PARTITIONS OF COUNTABLE POSETS.

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ABSTRACT. For a countable bounded principal ideal poset P and a natural number r, there exists a countable bounded principal ideal poset P' such that for an arbitrary r-colouring of the points (resp. two-chains) of P', a monochromatically embedded copy of P can be found in P'. Moreover, a best possible upper bound for the height of P' in terms of r and the height of P is given.

1. INTRODUCTION

If X is a set then $|X|$ denotes the cardinality of X. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the natural numbers. If k is a natural number, then $\binom{X}{k}$ denotes the set of k-element subsets of X and we write $k \cdot X := \{kx : x \in X\}$ when X is a set of natural numbers. If n is a natural number, we write $[n]$ for the set $\{1, \dots, n\}$.

For natural numbers k, l and r the associated Ramsey number [6], denoted by $R(k, l, r)$, is the smallest natural number N with the property that for any r-colouring $\chi : \binom{[N]}{k} \longrightarrow [r]$ of the k-element subsets of $[N]$, there is a $Y \in \binom{[N]}{l}$ such that χ is constant on $\binom{Y}{k}$.

If P is any poset and the sizes of the chains of P are bounded, then for every $p \in P$ the *level* $\ell(p)$ of p is the size (cardinality) of the largest chain in P of which p is the maximal element and the *height* $h(P)$ of P is the maximal level in P. For posets P and P' an injective map $\lambda : P \longrightarrow P'$ is called an *embedding* whenever $a \leq b$ iff $\lambda(a) \leq \lambda(b)$. If in addition $\ell(a) = \ell(b)$ implies $\ell(\lambda(a)) = \ell(\lambda(b))$ the embedding is called *level-preserving*.

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If s is a natural number, then an s-chain is a chain of size s. For posets P and P' and natural numbers r and s, write $P' \underset{(s,r)}{\longmapsto} P$ if for every r-colouring χ of the s-chains of P', there is an embedding $\lambda : P \longrightarrow P'$ such that χ is constant on the s-chains of $\lambda(P)$. It follows from the work of Nešetřil and Rödl [4, 5] that for every finite poset P and natural numbers r and s there is a finite poset P' such that $P' \underset{(s,r)}{\longmapsto} P$. Fouché [2] was able to show that one can find such a P' with $h(P') = R(s, h(P), r)$. For $s = 1$ this was done by Nešetřil and Rödl in [5]. This result is the best possible as far as the height of P' is concerned.

Call P a *bounded principal ideal (bpi)* poset if there is a natural number l such that $|[x]| \leq l$ for all $x \in P$, where $[x] := \{z \in P : z \leq x\}$. An alternative characterization of countable bpi posets appears in Proposition 3.1 below. (For examples of these posets, see also Theorem 2 and Section 4 of this paper.)

We prove analogues of the results of Fouché [2] for countable bounded principal ideal posets where one-chains (points) or two-chains are partitioned. Our main result is the following theorem.

Theorem 1. *For a countable bounded principal ideal poset P and numbers $r \in \mathbb{N}$ and $s \in \{1,2\}$, there exists a countable bounded principal ideal poset P' with $h(P') = R(s, h(P), r)$ such that $P' \underset{(s,r)}{\longmapsto} P$.*

It is clear that if $P' \underset{(s,r)}{\longmapsto} P$, then it necessarily follows that $h(P') \geq R(s, h(P), r)$. In this sense Theorem 1 is the best possible. It still is an open problem whether Theorem 1 also holds for all $s \geq 3$. The finitary arguments in [2] for finite posets and for $s \geq 3$ do not extend to the countable case.

If $k_1 < \cdots < k_t$ are natural numbers and X is any set, let $B_X(k_1, \ldots, k_t) := \bigcup_{i=1}^{t} \binom{X}{k_i}$ with a poset structure induced by the subset relation. We also write $B_n(k_1, \ldots, k_t)$ instead of $B_{[n]}(k_1, \ldots, k_t)$ if n is a natural number and write $B_{\mathbb{N}}$ for the poset based on the set of finite subsets of $\mathbb{N}$. We shall see (see Proposition 3.1) that any bpi poset can be embedded into $B_{\mathbb{N}}(k_1, \cdots, k_t)$ for some natural numbers $k_1, \cdots, k_t$ with $k_1 < \ldots < k_t$ and $t \geq 1$.

We shall deduce Theorem 1 from the following result.

Theorem 2. *For natural numbers $r, s, k_1, \ldots, k_t$ with $s \in \{1,2\}$ and $k_1 < \cdots < k_t$, there are natural numbers $K_1 < \cdots < K_m$ with $m = R(s,t,r)$ such that $B_{\mathbb{N}}(K_1, \ldots, K_m) \underset{(s,r)}{\longmapsto} B_{\mathbb{N}}(k_1, \ldots, k_t)$.*

As an application we show in Section 4 that the poset of finite-dimensional subspaces of a vector space of countably infinite dimension over a finite field has the same property as $B_{\mathbb{N}}$ in Theorem 2. We prove a similar result for the poset $\Pi_{\mathbb{N}}$ of partitions of $\mathbb{N}$ into finite blocks where almost all blocks are singletons.

It follows from Corollary 3 of Hindman's Finite Unions Theorem [3] that $B_{\mathbb{N}} \underset{(1,r)}{\longmapsto} B_{\mathbb{N}}$ for every natural number r. Using this result, we show in Section 4 that $\Pi_{\mathbb{N}} \underset{(1,r)}{\longmapsto} \Pi_{\mathbb{N}}$ for every natural number r.

2. Proof of Theorem 2

The following two results are needed for the proof of Theorem 2. Finitary versions of these results appear in [2].

Proposition 2.1. *For natural numbers m and t with $m \geq t$ and any sequence of natural numbers $k_1 < \cdots < k_t$ there are natural numbers $l_1 < \cdots < l_m$ such that for every sequence $1 \leq \alpha_1 < \cdots < \alpha_t \leq m$ there is an embedding*

$$\lambda : B_{\mathbb{N}}(k_1, \ldots, k_t) \longrightarrow B_{\mathbb{N}}(l_{\alpha_1}, \ldots, l_{\alpha_t}).$$

Proof. Assume that $t \geq 2$, the case $t = 1$ being trivial. For $i = 1, \ldots, t-1$ let $\delta_i = k_{i+1} - k_i$ and $v = \max\{\delta_i : i = 1, \ldots, t-1\}$. Define $l_1, \ldots, l_m$ such that

$$l_i = k_1 + (i-1)v \ , \ i = 1, \ldots, m.$$

For a given sequence $1 \leq \alpha_1 < \cdots < \alpha_t \leq m$, let $\lambda : B_{\mathbb{N}}(k_1, \ldots, k_t) \longrightarrow B_{\mathbb{N}}(l_{\alpha_1}, \ldots, l_{\alpha_t})$ be the map defined by

$$\lambda(Z) = 2 \cdot Z \cup \{2j + 1 : j = 1, \ldots, (\alpha_i - 1)v + k_1 - k_i\}$$

for every $Z \in B_{\mathbb{N}}$ with $|Z| = k_i$ and $i = 1, \ldots, t$. It follows that if $|Z| = k_i$ then $|\lambda(Z)| = l_{\alpha_i}$. Furthermore $Z \subseteq Z'$ if and only if $\lambda(Z) \subseteq \lambda(Z')$, since $\lambda(Z) \cap 2 \cdot \mathbb{N} = 2 \cdot Z$ for every Z and $(\alpha_i - 1)v + k_1 - k_i$ increases with i. □

Lemma 2.2. *For natural numbers r and $k_1 < \cdots < k_n$ there exist natural numbers $K_1 < \cdots < K_n$ such that for every r-colouring χ of the two-chains of $B_{\mathbb{N}}(K_1, \ldots, K_n)$ there exist maps*

$$\lambda : B_{\mathbb{N}}(k_1, \ldots, k_n) \longrightarrow B_{\mathbb{N}}(K_1, \ldots, K_n) \text{ and } \pi : \mathbb{N} \longrightarrow \mathbb{N}$$

where λ is an embedding and π is strictly increasing with the property that $\lambda(\sigma) \cap \pi(\mathbb{N}) = \pi(\sigma) := \{\pi(i) : i \in \sigma\}$ for $\sigma \in B_{\mathbb{N}}(k_1, \ldots, k_n)$, and such that for every $1 \leq i < j \leq n$ the two-chains of $\lambda(B_{\mathbb{N}}(k_i, k_j))$ are monochromatic.

Proof. The proof is by induction on n, the case $n = 1$ holding trivially. Assume the statement is true for $n = t - 1$ $(t \geq 2)$ and let r and $k_1 < \cdots < k_t$ be natural numbers. Let $K_1 < \cdots < K_{t-1}$ be natural numbers satisfying the conditions of the statement with respect to the numbers r and $k_1 < \cdots < k_{t-1}$.

Let $u = \binom{k_t}{k_{t-1}} K_{t-1}$. Then by repeated application of Ramsey's Theorem B [6] (the "finite version") there is a natural number K_t with the following property:

$P(K_t)$: For every r-colouring χ' of the elements of $B_{K_t}(K_1, \ldots, K_{t-1})$ there is a subset Y of $[K_t]$ with $|Y| = u$ such that for every $i = 1, 2, \ldots, t-1$ the set $\binom{Y}{K_i}$ is monochromatic under χ'.

Indeed, define the numbers $N_{t-1}, N_{t-2}, \ldots, N_0$ as follows:

$$N_{t-1} = u \qquad \text{and} \qquad N_j = R(K_{j+1}, N_{j+1}, r), \qquad j = 0, \ldots, t-2.$$

Finally, set $K_t = N_0$. Let χ' be any r-colouring of $B_{K_t}(K_1, \ldots, K_{t-1})$. Suppose, inductively, that for $1 \leq j \leq t-2$ a subset Y_j of $[K_t]$ with $|Y_j| = N_j$ exists such that for every $i \in [j]$ the colour χ' is constant on $\binom{Y_j}{K_i}$. Consider the restriction of χ' to $\binom{Y_j}{K_{j+1}}$. By the definition of N_j there exists a subset Y_{j+1} of Y_j with $|Y_{j+1}| = N_{j+1}$ such that χ' is constant on $\binom{Y_{j+1}}{K_{j+1}}$. Since Y_{j+1} is contained in Y_j, it follows from the induction hypothesis that for every $i \in [j+1]$ the colour χ' is constant on $\binom{Y_{j+1}}{K_i}$. The property $P(K_t)$ is witnessed by $Y = Y_{t-1}$.

In order to prove that the numbers $K_1, \ldots, K_t$ satisfy the conditions of the statement of the lemma with respect to the given numbers r and $k_1 < \cdots < k_t$, let χ be any r-colouring of the two-chains of $B_{\mathbb{N}}(K_1, \ldots, K_t)$. Let $\zeta' = \{x_1 < \cdots < x_{K_t}\}$ be a K_t-element subset of $\mathbb{N}$ and let χ' be the r-colouring of the elements of $B_{\zeta'}(K_1, \ldots, K_{t-1})$ induced by χ via

$$\chi'(\sigma) := \chi(\sigma \subset \zeta').$$

By the property $P(K_t)$ there is a subset $\tau'_{\zeta'} = \{x_{\alpha_{\zeta'}(1)} < \cdots < x_{\alpha_{\zeta'}(u)}\}$ and colours $c_{\zeta',i}$ such that $\binom{\tau'_{\zeta'}}{K_i}$ has colour $c_{\zeta',i}$ under χ'. Since there are only finitely many (say a) increasing functions $\alpha : \{1, \ldots, u\} \longrightarrow \{1, \ldots, K_t\}$, the function $\Omega : \zeta' \mapsto (\alpha_{\zeta'}(\cdot), c_{\zeta',1}, \ldots, c_{\zeta',t-1})$ is an ar^{t-1}-colouring of $\binom{\mathbb{N}}{K_t}$. By Ramsey's Theorem A [6] (the "infinite version") there is an infinite set $\mu = \{m_1 < m_2 < \ldots\}$ such that $\binom{\mu}{K_t}$ is Ω-monochromatic with colour

$(\alpha, c_1, \ldots, c_{t-1})$, say. This implies that for every $\zeta = \{m_{z_1}, \ldots, m_{z_{K_t}}\} \in \binom{\mu}{K_t}$ the subset $\tau = \{m_{z_{\alpha(1)}} < \cdots < m_{z_{\alpha(u)}}\}$ is such that for all $J \in \binom{\tau}{K_i}$ we have $\chi(J \subset \zeta) = c_i$.

Now consider the restriction of χ to the two-chains of $B_{\mathbb{N}}(K_1, \ldots, K_{t-1})$. By the induction hypothesis (and since the bijection $i \mapsto m_{iK_t}$ between $\mathbb{N}$ and $\mu_{K_t} := \{m_{K_t} < m_{2K_t} < \ldots\}$ induces an isomorphism between $B_{\mathbb{N}}(K_1, \ldots, K_{t-1})$ and $B_{\mu_{K_t}}(K_1, \ldots, K_{t-1})$) there exist maps

$$\lambda : B_{\mathbb{N}}(k_1, \ldots, k_{t-1}) \longrightarrow B_{\mathbb{N}}(K_1, \ldots, K_{t-1}) \text{ and } \pi : \mathbb{N} \longrightarrow \mathbb{N}$$

satisfying the conditions of the statement of the lemma and such that $\lambda(B_{\mathbb{N}}(k_1, \ldots, k_{t-1})) \subset B_{\mu_{K_t}}(K_1, \ldots, K_{t-1})$ and $\pi(\mathbb{N}) \subset \mu_{K_t}$.

Now extend λ to $B_{\mathbb{N}}(k_1, \ldots, k_t)$. This extension will also be denoted by λ. For every $\gamma \in \binom{\mathbb{N}}{k_t}$ let $Y_\gamma := \bigcup_{\sigma \in \binom{\gamma}{k_{t-1}}} \lambda(\sigma)$. By the definition of u it follows that $|Y_\gamma| < u$, say $Y_\gamma = \{m_{y_1} < \cdots < m_{y_v}\}$ with $v < u$.

Let $\alpha(0) = 0$ and $\delta_i := \alpha(i+1) - \alpha(i) - 1$, $i = 0, 1, \ldots, v-1$, where $\alpha(1) < \cdots < \alpha(v) < \cdots < \alpha(u)$ are as defined above. For every $\gamma \in \binom{\mathbb{N}}{k_t}$ define

$$\begin{aligned}\lambda(\gamma) = \{m_1 < \cdots < m_{\delta_0} < m_{y_1} < m_{y_1+1} < \ldots \\ < m_{y_1+\delta_1} < m_{y_2} < m_{y_2+1} < \ldots \\ < m_{y_v} < m_{y_v+1} < \cdots < m_{y_v+K_t-\alpha(v)}\}.\end{aligned}$$

To show that the extended λ has the required properties, we only have to consider subsets and two-chains involving subsets of size k_t. If $\tau \subset \gamma$ is any two-chain in $B_{\mathbb{N}}(k_1, \ldots, k_t)$ with $|\gamma| = k_t$, then $\tau \subseteq \sigma \subset \gamma$ for some σ with $|\sigma| = k_{t-1}$, and therefore $\lambda(\tau) \subseteq \lambda(\sigma) \subseteq Y_\gamma \subset \lambda(\gamma)$. Thus λ is monotone. To complete the proof that λ is an embedding, let $\sigma \not\subseteq \gamma$. If $|\gamma| < k_t$, then $\lambda(\sigma) \not\subseteq \lambda(\gamma)$ by choice of λ. If $|\gamma| = k_t$ let $y \in \sigma \backslash \gamma$. Then $\pi(y) \in \pi(\sigma) = \lambda(\sigma) \cap \pi(\mathbb{N}) \subseteq \lambda(\sigma)$, while $\pi(y) \notin Y_\gamma$, i.e., $\pi(y) \notin \lambda(\gamma)$. Thus also in this case $\lambda(\sigma) \not\subseteq \lambda(\gamma)$ and λ is an embedding.

If $\gamma \in B_{\mathbb{N}}(k_t)$ and $Y_\gamma = \{m_{y_1} < \cdots < m_{y_v}\}$, it follows from the construction of $\lambda(\gamma)$ that m_{y_i} is the $\alpha(i)$th element in $\lambda(\gamma)$. Therefore, since the colour is independent of the underlying K_t-element set, the two-chains in $\lambda(B_{\mathbb{N}}(k_j, k_t))$ are monochromatic of colour c_j.

Finally we show that $\lambda(\gamma) \cap \pi(\mathbb{N}) = \pi(\gamma)$ for every $\gamma \in B_{\mathbb{N}}(k_1, \ldots, k_t)$ with $|\gamma| = k_t$. If $m_y \in \lambda(\gamma) \cap \pi(\mathbb{N})$ then $y \equiv 0 (\text{mod } K_t)$ and therefore $m_y \in Y_\gamma$ because if $m_x \in \lambda(\gamma) \setminus Y_\gamma$ then $x \not\equiv 0 (\text{mod } K_t)$. Consequently for some

$\sigma \subset \gamma$ with $|\sigma| = k_{t-1}$, we have that $m_y \in \lambda(\sigma) \cap \pi(\mathbb{N}) = \pi(\sigma) \subset \pi(\gamma)$, i.e. $\lambda(\gamma) \cap \pi(\mathbb{N}) \subseteq \pi(\gamma)$. Conversely, let $m_y \in \pi(\gamma)$ and σ any subset of γ with $m_y \in \pi(\sigma)$ and $|\sigma| = k_{t-1}$. Then $m_y \in \pi(\sigma) = \lambda(\sigma) \cap \pi(\mathbb{N}) \subset \lambda(\gamma) \cap \pi(\mathbb{N})$. □

Proof of Theorem 2. For point-colourings ($s = 1$) let $m = R(1, t, r) = r(t-1) + 1$. For $k_1 < \cdots < k_t$ find $K_1 < \cdots < K_m$ satisfying the conditions of Proposition 2.1. Let χ be an r-colouring of the 1-chains of $B_{\mathbb{N}}(K_1, \ldots, K_m)$. Use Ramsey's theorem A to find an infinite subset β_1 of $\mathbb{N}$ such that all K_1-element subsets of β_1 have the same colour with respect to χ. Inductively find infinite sets $\beta_j \subseteq \beta_{j-1}$ such that all K_j-element subsets of β_j have the same colour with respect to χ. By the pigeonhole principle there are natural numbers $K_{\alpha_1} < \cdots < K_{\alpha_t}$ such that all K_{α_i}-element subsets of β_m have the same colour with respect to χ. By Proposition 2.1 the poset $B_{\mathbb{N}}(k_1, \ldots, k_t)$ can be embedded into $B_{\beta_m}(K_{\alpha_1}, \ldots, K_{\alpha_t})$ and we are done.

For two-chain colourings ($s = 2$) let $m = R(2, t, r)$ and let the natural numbers $L_1 < \cdots < L_m$ satisfy the conditions of Proposition 2.1 with respect to the given numbers $k_1 < \cdots < k_t$ and let $K_1 < \cdots < K_m$ satisfy the conditions of Lemma 2.2 with respect to the numbers r and $L_1 < \cdots < L_m$.

Let χ be any r-colouring of the two-chains of $B_{\mathbb{N}}(K_1, \ldots, K_m)$. Then there exists a (level-preserving) embedding $\lambda' : B_{\mathbb{N}}(L_1, \ldots, L_m) \longrightarrow B_{\mathbb{N}}(K_1, \ldots, K_m)$ such that for every $i, j \in [m]$ with $i < j$, the two-chains of $\lambda'(B_{\mathbb{N}}(L_i, L_j))$ are monochromatic with respect to χ. By Ramsey's theorem B there exist t numbers $1 \leq \alpha_1 < \cdots < \alpha_t \leq m$ such that the two-chains in $\lambda'(B_{\mathbb{N}}(L_{\alpha_1}, \ldots, L_{\alpha_t}))$ are monochromatic with respect to χ. If $\lambda'' : B_{\mathbb{N}}(k_1, \ldots, k_t) \longrightarrow B_{\mathbb{N}}(L_{\alpha_1}, \ldots, L_{\alpha_t})$ is a (level-preserving) embedding, then $\lambda = \lambda' \circ \lambda'' : B_{\mathbb{N}}(k_1, \ldots, k_t) \longrightarrow B_{\mathbb{N}}(K_1, \ldots, K_m)$ is an embedding with the property that the two-chains in $\lambda(B_{\mathbb{N}}(k_1, \ldots, k_t))$ are monochromatic with respect to χ. □

3. Proof of Theorem 1

The following proposition is an infinitary analogue of Lemma 2.1 of [2].

Proposition 3.1. *For every countable bounded principal ideal poset P of height t there are natural numbers $k_1 < \cdots < k_t$ and a level-preserving embedding $\lambda : P \longrightarrow B_{\mathbb{N}}(k_1, \ldots, k_t)$.*

Proof. Let π be an injection (not necessarily order-preserving) from P into the odd numbers. Let

$$\lambda'(p) := \pi([p]) = \pi(\{q \in P : q \leq p\}).$$

Let $j_u := \max\{|\lambda'(p)| : \ell(p) = u\}$ and let $k_i := \sum_{u=1}^{i} j_u$ where $i = 1, \ldots, t$. For p with $\ell(p) = i$, let

$$\lambda(p) := \lambda'(p) \cup 2 \cdot \{1, \ldots, k_i - |\lambda'(p)|\}.$$

Now if $p < q$, then $\lambda'(p) \subseteq \lambda'(q)$ and $\ell(p) < \ell(q)$, which in turn implies

$$k_{\ell(p)} \leq k_{\ell(q)} - j_{\ell(q)} \leq k_{\ell(q)} - |\lambda'(q)|.$$

Thus $\lambda(p) \subseteq \lambda(q)$. On the other hand if $p \not\leq q$, then $\pi(p) \in \lambda(p)$ but $\pi(p) \notin \lambda(q)$ and $\lambda(p) \not\subseteq \lambda(q)$. Thus λ is an embedding into $B_{\mathbb{N}}(k_1, \ldots, k_t)$ and since $\lambda(p)$ has $k_{\ell(p)}$ elements, λ is level-preserving. □

Proof of Theorem 1. Let P be any countable bounded principal ideal poset of height t. By Proposition 3.1 there is a level-preserving embedding $\lambda : P \longrightarrow B_{\mathbb{N}}(k_1, \ldots, k_t)$ for natural numbers $k_1 < \cdots < k_t$. By Theorem 2 there exist natural numbers $K_1 < \cdots < K_m$ with $m = R(2, t, r)$ such that $B_{\mathbb{N}}(K_1, \ldots, K_m) \underset{(2,r)}{\longmapsto} B_{\mathbb{N}}(k_1, \ldots, k_t)$. Hence $B_{\mathbb{N}}(K_1, \ldots, K_m) \underset{(2,r)}{\longmapsto} P$. □

4. Applications

Let V be the vector space of countably infinite dimension over the finite field $\mathbb{F}_q$, and $L_{\mathbb{N}}(q)$ the poset of the finite-dimensional subspaces of V.

If $k_1, \ldots, k_t$ are natural numbers with $k_1 < \cdots < k_t$, define the countable bounded principal ideal poset $L_{\mathbb{N}}(q)(k_1, \ldots, k_t)$ to be the induced subposet of $L_{\mathbb{N}}(q)$ consisting of the subspaces of $L_{\mathbb{N}}(q)$ with dimension k_i, $i = 1, \ldots, t$.

Proposition 4.1. *For natural numbers $r, s, k_1, \ldots, k_t$ with $s \in \{1, 2\}$ and $k_1 < \cdots < k_t$, there exist natural numbers $K_1 < \cdots < K_m$ with $m = R(s, t, r)$ such that $L_{\mathbb{N}}(q)(K_1, \ldots, K_m) \underset{(s,r)}{\longmapsto} L_{\mathbb{N}}(q)(k_1, \ldots, k_t)$.*

Proof. By Proposition3.1 there exist natural numbers $k'_1 < \cdots < k'_t$ such that $L_{\mathbb{N}}(q)(k_1, \ldots, k_t)$ can be embedded into $B_{\mathbb{N}}(k'_1, \ldots, k'_t)$. By Theorem 2 there exist natural numbers $K_1 < \cdots < K_m$ with m as required and such that $B_{\mathbb{N}}(K_1, \ldots, K_m) \underset{(s,r)}{\longmapsto} B_{\mathbb{N}}(k'_1, \ldots, k'_t)$. Finally, $B_{\mathbb{N}}(K_1, \ldots, K_m)$ can be embedded into $L_{\mathbb{N}}(q)(K_1, \ldots, K_m)$: Let $v_1, v_2, \ldots$ be any fixed basis of V. For $\sigma = \{s_1, \ldots, s_i\} \in B_{\mathbb{N}}(K_1, \ldots, K_m)$ define a poset embedding $\mu : B_{\mathbb{N}}(K_1, \ldots, K_m) \longrightarrow L_{\mathbb{N}}(q)(K_1, \ldots, K_m)$ by letting $\mu(\sigma)$ be the subspace of V generated by $v_{s_1}, \ldots, v_{s_i}$. □

Let $\Pi_{\mathbb{N}}$ be the poset consisting of all partitions of $\mathbb{N}$ into finite blocks, where almost all blocks are singletons, with the usual refinement relation (see, for example, [1], p.13).

For $b \in \Pi_{\mathbb{N}}$ the notation $b = b_1|\dots|b_t$ implies that blocks $b_1, \dots, b_t$ are the only blocks having cardinality greater than one. Blocks with cardinality one are omitted.

For any $b \in \Pi_{\mathbb{N}}$ and $b = b_1|\dots|b_t$, let $r(b)$, the *rank* of b, be defined by

$$r(b) = |b_1| + \cdots + |b_t| - t.$$

Lemma 4.2. *$\Pi_{\mathbb{N}}$ is a ranked poset with rank function r.*

Proof. For $b = b_1|\dots|b_t \in \Pi_{\mathbb{N}}$, let $B = b_1 \cup \dots \cup b_t$, and let Π_B be the poset consisting of all partitions of B. (Note that Π_B is isomorphic to $\Pi_{|B|}$.) Then Π_B can be embedded into $\Pi_{\mathbb{N}}$ (by appending $\mathbb{N} \backslash B$ as singleton blocks to every element of Π_B). Every saturated chain with b as maximal element in $\Pi_{\mathbb{N}}$ is the image of a unique saturated chain in Π_B with $b = b_1|\dots|b_t$ as maximal element. Since Π_B is a ranked poset [1], every saturated chain from ϵ to any fixed $b \in \Pi_{\mathbb{N}}$ is of the same length where ϵ is the partition of $\mathbb{N}$ consisting only of singleton blocks. $\square$

For natural numbers $k_1 < \cdots < k_t$, let $\Pi_{\mathbb{N}}(k_1, \dots, k_t)$ denote the induced sub-poset of $\Pi_{\mathbb{N}}$ consisting of all partitions of $\mathbb{N}$ into finite blocks with rank $k_i, i = 1, \dots, t$. Obviously, $\Pi_{\mathbb{N}}(k_1, \dots, k_t)$ is a countable bounded principal ideal poset.

Proposition 4.3. *For natural numbers $r, s, k_1, \dots, k_t$ with $s \in \{1, 2\}$ and $k_1 < \cdots < k_t$, there exist natural numbers $K_1 < \cdots < K_m$ with $m = R(s, t, r)$ such that $\Pi_{\mathbb{N}}(K_1, \dots, K_m) \underset{(s,r)}{\longmapsto} \Pi_{\mathbb{N}}(k_1, \dots, k_t)$.*

Proof. The proof is analogous to that of Proposition4.1 where we now use a poset embedding $\mu : B_{\mathbb{N}}(K_1, \dots, K_m) \longrightarrow \Pi_{\mathbb{N}}(K_1, \dots, K_m)$. For this purpose, we identify $B_{\mathbb{N}}(K_1, \dots, K_m)$ with $B_{\{2n-1:n \in \mathbb{N}\}}(K_1, \dots, K_m)$ via $\sigma \mapsto \{2n - 1 : n \in \sigma\}$. For $\sigma = \{s_1, \dots, s_i\} \in B_{\mathbb{N}}(K_1, \dots, K_m)$ we define $\mu(\sigma) \in \Pi_{\mathbb{N}}(K_1, \dots, K_m)$ by $\mu(\sigma) = s_1\ 2s_1|s_2\ 2s_2|\dots|s_i\ 2s_i$. This defines a poset embedding which is rank-preserving. $\square$

As an application of the construction in Proposition 3.1 we prove the following result.

Proposition 4.4. *For every natural number r, $\Pi_{\mathbb{N}} \underset{(1,r)}{\longmapsto} \Pi_{\mathbb{N}}$.*

Proof. Let $\chi : \Pi_{\mathbb{N}} \longrightarrow [r]$ be any r-colouring of the points of $\Pi_{\mathbb{N}}$. By the construction in the proof of Proposition4.3, there is an embedding $\alpha_n : B_{\mathbb{N}}(1,\ldots,n) \longrightarrow \Pi_{\mathbb{N}}(1,\ldots,n)$ such that α_{n+1} is an extension of α_n for every n. This means that there exists an embedding $\alpha : B_{\mathbb{N}} \longrightarrow \Pi_{\mathbb{N}}$ which is an extension of α_n for every n.

By Hindman's Finite Unions Theorem ([3], Corollary 3.3) there exists an embedding $\beta : B_{\mathbb{N}} \longrightarrow B_{\mathbb{N}}$ such that $\alpha\beta(B_{\mathbb{N}})$ is monochromatic under χ.

Let $\pi : \Pi_{\mathbb{N}} \longrightarrow \mathbb{N}$ be any injective map with $\pi(b)$ odd for every $b \in \Pi_{\mathbb{N}}$ and π_n the restriction of π to $\Pi_{\mathbb{N}}(1,\ldots,n)$ for every natural number n. By the construction in the proof of Proposition3.1, for every natural number n there exist natural numbers $k_1 < \cdots < k_n$ and an embedding $\gamma_n : \Pi_{\mathbb{N}}(1,\ldots,n) \longrightarrow B_{\mathbb{N}}(k_1,\ldots,k_n)$ such that γ_{n+1} is an extension of γ_n for every n. Therefore there exists an embedding $\gamma : \Pi_{\mathbb{N}} \longrightarrow B_{\mathbb{N}}$ which is an extension of γ_n for every n. It follows that $\alpha\beta\gamma : \Pi_{\mathbb{N}} \longrightarrow \Pi_{\mathbb{N}}$ is an embedding such that $\alpha\beta\gamma(\Pi_{\mathbb{N}})$ is monochromatic under χ. □

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