, denoted deg(v), is $|N(v)|$. The *maximum degree* (*minimum degree*, respectively) of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively).

A set $D \subseteq V$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The *domination number*, denoted $\gamma(G)$, is the minimum cardinality amongst all dominating sets of G.

A set $D \subseteq V$ is *independent* if no two vertices of D are adjacent. An *independent dominating set* is a dominating set that is also independent. The *independent domination number*, denoted $i(G)$, is the minimum cardinality amongst all independent dominating sets of G. Since any independent dominating set is a dominating set, it follows that $\gamma(G) \leq i(G)$ for any graph G. Since any maximal independent set of a graph of order p containing a vertex of degree $\Delta(G)$ contains at most $p - \Delta(G)$ vertices and every maximal independent set is also dominating, $i(G) \leq p - \Delta(G)$. Thus, $\gamma(G) \leq i(G) \leq p - \Delta(G)$.

Graphs G of order p for which equality holds in the bounds $\mu(G) \leq p - \Delta(G)$ where $\mu \in \{\gamma, i\}$ were studied in [1, 3].

A set $D \subseteq V$ is a *weak dominating set* if every vertex u not in D is adjacent to a vertex v in D where $\deg(v) \leq \deg(u)$. The *weak domination number*, denoted $\gamma_w(G)$, is the minimum cardinality amongst all weak dominating sets of G.

A set $D \subseteq V$ is a *strong dominating set* if every vertex u not in D is adjacent to a vertex v in D where $\deg(v) \geq \deg(u)$. The *strong domination number*, denoted by $\gamma_{st}(G)$, is the minimum cardinality amongst all strong dominating sets of G.

The *independent weak domination number*, denoted $i_w(G)$, is the minimum cardinality amongst all weak dominating sets which are also independent. The *independent strong domination number*, denoted $i_{st}(G)$, is defined similarly.

We will call a weak dominating set S (similarly a strong dominating set, an independent weak dominating set and an independent strong dominating set) a *γ_w-set* (similarly a *γ_{st}-set*, an *i_w-set* and an *i_{st}-set*) if $|S| = \gamma_w(G)$ (similarly $|S| = \gamma_{st}(G)$, $|S| = i_w(G)$ and $|S| = i_{st}(G)$).

Let S be a weak dominating set with $x \in S$. A vertex $y \in N[x]$ is a *weak private neighbor of x with respect to S* if $\deg(y) \geq \deg(x)$ and for all $z \in S - \{x\}$, if $y \in N(z)$, then $\deg(y) < \deg(z)$.

The concepts of weak and strong domination were introduced by Sampathkumar and Pushpa Latha in [7] and further studied in [4, 5, 6].

Since any independent weak (independent strong, respectively) dominating set must be a weak (strong, respectively) dominating set, we have the following result.

Proposition 1 *For any graph G,* $\gamma_w(G) \leq i_w(G)$ *and* $\gamma_{st}(G) \leq i_{st}(G)$.

Vizing [9] showed the following bound on the size of a graph with given domination number.

Theorem 2 *If G is a graph with p vertices, q edges, and domination number* $\gamma(G) \geq 2$, *then*

$$q \leq \frac{(p-\gamma(G))(p-\gamma(G)+2)}{2}.$$

Sanchis [8] improved this bound for certain values of the domination number.

Theorem 3 *If G is a graph with p vertices, q edges, domination number* $3 \leq \gamma(G) \leq p/2$, *and no isolated vertices, then*

$$q \leq \frac{(p-\gamma(G))(p-\gamma(G)+1)}{2}.$$

In this paper we will give similar bounds on the number of edges in terms of these other domination parameters. In Section 2, we prove an upper bound on the number of edges in terms of the number of vertices and the weak domination number as well as another upper bound involving the independent weak domination number. In Section 3, the upper bounds on the number of edges proved are in terms of the number of vertices and the strong domination number or the independent strong domination number. In both sections we characterize all graphs whose number of edges achieve these upper bounds.

2 Weak domination

We begin this section by exhibiting a bound on the size of the graph in terms of the weak domination number. We need the following definitions in order to characterize all graphs which achieve this bound. A *split graph* is a graph whose vertex set can be partitioned into two sets I and K where I is an independent set and K induces a complete graph. A *complete split graph* is a split graph where all possible edges appear between the sets I and K.

Theorem 4 *If G is a graph with p vertices and q edges, then*

$$q \leq \frac{(p - \gamma_w(G))(p + \gamma_w(G) - 1)}{2}.$$

Furthermore, equality is achieved if and only in G is a complete split graph.

Proof. Let G be a $(p, q)-$graph. Also let D be a γ_w-set of G.

Claim. Any vertex in D has degree at most $p - \gamma_w(G)$.
Proof of claim. Let $x \in D$ have the largest degree of any vertex in D and suppose that $\deg(x) \geq p - \gamma_w(G) + 1$. Since $|V - D| = p - \gamma_w(G)$, the vertex x is adjacent to some other vertex in D, say y. By the way we chose x we know that $\deg(x) \geq \deg(y)$. Since x cannot be its own weak private neighbor (since it is weakly dominated by y), x must have a weak private neighbor $w \in V - D$ with $\deg(w) \geq \deg(x) \geq p - \gamma_w(G) + 1$. Since $\deg(w) \geq p - \gamma_w(G) + 1 > |V - D - \{w\} \cup \{x\}|$, then w must be adjacent to another vertex in D, say $z \neq x$. But since w is a weak private neighbor of x, it must be true that $\deg(z) > \deg(w) \geq \deg(x)$, contradicting the choice of x. Thus, any vertex in D has degree at most $p - \gamma_w(G)$ as claimed.

Since each of the $\gamma_w(G)$ vertices in D have degree at most $p - \gamma_w(G)$ and each of the $p - \gamma_w(G)$ vertices in $V - D$ have degree at most $p - 1$, it is true that $q \leq \frac{(p-\gamma_w(G))\gamma_w(G)+(p-1)(p-\gamma_w(G))}{2} = \frac{(p-\gamma_w(G))(p+\gamma_w(G)-1)}{2}$.

It is easily seen that if G is a complete split graph with $V = K \cup I$, where K induces a clique, I is an independent set and each vertex of I is adjacent to every vertex in K, then $\gamma_w(G) = |I|$, $|K| = p - \gamma_w(G)$ and $q = \frac{(p-\gamma_w(G))(p+\gamma_w(G)-1)}{2}$. Also, if this equality holds, the above inequalities become equalities. Thus, if G is a graph where this equality holds and D is a γ_w-set, then every vertex in D must have degree $p - \gamma_w(G)$ and every vertex in $V - D$ must have degree $p - 1$. Hence, G is the graph with $V = K \cup D$ where K is a clique containing $p - \gamma_w(G)$ vertices, D is an independent set and each vertex of D is adjacent to every vertex in K. □

We now consider a bound on the size of a graph in terms of the independent weak domination number. Again, we characterize all graphs which achieve this bound.

Theorem 5 *If G is a graph with p vertices and q edges, then*

$$q \leq \frac{(p - i_w(G))(p + i_w(G) - 1)}{2}.$$

Furthermore, equality is achieved if and only in G is a complete split graph.

Proof. Let G be a $(p, q)-$graph. Also let D be an i_w-set of G. Clearly, any vertex in D has degree at most $p - i_w(G)$ since any vertex in D is not adjacent to any of the $i_w(G)$ vertices in D. Also, each vertex in $V - D$ has degree at most $p-1$. Thus, $q \leq \frac{(p-i_w(G))(p-1)+i_w(G)(p-i_w(G))}{2} = \frac{(p-i_w(G))(p+i_w(G)-1)}{2}$.

We get equality above if and only if D is independent, each vertex in D has degree $p - i_w(G)$, and each vertex in $V - D$ has degree $p-1$. Hence, $q = \frac{(p-i_w(G))(p+i_w(G)-1)}{2}$ if and only if G is a complete split graph with $V = K \cup D$ where K is a clique containing $p - i_w(G)$ vertices, D is an independent set and each vertex of D is adjacent to every vertex in K. □

3 Strong domination

Domke, Hattingh, Markus, and Ungerer [2] noted that for any graph G with p vertices and maximum degree $\Delta(G)$, $\gamma_{st}(G) \leq i_{st}(G) \leq p - \Delta(G)$. They also proved the following theorems giving necessary and sufficient conditions for equality to hold.

Theorem 6 *Let G be a graph with p vertices. Then $i_{st}(G) = p - \Delta(G)$ if and only if for every vertex v of degree $\Delta(G)$, $V - N(v)$ is an independent set.*

Theorem 7 *Let G be a graph with p vertices. Then $\gamma_{st}(G) = p - \Delta(G)$ if and only if for every vertex v of degree $\Delta(G)$ the following two conditions hold:*

1. *$V - N(v)$ is an independent set, and*
2. *If $u \in N(v)$ is adjacent to vertices x and y in $V - N[v]$, then* $\deg(u) < \max\{\deg(x), \deg(y)\}$.

In this section we first find bounds on the size of a graph in terms of the strong domination number. We break this result into two cases: $\gamma_{st}(G) \geq 3$ and $\gamma_{st}(G) \leq 2$.

Theorem 8 *If G is a graph with p vertices, q edges, and strong domination number $\gamma_{st}(G) \geq 3$, then*

$$q \leq \frac{(p - \gamma_{st}(G))(p-1)}{2}.$$

Proof. Let G be a (p,q)–graph with strong domination number $\gamma_{st}(G) \geq 3$. We know that $\gamma_{st}(G) \leq p - \Delta(G)$ or $\Delta(G) \leq p - \gamma_{st}(G)$.

Case 1. Suppose $\Delta(G) \leq p - \gamma_{st}(G) - 1$. Then $q \leq \frac{p}{2}(p - \gamma_{st}(G) - 1) = \frac{p}{2}(p - \gamma_{st}(G)) - \frac{p}{2} < \frac{p}{2}(p - \gamma_{st}(G)) - \frac{(p-\gamma_{st}(G))}{2} = \frac{(p-\gamma_{st}(G))(p-1)}{2}$.

Case 2. Suppose $\Delta(G) = p - \gamma_{st}(G)$. Let v be a vertex of degree $\Delta(G)$. By Theorem 7, conditions 1 and 2 hold for v. Since $V - N(v)$ is an independent set, then for any vertex $w \in V - N[v]$, $\deg(w) \leq |N(v)| = \Delta(G) = p - \gamma_{st}(G)$.

Case 2.1. Suppose there are vertices $x, y \in V - N[v]$ where $\deg(x) = \deg(y) = p - \gamma_{st}(G)$. Hence, x and y are adjacent to every vertex in $N(v)$, and every vertex in $N(v)$ is adjacent to x and y in $V - N[v]$. So, for any vertex $u \in N(v)$, $\deg(u) < \max\{\deg(x), \deg(y)\} = p - \gamma_{st}(G)$. Hence, $q \leq \frac{\gamma_{st}(G)(p-\gamma_{st}(G))+(p-\gamma_{st}(G))(p-\gamma_{st}(G)-1)}{2} = \frac{(p-\gamma_{st}(G))(p-1)}{2}$.

Case 2.2. Suppose at most one vertex in $V - N[v]$ has degree $p - \gamma_{st}(G)$. Let k be the number of vertices in $N(v)$ which have two or more neighbors in $V - N[v]$. Hence, there are k vertices in $N(v)$ with at most $|V - N[v]| = \gamma_{st}(G) - 1$ neighbors in $V - N[v]$. The other $p - \gamma_{st}(G) - k$ vertices in $N(v)$ have at most one neighbor in $V - N[v]$. If a vertex in $N(v)$ has two or more neighbors in $V - N[v]$, then by condition 2 of Theorem 7 its degree must be at most $\Delta(G) - 1 = p - \gamma_{st}(G) - 1$. Otherwise, if a vertex in $N(v)$ has at most one neighbor in $V - N[v]$, then its degree must be at most $\Delta(G) = p - \gamma_{st}(G)$. Now, the sum of the degrees of the vertices in $V - N[v]$ is equal to the number of edges between $V - N[v]$ and $N(v)$ since $V - N[v]$ is independent. Since there are k vertices in $N(v)$ with at most $\gamma_{st}(G) - 1$ neighbors in $V - N[v]$ and there are $p - \gamma_{st}(G) - k$ vertices in $N(v)$ which have at most one neighbor in $V - N[v]$, then $\sum_{w \in V - N[v]} \deg(w) \leq k(\gamma_{st}(G) - 1) + (p - \gamma_{st}(G) - k)$. Now,

$$\begin{aligned}
2q &= \textstyle\sum_{w \in V(G)} \deg(w) \\
&= \deg(v) + \textstyle\sum_{w \in N(v)} \deg(w) + \sum_{w \in V - N[v]} \deg(w) \\
&\leq [p - \gamma_{st}(G)] + [k(p - \gamma_{st}(G) - 1) + (p - \gamma_{st}(G) - k)(p - \gamma_{st}(G))] \\
&\qquad + [k(\gamma_{st}(G) - 1) + (p - \gamma_{st}(G) - k)] \\
&= (p - \gamma_{st}(G))(p - 1) - (\gamma_{st}(G) - 3)(p - \gamma_{st}(G) - k).
\end{aligned}$$

Since $\gamma_{st}(G) \geq 3$ and $p - \gamma_{st}(G) - k \geq 0$, then $q \leq \frac{1}{2}(p - \gamma_{st}(G))(p - 1)$. □

A *perfect matching* is a spanning subgraph of a graph where all vertices have degree one. In order to characterize all graphs for which equality holds in the bound of Theorem 8, we need to define three types of graphs.

The first type of graph G_1 is defined as follows. Let $3 \leq m < n$ be integers, let H be an $(n - 1 - m)$-regular graph, and let I be an independent set of size m. Let $G_1 = I + H$. Note that every vertex of I has degree

$n = \Delta(G_1)$, while every vertex of $V(H)$ has degree $n-1$. Note also that G_1 has order $p = m + n$. Since every vertex v of maximum degree has $V - N(v)$ independent, and every vertex $u \in N(v)$ has $\deg(u) < \Delta(G_1)$, by Theorem 7, $\gamma_{st}(G_1) = p - \Delta(G_1) = m$. Furthermore, $2q = \gamma_{st}(G_1)\Delta(G_1) + \Delta(G_1)(\Delta(G_1) - 1) = \Delta(G_1)(\gamma_{st}(G_1) + \Delta(G_1) - 1) = (p - \gamma_{st}(G_1))(p-1)$.

The second type of graph G_2 is defined as follows. Let $n \geq 2$ be an even integer and let K be a complete graph of order n without a perfect matching. Let $\{v, x, y\}$ be an independent set and let $G_2 = (K + \{v, x\}) \cup \{y\}$. Then G_2 has order $p = n + 3$, $\deg(v) = \deg(x) = n = p - 3$, $\deg(u) = n = p - 3$ for all $u \in V(K)$, while y is an isolated vertex. Clearly, $\gamma_{st}(G_2) = 3$, while $2q = (p-3)(p-1) = (p - \gamma_{st}(G_2))(p-1)$.

The third type of graph G_3 is defined as follows. Let $k \geq 4$ be an integer, let H be any $(k-4)$-regular graph of order k, let $p \geq k + 5$ be an integer such that $p - k - 3$ is even and let K be a complete graph of order $p - k - 3$ without a perfect matching. Construct the graph G_3 by joining H and K, joining new vertices v and x with every vertex of $V(J)$ and joining the new vertex y with every vertex of $V(H)$. Note that every vertex of $V(H)$ has degree $p-4$, every vertex of $V(K)$ has degree $p-3$, $\deg(v) = \deg(x) = p-3$, while $\deg(y) = k$.

Since every vertex v of maximum degree has $V - N(v)$ independent, and every vertex $u \in N(v)$ adjacent to two vertices of $V - N[v]$ has $\deg(u) < \Delta(G_3)$, by Theorem 7, $\gamma_{st}(G_3) = p - (p-3) = 3$. Furthermore, $2q = k + k(p-4) + (p-k-1)(p-3) = k + kp - 4k + p^2 - 3p - kp + 3k - p + 3 = p^2 - 4p + 3 = (p-3)(p-1) = (p - \gamma_{st}(G_3))(p-1)$.

Theorem 9 *If G is a graph with p vertices, q edges, and strong domination number $\gamma_{st}(G) \geq 3$, then $q = \frac{(p-\gamma_{st}(G))(p-1)}{2}$ if and only if G is either of type 1 or of type 2 or of type 3.*

Proof. The discussion preceding the statement of the theorem showed that if G is either of type 1 or of type 2 or of type 3, then $q = \frac{(p-\gamma_{st}(G))(p-1)}{2}$.

Conversely, suppose $\gamma_{st}(G) \geq 3$ and $q = \frac{(p-\gamma_{st}(G))(p-1)}{2}$. If $\Delta(G) \leq p - \gamma_{st}(G) - 1$, then by the argument in case 1 in the proof of Theorem 8, $q < \frac{(p-\gamma_{st}(G))(p-1)}{2}$, a contradiction. Hence, $\Delta(G) = p - \gamma_{st}(G)$. Let v be a vertex of maximum degree.

Case 1. Suppose there are vertices $x, y \in V - N[v]$ where $\deg(x) = \deg(y) = \Delta(G)$. As in case 2.1 of Theorem 8, every vertex $u \in N(v)$ has $\deg(u) < \Delta(G)$. In order for equality to hold, all $\gamma_{st}(G)$ vertices in $V - N(v)$ must have degree $\Delta(G) = p - \gamma_{st}(G)$ and all vertices in $N(v)$ must have degree $\Delta(G) - 1 = p - \gamma_{st}(G) - 1$. Let $I = V - N(v)$ and let $H = \langle N(v) \rangle$.

Let $m = |I|$ and let $n = p(H)$. Then $3 \leq m = \gamma_{st}(G) = p - \Delta(G)$ and $n = \Delta(G)$. If $m \geq n$ and $u \in N(v)$, then $\deg(u) \geq p - \Delta(G) \geq \Delta(G)$, which is a contradiction, since $\deg(u) = \Delta(G) - 1$. Thus, $m < n$. Note also that H is regular of degree $n - 1 - m$. Hence, G is of type 1.

Case 2. Suppose at most one vertex in $V - N[v]$ has degree $\Delta(G)$, and let k be the number of vertices in $N(v)$ which have two or more neighbors in $V - N[v]$. As in case 2.2 of Theorem 8, $(\gamma_{st}(G) - 3)(p - \gamma_{st}(G) - k) = 0$ which means that either $\gamma_{st}(G) = 3$ or $k = p - \gamma_{st}(G) = \Delta(G)$. In the latter case, every vertex in $N(v)$ has two or more neighbors in $V - N[v]$. So, all vertices in $V - N[v]$ have degree $\Delta(G) = p - \gamma_{st}(G)$. Thus, if there is at most one vertex in $V - N[v]$ with degree $p - \gamma_{st}(G)$, then $|V - N[v]| \leq 1$ or $\gamma_{st}(G) \leq 2$, a contradiction. Hence, $\gamma_{st}(G) = 3$.

Let $V - N[v] = \{x, y\}$. Now, $|N(v)| = \Delta(G) = p - \gamma_{st}(G) = p - 3$. Let $k = |N(x) \cap N(y)|$. Since these k vertices in $N(v)$ have two neighbors in $V - N[v]$, by condition 2 of Theorem 7, their degrees can be at most $\Delta(G) - 1 = p - 4$. The other $p - 3 - k$ vertices can have degree at most $\Delta(G) = p - 3$. By an argument similar to case 2.2 in the proof of Theorem 8, $\deg(x) + \deg(y) = p - 3 + k$. So, $2q = \deg(v) + \sum_{w \in N(v)} \deg(w) + \sum_{w \in V - N[v]} \deg(w) = \deg(v) + \sum_{w \in N(x) \cap N(y)} \deg(w) + \sum_{w \in (N(x) - N(y)) \cup (N(y) - N(x))} \deg(w) + \deg(x) + \deg(y) \leq p - 3 + k(p-4) + (p - 3 - k)(p - 3) + p - 3 + k = p^2 - 4p + 3 = (p-1)(p-3)$. Thus, $\frac{(p-3)(p-1)}{2} = q \leq \frac{(p-1)(p-3)}{2}$, and equality holds in each of the bounds on the degrees of the vertices of $N[v]$.

Suppose first that $k = 0$. Then $\deg(u) = p - 3$ for all $u \in N(v)$, and $\deg_{\langle N(v) \rangle}(u) = p - 5$, so that for each vertex $u \in N(v)$, there is exactly one vertex of $N(v)$ which is not adjacent to u. Thus, if we let $K = \langle N(v) \rangle$, then K is a complete graph without a perfect matching, and has order $p - 3 \geq 2$, an even number.

Since $\deg(x) + \deg(y) = p - 3 + k = p - 3$, either $N(x) \neq \emptyset$ and $N(y) = \emptyset$ or $N(x) \neq \emptyset$ and $N(y) \neq \emptyset$ or $N(x) = \emptyset$ and $N(y) \neq \emptyset$. Since the first and third possibilities are similar, we only consider the first and second possibilities. If $N(x) \neq \emptyset$ and $N(y) = \emptyset$, then, if we let $n = p - 3$, we see that G is a graph of type 2.

Suppose, therefore, $N(x) \neq \emptyset$ and $N(y) \neq \emptyset$. Let $u \in N(x)$ and suppose $w \in N(v)$ is not adjacent to u. If $w \in N(y)$, then $\{u, w\}$ is a strong dominating set of G, which is a contradiction. Thus, $w \in N(x)$. Let $\ell \in N(y)$. Then $\{u, \ell\}$ is a strong dominating set of G, which is a contradiction.

We may conclude that $k \geq 1$. Let $u \in N(x) \cap N(y)$. If $\deg(x) \leq p - 4$ and $\deg(y) \leq p - 4$, then $\{u, v\}$ is a strong dominating set of G, which is a contradiction. Thus, without loss of generality, assume that $\deg(x) = p - 3$.

Since we are assuming that at most one of x and y has degree $p-3$, we have $\deg(y) \leq p-4$. Notice that $k = \deg(y)$. Let $H = \langle N(y) \rangle$ and let $K = \langle N(v) - N(y) \rangle$. If $u \in V(K)$ and $\ell \in V(H)$, and u and ℓ are nonadjacent, then $\{u, \ell\}$ is a strong dominating set of G, which is a contradiction. Thus, every vertex of K is adjacent to every vertex of H, and each vertex of K is nonadjacent to exactly one vertex of K. Thus, K is a complete graph without a perfect matching and has order $p-k-3$, an even number. Each vertex of H has degree $p-4$ and has degree $k-4$ in H. Hence, G is of type 3.

□

The previous results involved graphs with larger strong domination number. The following result covers the cases where the strong domination number is smaller.

Proposition 10 *If G is a graph with p vertices, q edges, and strong domination number $\gamma_{st}(G) \leq 2$, then*

$$q \leq \frac{(p-\gamma_{st}(G))p}{2}.$$

Furthermore, equality holds if and only if G is either a complete graph or a complete graph minus a perfect matching.

Proof. Let G be a (p,q)-graph, and strong domination number $\gamma_{st}(G) \leq 2$. We know that $\Delta(G) \leq p - \gamma_{st}(G)$, so $q \leq \frac{\Delta(G)\cdot p}{2} \leq \frac{(p-\gamma_{st}(G))p}{2}$.

If $\gamma_{st}(G) = 1$, then $q = \frac{(p-1)p}{2}$ if and only if $G = K_p$, where $\gamma_{st}(K_p) = 1$.

If $\gamma_{st}(G) = 2$, then $\Delta(G) \leq p-2$. So, $q = \frac{(p-2)p}{2}$ if and only if G is regular of degree $p-2$. This is the graph obtained from K_p by removing a perfect matching. It is true that the strong domination number of such a graph is two. □

We now give a bound on the number of edges in a graph in terms of the independent domination number.

Theorem 11 *If G is a graph with p vertices and q edges, then*

$$q \leq \frac{(p-i_{st}(G))p}{2}.$$

Furthermore, equality is achieved if and only if G is a complete r-partite graph where each of the partite sets has $\frac{p}{r}$ vertices.

Proof. Let G be a (p,q)–graph. Then, for every vertex v in G, $\deg(v) \leq \Delta(G) \leq p - i_{st}(G)$. Therefore, $q = \frac{1}{2}\sum_{v\in V}\deg(v) \leq \frac{(p-i_{st}(G))p}{2}$.

If G is a complete r-partite graph where each partite set contains $\frac{p}{r}$ vertices, then G is a regular graph where every vertex has degree $p - \frac{p}{r}$. Now, any independent strong dominating set must be a subset of one of the partite sets, say V_1. The only way to independently dominate the vertices of V_1 is to include the entire partite set V_1 in an independent strong dominating set. Hence, $i_{st}(G) = |V_1| = \frac{p}{r}$. So, the degree of any vertex of G is $p - \frac{p}{r} = p - i_{st}(G)$. Thus, the total number of edges of G is $q = \frac{1}{2}p(p - i_{st}(G))$ and the equality holds.

Now, suppose G is a graph with p vertices and $q = \frac{(p-i_{st}(G))p}{2}$ edges. Since, for any vertex v, $\deg(v) \leq \Delta(G) \leq p - i_{st}(G)$, then every vertex must have $\deg(v) = p - i_{st}(G) = \Delta(G)$. Hence, by Theorem 6, $V - N(v)$ is an independent set of size $i_{st}(G)$. Also, for any vertex $x \in V - N(v)$, since $V - N(v)$ is independent and $\deg(x) = p - i_{st}(G) = |N(v)|$, then $N(x) = N(v)$. Thus, G is a complete r-partite graph where $r = \frac{p}{i_{st}(G)}$. □

References

[1] G.S. Domke, J.E. Dunbar and L.R. Markus, Gallai-type theorems and domination parameters, *Discrete Math.* **167/168** (1997) 237-248.

[2] G.S. Domke, J.H. Hattingh, L.R. Markus and E. Ungerer, On parameters related to strong and weak domination in graphs, submitted.

[3] O. Favaron and C.M. Mynhardt, On equality in an upper bound for domination parameters of graphs, *J. Graph Theory* **24** (1997) 221-231.

[4] J.H. Hattingh and M.A. Henning, On strong domination in graphs, *J. Combin. Math. and Combin. Computing* **26** (1998) 73 - 82.

[5] J.H. Hattingh and R.C. Laskar, On weak domination in graphs, *Ars Combin.* **49** (1998) 205 - 216.

[6] J.H. Hattingh and D. Rautenbach, On weak domination in graphs II, submitted.

[7] E. Sampathkumar and L. Pushpa Latha, Strong, weak domination and domination balance in a graph, *Discrete Math.* **161** (1996) 235-242.

[8] L.A. Sanchis, Maximum number of edges in connected graphs with a given domination number, *Discrete Math.* **87** (1991) 65-72.

[9] V.G. Vizing, A bound on the external stability number of a graph, *Dokl. Akad. Nauk.* **164** (1965) 729-731.