ON MINIMIZING THE EFFECTS OF FIRE OR A VIRUS ON A NETWORK

S. Finbow, B. Hartnell, Q. Li and K. Schmeisser
Saint Mary's University, Halifax, Canada

Dedicated to Professor E. J. Cockayne on the occasion of his 60th birthday

ABSTRACT. The following problem was introduced at a conference in 1995. Fires start at F nodes of a graph and D defenders (firefighters) then protect D nodes not yet on fire. Then the fires spread to any neighbouring unprotected nodes. The fires and the firefighters take turns until the fires can no longer spread. We examine two cases: when the fires erupt at random and when they start at a set of nodes which allows the fires to maximize the damage. In the random situation, for a given number of nodes, we characterize the graphs which minimize the damage when D = F = 1 and we show that the star is an optimal graph for D = 1 regardless of the value of F. In the latter case, optimal graphs are given whenever D is at least as large as F.

1. Introduction

The subject of this paper was originally introduced at a conference [2] in 1995. In its simplest form the problem is the following attempt to model firefighting or virus control on a network. A fire (or virus) breaks out at a vertex of a graph. The fire and the firefighter then alternate moves on the graph. A move for the firefighter (defender) is to choose any vertex not yet on fire (affected by the virus). Once a vertex has been chosen by the firefighter it is considered protected or safe from any future moves of the fire. After the firefighter's move, the fire makes its move by spreading to all vertices which are adjacent to the ones on fire, except for those that are protected. The firefighter and the fire alternate taking turns until the fire can no longer spread. The object of the firefighter is to have the fire damage as few of the vertices of the graph as possible where the damage is the total number of vertices that have been set on fire by the time the fire can no longer spread. The general problem is for F fires to start at a set of F vertices of the graph and then D defenders or firefighters to each choose a vertex not yet affected by the fires to protect. As before the fires and defenders alternate moves until the fires can spread no further. Our objective is to consider two situations. The first is when the fires erupt or break out at random and the other is when they start at vertices which allow the fires to maximize the damage (measured as the number of vertices eventually set on fire) to the graph. In both cases, we wish to determine which graphs minimize the damage. That is, we are interested in the design of optimal graphs. In related work, MacGillivray and Wang, in [3], considered actual algorithms for the firefighter in order to keep the damage to a minimum. Not surprisingly, for a general graph the problem was shown to be NP-complete even for F=D=1. In [4], the problem was considered when the graph was a grid.

A related problem that was considered in 1978 by Gunther and Hartnell [1] turns out to be very helpful. This problem was posed in terms of an underground resistance movement and addressed the question of how to establish a communications network among the members which minimizes the effects of treachery or subversion of a particular member or members, followed by the consequent betrayal of other members. The resistance movement is modeled as a graph where the vertices represent the individual members of the movement and the edges represent the lines of communication between various members. If a vertex is subverted then any neighbours are betrayed. In section 4 the relevance of this problem to the current paper will be explained.

In what follows, for a vertex v of a graph G, N[v] will represent the set consisting of the vertex v and all vertices that are adjacent to v whereas N(v) will be N[v] – {v}. Similarly, $N_2[v]$ is the set of vertices consisting of N[v] as well as any vertex adjacent to a vertex of N[v] and $N_2(v)$ is $N_2[v]$ – {v}. We shall use deg(v) to represent the number of vertices adjacent to v and d(u, v) to be the length of a shortest path between vertices u and v. A leaf is a vertex of degree one and its unique neighbour is called a stem.

In the next section we consider the fires starting at random and in the following section the fires start at the set of vertices which allow maximum damage to the graph.

2. Random Fires

In this section, we will consider fires starting at random on a graph. Given a graph on p vertices and F fires, we will assume that each of the $\binom{P}{F}$ possible subsets of F vertices are equally likely to be chosen by the fires. The problem is to determine which graphs are optimal given p vertices, F fires and D defenders. A graph G is optimal if the sum, over all subsets X of size F of V(G) of the number of vertices damaged when the fires break out at X is minimum over all graphs with the same number of vertices as G. That is, which graphs allow the defenders to minimize the number of vertices destroyed. Our first result characterizes the graphs which achieve this minimum in the special case that there is exactly one fire and one defender.

Theorem 2.1 For p vertices and F = D = 1, where the fire starts at random, a connected graph G on p vertices is optimal if and only if G is a tree where (i) each vertex of G has at most 2 neighbours of degree greater than one and (ii) each vertex has at most one neighbour of degree more than two.

Proof: For any graph G on p vertices, since the fire is equally likely to start at any vertex, the expected damage is at least as large as $\sum_{v \in V(G)} (\deg(v) + 1 - 1)/p = \frac{2|E(G)|}{p}$, since the fire would destroy all the neighbours except for one that the defender could save. Also we note that G must be a tree in order to minimize the number of edges and hence the damage. Observe that if G is a tree and satisfies conditions (i) and (ii) then the damage on G can be limited to this minimum. In particular, say the fire starts on a vertex v with at most two neighbours of degree greater than one, say u_1 and u_2, where at most one of these, say u_2, has degree more than two. The defender should first choose u_2 and then, on the next move, the unique neighbour of u_1 not yet affected by the fire. This strategy establishes the if part of the theorem. Next note that if some vertex, say v, of G had at least 3 neighbours of degree greater than one, then the fire would spread on its third move to at least one vertex in $N_2[v] - N[v]$ regardless of where the defender had moved. Similarly, if v had more than one neighbour, say u_1 and u_2, of degree more than two, then the defender can protect one of these, say u_1, on the first move. But then on the second move, when the fire has spread to u_2, the defender cannot protect all of the neighbours of u_2. Thus the only if direction follows. □

We next consider an arbitrary number of fires where there is only one defender.

Theorem 2.2 For p vertices and $F \geq D = 1$, where the fires start at random, a star on p vertices is optimal.

Proof: First note that the expected damage on the star with p = k vertices, where there are F fires, is $\left[F\binom{k-1}{F} + (k-1)\binom{k-1}{F-1}\right] \div \binom{k}{F}$ since either the F fires start only on leaves (resulting in a total of F vertices on fire) or one of the fires start at the centre of the star (resulting in a total damage of k − 1). We proceed by induction on p and F. By Theorem 2.1 a star is optimal on p vertices with F = D = 1. Assume the result holds for all F with $1 \leq F < f$ and consider F = f. The result certainly holds for all $p \leq F + D$. Assume it is true for all $p \leq k$ and consider p = k + 1.

Given a graph G on k + 1 vertices, find any spanning tree, say T, of G and let x be a leaf of T with N(x) = y in T. The expected damage from F fires on G is at least as large as the expected damage from F fires on T. The expected damage from F fires on T is at least the expected damage from F fires on T – x, plus a damage of 1 every time a fire is started at y but not x (keeping in mind the observation that defenders should choose non-leaf vertices) plus the expected damage from F – 1 fires on T – x, plus 1 for every time F – 1 fires are started on T – x (since this implies a fire was also started at x). By the induction hypothesis this means the sum of the damage over all possible subsets of vertices that the F fires start at is

$$\geq F\binom{k-1}{F} + (k-1)\binom{k-1}{F-1} + \binom{k-1}{F-1}$$

$$+ (F-1)\binom{k-1}{F-1} + (k-1)\binom{k-1}{F-2} + \binom{k}{F-1}$$

$$= F\left[\binom{k-1}{F} + \binom{k-1}{F-1}\right] + (k-1)\left[\binom{k-1}{F-1} + \binom{k-1}{F-2}\right] + \binom{k}{F-1}$$

$$= F\binom{k}{F} + (k-1)\binom{k}{F-1} + \binom{k}{F-1}$$

$$= F\binom{k}{F} + k\binom{k}{F-1}$$

Thus the expected damage for the graph G is at least as large as for the star and the result holds for all p by induction. Since it is true for F whenever it is valid for F – 1 and smaller, it follows by induction that the result is valid for all F. □

3. Worst Case : F = D for small values

We now consider the situation where, for a given graph and F fires, the fires start at a set of vertices which will result in the most damage. For a given number of vertices we wish to determine which connected graphs minimize the damage. That is, we wish to determine, for each p, F and D, the connected graphs on p vertices defended by D firefighters, for which the maximum amount of damage caused by F fires is minimized.

LEMMA 3.1 The path or cycle on p vertices are the unique optimal connected graphs for $F = D = 1$.

Proof: Any other connected graph has a vertex of degree 3 or more. If the fire started at such a vertex it would destroy at least 3 vertices. But wherever the fire starts on the path or cycle the defender can protect a neighbour and on its second move stop the fire thus limiting the damage to 2. □

We next examine the case when $F = 2 = D$. For G a connected graph, let B(G) be the maximum number of vertices that can be damaged when two fires are started, and two defenders defend optimally.

LEMMA 3.2 If G is a tree and there exist distinct vertices $u, v \in V(G)$ such that $\deg(u) + \deg(v) \geq 8 - d(u, v)$, and $d(u, v) \leq 3$, then $B(G) \geq 5$.

Proof: Note that if the fires start at u and v, then the damage would be at least the vertices u and v, plus all their neighbours, minus the neighbours they have in common, minus the two vertices defended. That is to say: $B(G) \geq 2 + \deg(u) + \deg(v) - | N[u] \cap N[v] | - 2$. But, in a tree, $| N[u] \cap N[v] | = 3 - d(u, v)$ when $d(u, v) \leq 3$, so we have $B(G) \geq \deg(u) + \deg(v) - 3 + d(u,v)$ implying $B(G) \geq 5$.□

THEOREM 3.3 Let G be a connected graph. If $| V(G) | \geq 10$, then $B(G) \geq 4$. Furthermore, if $| V(G) | = 10$, then $B(G)=4$ if and only if G is one of the graphs shown in Figure 1 and, for $| V(G) | \geq 11$, $B(G) = 4$ if and only if G is a path or cycle.

Proof: It is easy to verify that if G is a path or cycle and $| V(G) | \geq 6$, then $B(G) = 4$.

Now consider the only if direction.

Case 1: If $v \in V(G)$ and $\deg(v) \geq 5$ then either G has a star as a spanning tree or v must have a neighbour of degree two or more (and there is a spanning tree with this property). In either case it follows that B(G) is at least 5 by Lemma 3.2.

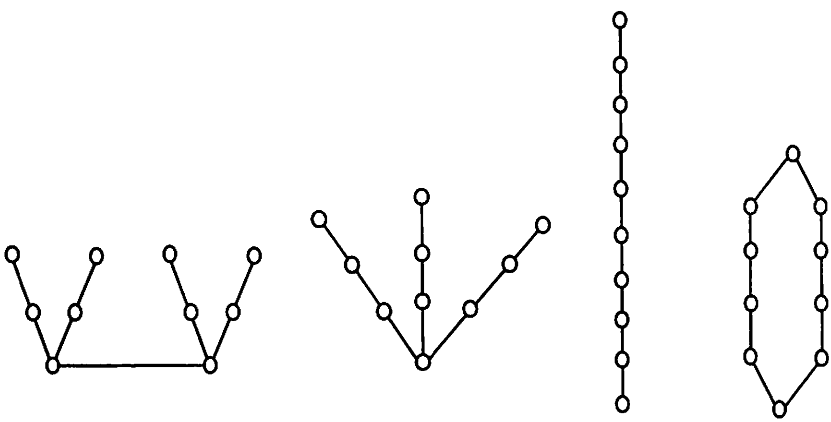
Case 2: If $v \in V(G)$ and $\deg(v) = 4$ then we have $N(v) = \{u_1, u_2, u_3, u_4\}$. Either the result holds or from Lemma 3.2, considering the corresponding spanning tree, we must have $\deg(u_i) \leq 2$, for i=1,2,3,4 and any neighbours of the u_i's (other than v) must be of degree 1. This is a contradiction since it implies $| V(G) | \leq 9$.

Case 3: If $v \in V(G)$ and $deg(v) = 3$ then we have $N(v) = \{u_1, u_2, u_3\}$. Either the result holds or from Lemma 3.2 (using a spanning tree) we have $deg(u_i) \le 3$. Furthermore we observe that at most one of the u_i's can be of degree three. In the following cases we use a spanning tree with the given degrees so as to utilize Lemma 3.2.

Case 3a: Say $deg(u_1) = 3$. Then either the result holds or, from Lemma 3.2, u_2 and u_3 are of degree at most two, and any neighbours of u_2 and u_3, other than v, must be of degree one. By symmetry u_1 has neighbours with the same restrictions as the neighbours of v. But this would then force $|V(G)| \le 10$. For equality we must have H_1 as shown in Figure 1.

Case 3b: If $deg(u_i) = 2$, then $N(u_i) \subseteq \{v, x_i\}$. Then either the result holds or, from Lemma 3.2, $deg(x_i) \le 2$. If $deg(x_i)=2$, then $N(x_i) \subseteq \{u_i, y_i\}$. Also from the lemma, $deg(y_i) = 1$ or the result holds. So we can conclude that $B(G) \le 4$ means $|V(G)| \le 10$. For equality we must have H_2 as shown in Figure 1.

By the above, we are left with the case of the maximum degree equal to 2. This concludes the proof. □



H_1 H_2 P_{10} C_{10}

Figure 1

4. Related Results

As indicated in the introduction, the results from [1] are useful. We give a brief explanation of their relevance in this section. The problem is, given two natural numbers b and p, to find a connected graph G of order p in which the maximum number of betrayals resulting from b subversions is minimum and, to find a formula for this number. In particular, given a graph G on p vertices and an integer b, the expression K(G, b) is defined to be the maximum $\{|N[S]|\}$ where the maximum is taken over all possible subsets S of b vertices. The expression K(p, b) is defined to be the minimum {K(G, b)} where the minimum is taken over all connected graphs on p vertices.

The significance of this result for our present problem is that it gives a lower bound on the damage by F fires when there are D defenders. That is, we know that regardless of the set S of vertices where the F fires start, for $|S| = b = F$, and any connected graph on p vertices, that at least K(p,b) – D vertices must be destroyed regardless of the choice of graph. Of course, some of the fires may continue to spread after the D defenders have made their first two moves increasing this number. In general, the formula for K(p, b) is given by the following theorem. The expression [x] is the greatest integer function.

Theorem 4.1 [1]

For a fixed p and b the following hold:

(i) $K(p, b) = p$, for $p \leq 2b + 1$.

(ii) For $b \leq 3$, $K(p, b) = 3b$ for $p \geq 4b$.
For $b \geq 4$, $K(p, b) = 3b$ for $p \geq 5b - 4$.

(iii) CASE 1: $b = 3t + 1$.

(a) $K(p, b) = (6t + 3) + (a - 1)$ where $p = (6t + 3) + (2a - 1)$ or $p =(6t + 3) + (2a)$ and $1 \leq a \leq 2t + 1$.

(b) $K(p, b) = (8t + 4) + [x/5]$ where $p = 10t + 6 + x$ and $0 \leq x < 5t - 5$.

CASE 2: $b = 3t + 2$.

(a) $K(p, b) = (6t + 5) + (a - 1)$ where $p = (6t + 5) + (2a - 1)$ or $p = (6t + 5) + (2a)$ and $1 \leq a \leq 2t + 2$.

(b) $K(10t + 10, b) = 8t + 6$.

(c) $K(p, b) = (8t + 7) + [x/5]$ where $p = 10t + 11 + x$ and $0 \leq x < 5t - 5$.

CASE 3: $b = 3t + 3$.

(a) $K(p, b) = (6t + 7) + (a - 1)$ where $p = (6t + 7) + (2a - 1)$ or $p = (6t + 7) + (2a)$ and $1 \leq a \leq 2t + 3$.

(b) $K(10t + 14, b) = K(10t + 15, b) = 8t + 9$.
(c) $K(p, b) = (8t + 10) + [x/5]$ where $p=10t + 16 + x$ and $0 \le x < 5t - 5$.

It should be noted that there are three types of trees, see Figure 2, that play a special role in [1]. One of them is the path on n vertices, denoted by P_n, while the other two are called the superstar and the fifth column. In [1] it is shown that the path achieves the values given in (ii) of Theorem 4.1. The superstar on 2n + 1 vertices, denoted by S_n, is the tree formed by taking a star with n leaves and attaching a leaf to each of the original leaves. The tree S'_n is on 2n + 2 vertices and is formed by attaching one more leaf to S_n where that leaf is attached to the central vertex of degree n. These graphs are shown in [1] to attain the values given in part (a) of each of the three cases of (iii) of Theorem 4.1. The fifth column, denoted by F_n, consists of n P_5's, where the centre vertex of each P_5 is joined to the centre vertex of an adjacent P_5 in such a manner that these n vertices form a path. In [1], it is observed that in addition to the optimal trees on 5n and 5n + 5 vertices (namely, F_n and F_{n+1}) one can form intermediate trees on 5n + x vertices, for $1 \le x \le 4$, by taking n P_5's and one additional path of the appropriate length. These are the graphs that achieve the values shown in parts (b) and (c) of the cases of (iii) in Theorem 4.1. As a specific example, in [1] it is shown that for p = 7 and b = 2 that the graph on 7 vertices which minimizes | N[S] |, regardless of the choice of the 2 vertices in S, is S_3. That is, | N[S] | is 5. Thus we can conclude that regardless of which graph on 7 vertices two fires start on, that the very best one could do with D firefighters is to restrict the damage to 5 – D. Since we can manage to contain the fires to vertices in N[S] in this example, we have achieved the optimal graph for p = 7 and b = 2. Observe that if the graph on 7 vertices had been the path, that | N[S] | would be 6 and the damage would be 6 – D.

In general, (i) of Theorem 4.1 [1] shows that if $p \le 2F + 1$, (noting that the F fires play the role of the b vertices in S) then regardless of how we design a connected graph on p vertices, all vertices of the graph are in N[S] for some choice of S where S has F members. Hence, the damage will be p – D regardless of the choice of graph.

At the other extreme (see (ii) of Theorem 4.1), for $F \le 3$ and $p \ge 4F$, or $F \ge 4$ and $p \ge 5F - 4$, then, if $D \ge F$, we can do no better than by taking the path with a resulting damage of 3F – D. Since indeed the fires can be contained to N[S], the path is optimal.

The more interesting cases depend on which modulo class mod 3 the number F is as indicated by Theorem 4.1.

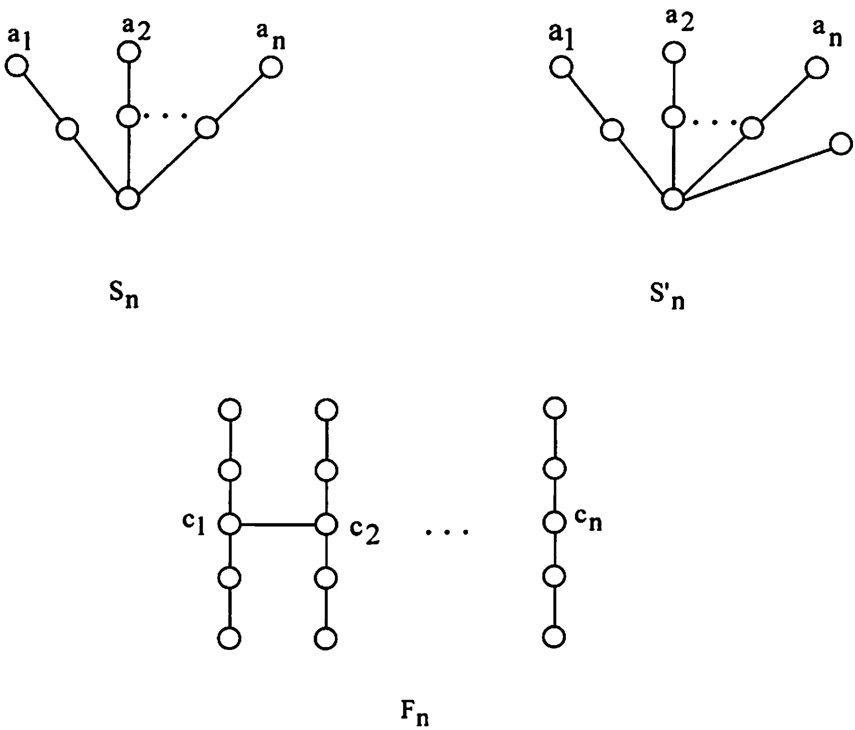


Figure 2

5. Worst Case: general F and D

The following lemma is useful in what follows.

Lemma 5.1 Given a graph G and F fires we may assume the fires do not start on leaves unless there is no other option.

Proof: If a fire started at a leaf, it would do at least as much damage by starting at the corresponding stem. □

Lemma 5.2 If S_m is a superstar on $2m + 1$ vertices, $F < m$, $D + F \leq 2m + 1$ and $D \geq \left\lceil \frac{m-(F-1)}{2} \right\rceil$, then the maximum damage caused to this graph by F fires when there are D defenders is $|N[F]| - D$.

Proof: By Lemma 5.1 we may assume that the fires begin only at stems or at the centre and stems. In the former case, using one defender at the centre and the rest on leaves corresponding to the stems chosen by the fires the damage is restricted to $| N[F] | - D$. In the latter situation, assume the fires start at the centre and stems $S_1, S_2, \ldots, S_{F-1}$. The defenders would first choose D stems, say $S_F, S_{F+1}, \ldots, S_{F+D-1}$, not chosen by the $F - 1$ fires. On their second move, the defenders would choose the leaves $l_{F+D}, l_{F+D+1}, \ldots, l_m$. The result follows. □

Corollary 5.3 Let $F < m$, $D + F \leq 2m + 1$ and $D \geq \left\lceil \frac{m - (F-1)}{2} \right\rceil$.

(a) For $F \leq 3$ and $2F + 1 < p \leq 4F - 1$, for $p=2m + 1$, S_m is an optimal graph while for
$p = 2m + 2$ S'_m is optimal.

(b) For $F \geq 4$,
(1) if $F=3t + 1$ and $6t + 3 < p \leq 10t + 5$, or
(2) if $F=3t + 2$ and $6t + 5 < p \leq 10t + 9$, or
(3) if $F= 3t + 3$ and $6t + 7 < p \leq 10t + 13$, the superstar graph on p vertices is optimal.

Proof: By Theorem 4.1 we can do no better than restrict the damage to the number given by Case 1 (a), Case 2 (a) or Case 3 (a) of part (iii). Hence we have an optimal graph for these values of p and F. □

Lemma 5.4 If F_n is attacked by F fires and there are D defenders where $F + D \leq 5n$, $F > \left\lceil \frac{n}{5} \right\rceil$ and $D \geq \left\lceil \frac{n}{5} \right\rceil$, then the damage caused is limited to $| N[F] | - D$.

Proof: The strategy is as follows. Use 1 defender per column (unless there are no fires on that column and none on adjacent centres of columns). If the center vertex is not attacked directly by a fire, but a neighbour (perhaps from an adjacent column) is, then defend at the center vertex. If a fire attacks the centre, then defend at a neighbour on that column that is not on fire. If both stems of that column are on fire as well as the centre, then defend at a leaf. Any additional defenders will protect vertices adjacent to the fire. Observe that no fire can spread to another column. Furthermore each fire on a column is defended against on the first move of the defenders. The only possible way for a vertex in the second neighbourhood to be damaged is if the centre of that column is set on fire and a neighbouring stem is not. But in this situation the defender for that column can protect the leaf on its second move. □

Corollary 5.5 For $F \geq 4$ where $F > \left\lceil \frac{n}{5} \right\rceil$, $D \geq \left\lceil \frac{n}{5} \right\rceil$ and $F + D \leq 5n$, if

(a) $F=3t + 1$ and $10t + 6 \leq p \leq 15t$, or

(b) $F=3t + 2$ and $10t + 10 \leq p \leq 15t + 5$, or

(c) $F=3t + 3$ and $10t + 14 \leq p \leq 15t + 10$, the graph based on F_n is optimal.

Proof: By Theorem 4.1 we can do no better than restrict the damage to the number given in Case 1 (b), Case 2 (b) or (c) or Case (3) (b) or (c) of part (iii). Hence we have an optimal graph for these values of p and F. □

We summarize the above results in the following theorem.

Theorem 5.6 Given p vertices, F fires and D defenders, the trees on p vertices which will minimize the damage when F fires start at a set of vertices which will do the most damage, are as follows.

(i) If $p \leq 2F + 1$, then all trees on p vertices are optimal.

(ii) (a) For $F \leq 3$ and $2F + 1 < p \leq 4F - 1$, for $p=2m + 1$ S_m is an optimal graph while for $p=2m + 2$ S'_m is optimal for $D \geq \left\lceil \frac{m-(F-1)}{2} \right\rceil$.

(b) For $F \leq 3$ and $p \geq 4F$, then the path on p vertices is optimal.

(c) For $F \geq 4$ and $2F + 1 < p < 5F - 4$, there are three cases:

(1) $F=3t + 1$ (hence $6t + 3 < p < 15t + 1$): For $6t + 3 < p \leq 10t + 5$, the superstar graph on p vertices is optimal. For $10t + 6 \leq p \leq 15t$, the graph based on F_n is optimal.

(2) $F=3t + 2$ (hence $6t + 5 < p < 15t + 6$): For $6t + 5 < p \leq 10t + 9$, the superstar graph on p vertices is optimal. For $10t + 10 \leq p \leq 15t + 5$, the graph based on F_n is optimal.

(3) $F=3t + 3$ (hence $6t + 7 < p < 15t + 11$): For $6t + 7 < p \leq 10t + 13$, the superstar graph on p vertices is optimal. For $10t + 14 \leq p \leq 15t + 10$, the graph based on F_n is optimal.

(iii) For $F \geq 4$ and $p \geq 5F - 4$, the path on p vertices is optimal for $D \geq F$.

Acknowledgements: The second author would like to thank A. Hynick and G. Miller for their role in discussions in the early stages of working on this problem.

References

[1] G. Gunther and B. L. Hartnell, "*On Minimizing the effects of betrayals in a resistance movement*", Proc. Eighth Manitoba Conference on Numerical Math. and Computing, 1978, 285-306.

[2] B. L. Hartnell, "*Firefighter! An application of Domination*", presentation at the Twentieth Conference on Numerical Mathematics and Computing, held at the University of Manitoba (Sept., 1995).

[3] G. MacGillivray and P. Wang, "*On Network Virus Control*", submitted.

[4] P. Wang and S. A. Moeller, "*Fire Control on Graphs*", submitted.