Small Irredundance Numbers for Queens Graphs

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Abstract

We prove some general results on irredundant sets of queens on chessboards, and determine the irredundance numbers of the queens graph Q_n for $n = 5, 6$.

To Ernie, a dominator of fame,
For whom chess is much more than a game,
We dedicate this piece
And request do not cease
To endeavour irredundance to tame.

1 Introduction

The *queens graph* Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are *adjacent* if they are on the same *line*, that is, in the same row, column or diagonal. Note that a square of Q_n has four lines: its row, column and two diagonals. If a queen is placed on a square of Q_n, the *lines of the queen* are the lines of the square. A queen on square x of Q_n *covers* a square y if $x = y$ or x and y are adjacent. A set D of squares is a *dominating set* of Q_n if every square of Q_n is either in D or adjacent to a square in D, *i.e.*, if a set of queens, one on each square in D, covers the rest of the board. If no two squares of the dominating set D are adjacent, then D is an *independent dominating set.* If each queen on a set X of squares covers a square which is not covered by a queen on any other square in X, then X is an *irredundant set* of Q_n.

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The *domination number* $\gamma(Q_n)$ (*independent domination number* $i(Q_n)$, *irredundance number* $ir(Q_n)$) of Q_n is the minimum size amongst all dominating (independent dominating, maximal irredundant) sets of Q_n. It is easily seen that any minimal dominating set of a graph is maximal irredundant, and that $ir(Q_n) \leq \gamma(Q_n) \leq i(Q_n)$ for all n (see [12, p. 58]).

Domination on queens graphs has received considerable attention in the 1980s and 1990s. Recent upper bounds, for example, can be found in [4, 16], and other recent papers include [3, 5, 6]. Surveys on the queens domination problem and other combinatorial problems on chessboards are given in [7, 10, 13].

For some of the chessboard graphs, for example the rooks and bishops graphs, formulas for ir are easy to obtain (and are listed in [13]), while some small values of ir for kings graphs are also known [9]. However, apart from the trivial values $ir(Q_1) = ir(Q_2) = ir(Q_3) = 1$ and $ir(Q_4) = 2$, no other exact values of $ir(Q_n)$ are known. The only other results on irredundance in queens graphs concern the upper irredundance number $IR(Q_n)$, that is, the maximum cardinality amongst all irredundant sets of Q_n. Values of $IR(Q_n)$ for $n \leq 8$ are given in [13], while bounds can be found in [2, 14]. The best known lower bound for $IR(Q_n)$, which greatly improves all previously known lower bounds, was obtained by Kearse [14]. Kearse showed that $IR(Q_{k^3}) \geq 6k^3 - 29k^2 - O(k)$, and from this it follows that $IR(Q_n) \geq 6n - O(n^{\frac{2}{3}})$.

As was shown in [1], the irredundance number of any graph is bounded below by $ir(G) \geq (\gamma(G)+1)/2$. This bound, together with the lower bound $\gamma(Q_n) \geq (n-1)/2$ of P. Spencer, as cited in [7, 15], shows that $ir(Q_n) \geq (n+1)/4$.

In this paper we determine the irredundance numbers of the queens graphs Q_5 and Q_6. (This will show that the above-mentioned lower bound for $ir(Q_n)$ is not best possible.) These are the first missing values in the table given in [13], and we prove the results without using a computer in the hope that the methods may eventually be used to obtain more general results. (It should be possible to determine $ir(Q_n)$ for small values of n by computer.)

Definitions pertaining to domination not given here can be found in [12]. We need some further definitions. The *closed neighbourhood* $N[v]$ of the vertex v in a graph $G = (V, E)$ (or of a square on a chessboard) consists of v and the set of vertices adjacent to v. The *closed neighbourhood* of a set $S \subseteq V$ is defined by $N[S] = \cup_{v \in S} N[v]$, and the *private neighbourhood* of $v \in S$ by $pn(v, S) = N[v] - N[S - \{v\}]$. A vertex in $pn(v, S)$ is called a *private neighbour*, abbreviated to *pn*, of v. Note that a vertex can be its own private neighbour. Thus a set S of vertices is irredundant if for every vertex $v \in S$, v has at least one pn. An irredundant set S is *maximal irredundant* if for every vertex $u \in V - S$, the set $S \cup \{u\}$ is not irredundant, which means that there exists at least one vertex $w \in S \cup \{u\}$ which does not have a pn.

If a vertex u is added to a set S and it "destroys" all the pns of some vertex w in S (*i.e.*, $pn(w,S) \neq \phi$ and $pn(w, S \cup \{u\} = \phi)$, we call u an *annihilator*, and say that u *annihilates* w. For $U \subseteq V - S$ we say that S is *U-annihilated* if every $u \in U$ annihilates some $w \in S$.

If $u \in V - S$ has no pns with respect to $S \cup \{u\}$, *i.e.*, if $pn(u, S \cup \{u\}) = \phi$, we say u is *pn-free* with respect to S. We say a vertex v (or a square in the case of chessboards) is *open* (with respect to S) if it is not dominated by S. We denote the open vertices (squares) with respect to S by R_S or simply by R if confusion is unlikely. The following result gives a useful characterisation of those irredundant sets that are maximal irredundant.

Theorem 1 [8] *An irredundant set S of G is maximal irredundant if and only if for each $v \in N[R]$ there exists $s_v \in S$ such that $\phi \neq pn(s_v, S) \subseteq N[v]$, that is, S is $N[R]$-annihilated.*

The following simple results will be useful later.

Proposition 2 *If S is a maximal irredundant set in a graph G and $|S| < i(G)$, then S is not independent.*

Proof. Suppose to the contrary that S is independent. Since $|S| < i(G)$, S is a proper subset of some maximal independent set of G, that is, there exists a vertex v such that $S \cup \{v\}$ is independent. But then $v \in R_S$ and for any $s \in S$, $s \in pn(s, S \cup \{v\})$, *i.e.*, v does not annihilate any vertex in S, a contradiction. ■

Proposition 3 *Let S be a maximal irredundant set of G with $|S| = \gamma(G) - k$, where $k \geq 1$. Then there does not exist a set $Y \subseteq V - S$ with $|Y| \leq k$ such that Y dominates R.*

Proof. Suppose to the contrary that such a set $Y \neq \phi$ does exist. Then $X = S \cup Y$ dominates G and $|X| \leq \gamma(G)$, *i.e.*, $|X| = \gamma(G)$. Therefore X is a minimal dominating set of G containing S as proper subset. But then X is maximal irredundant, contradicting the maximality of S. ■

2 Irredundance in the queens graph

We now return to chessboards and consider non-dominating irredundant sets of queens. Beginning at the lower left corner of the board, we number the rows and columns $1, 2, ..., n$. A square in column i and row j has coordinates (i, j). Note that two squares (i, j) and (k, l) are on the same diagonal if and only if $|i - k| = |j - l|$. We say a square of Q_n is *occupied* if it contains a queen; otherwise it is *unoccupied*. We begin with a simple result about the intersection of the lines of two adjacent squares on Q_n.

Proposition 4 (a) *If two squares q and q' with respective lines l_i and l'_i, $i = 1, ..., 4$, are on the same line, say $l_4 = l'_4$, then the lines l_i, $i = 1, 2, 3$, intersect the lines l'_i, $i = 1, 2, 3$, in at most six squares.*

(b) *If three squares are on the same line l, then there are at most two squares not on l where lines from all three these squares intersect.*

Proof. (a) We may assume without loss of generality that the lines are numbered such that l_i is parallel to (but does not coincide with) l'_i, $i = 1, 2, 3$. Thus for each i, l_i can only intersect the lines l'_j, $j \neq i$, in a square on the board, and the result follows.

(b) This is a simple exercise in geometry. ■

We now use Proposition 3 to obtain some properties of the squares of Q_n left open by a non-dominating maximal irredundant set of queens on the board. The idea is to show that if fewer than six rows and columns (in total) contain open squares, then "enough" diagonals contain open squares.

Proposition 5 *If X is a maximal irredundant set of queens on Q_n with $|X| < \gamma(Q_n)$, then R contains*

(a) *exactly four squares; their coordinates are (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_2), where $|x_1 - x_2| \neq |y_1 - y_2|$, or*

(b) *squares in (without loss of generality) exactly two rows and at least three columns, and if R is contained in exactly three columns, the squares with coordinates (say) (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_3, y_2) are open, where $|x_1 - x_2| \neq |y_1 - y_2|$ or $|y_1 - y_2| \neq |x_2 - x_3|$, or*

(c) *three squares, no two of which are in the same row or column.*

Proof. (The three possibilities (a), (b) and (c) are illustrated in Figure 1 for Q_5.) If there is only one row (column, diagonal) containing open squares, then a queen placed on any square in this row (column, diagonal) dominates the row (column, diagonal), contradicting Proposition 3. Hence we may assume without loss of generality that the squares with coordinates (x_1, y_1) and (x_2, y_2) are open.

Suppose (c) does not hold, and suppose firstly that there are nevertheless at least three rows and at least three columns containing open squares. Say (x_3, y_3) with $x_3 \notin \{x_1, x_2\}$ is open. Since (c) does not hold, $y_3 \in \{y_1, y_2\}$; without loss of generality let $y_3 = y_1$. Since there are at least three rows containing open squares, there is an open square (x_4, y_4) with $y_4 \notin \{y_1, y_2\}$, and since (c) does not hold, $x_4 \in \{x_1, x_2\}$. If $x_4 = x_1$, then the open squares (x_1, y_4), (x_2, y_2) and (x_3, y_1) contradict the fact that (c) does not hold, and so $x_4 = x_2$. Repeating this argument for any further open squares, we see that each open square is either in column x_2 or in row y_1. But then a queen placed on (x_2, y_1) dominates all open squares, contradicting Proposition 3.

Therefore we may assume that there are either exactly two rows and two columns containing open squares, or, without loss of generality, exactly two rows and at least three columns containing open squares.

In the former case, the open rows (columns) are y_1 and y_2 (x_1 and x_2). Note that a queen placed on (x_i, y_j), $i, j \in \{1, 2\}$, dominates all open squares in row j and column i. Thus the square in the opposite corner of the rectangle formed by these lines is open, for otherwise Proposition 3 is contradicted. Moreover, if $|x_1 - x_2| = |y_1 - y_2|$, then a queen on (x_i, y_j) dominates the opposite square, also contradicting Proposition 3. Since we assume exactly two rows and two columns contain open squares, there are no other open squares. Thus (a) holds.

Suppose exactly the two rows y_1 and y_2 and at least the three columns x_1, x_2 and x_3 contain open squares. If y_1 contains only one open square (x_1, y_1), then a queen on (x_1, y_2) dominates R, a contradiction. It follows that each row contains at least two open squares. Suppose all the open squares lie in the columns x_1, x_2 and x_3. Without loss of generality we may assume that x_2 contains two open squares, *i.e.*, (x_2, y_1) and (x_2, y_2) are open. Since each y_i contains at least two open squares and since (x_1, y_1) is assumed to be open, we may also assume without loss of generality that (x_3, y_2) is open. If $|x_1 - x_2| = |x_2 - x_3| = |y_1 - y_2|$, then a queen on (x_2, y_2) dominates all possible open squares, contradicting Proposition 3. Hence (b) holds. ■

In determining the irredundance numbers of Q_5 and Q_6 we do not need the full power of Proposition 5, but for larger boards this may well be necessary. (In the case of Proposition 5(b) we can also obtain stronger conditions on the relative positions of the open squares.) We now show that $ir(Q_5) = 3$ ($= \gamma(Q_5)$ - see [13]).

Lemma 6 *Suppose X is a maximal irredundant set of two queens on Q_5.*

(a) *There are at most seven pn-free squares.*

(b) *Each queen has at least two pns.*

(c) *Each queen can be annihilated from at most seven squares.*

Proof. Since $|X| < \gamma(Q_5) = i(Q_5) = 3$, it follows from Proposition 2 that the two queens q and q' lie on the same row, column or diagonal. Denote this line by L; without loss of generality we may assume that L is not a column. We first note the following:

(i) Each line of q, except L, is intersected by at most two lines of q' (see the proof of Proposition 4(a)).

(ii) The queen q on line l consisting of k squares can be annihilated from at most $k-1$ squares on l, while if q is not on l, she can be annihilated from at most k squares on l.

(iii) If q has at least two pns p_1 and p_2 in row (column) 1 (or 5), then q can be annihilated from at most three squares not in this row (column). This follows from Proposition 4(a) applied to p_1 and p_2, and the symmetry of the common neighbours of p_1 and p_2. (There is no room on Q_5 for the other three common neighbours.)

(iv) Similarly, if q has pns on squares $(i, 1)$ and $(i, 5)$ (or $(1, i)$ and $(5, i)$), where $i \in \{2, 3, 4\}$, then q can be annihilated from at most two squares not on i.

(a) The pn-free squares relative to X are the dominated but unoccupied squares that are not in line with the open squares, which are arranged as stated in Proposition 5. There are at least two rows and two columns containing these open squares. This leaves at most nine squares, *i.e.*, at most seven unoccupied squares. See Figure 1, in which the open squares are shown by dots and the queens are not shown.

(b) Since L is not a column, it follows from (i) that each queen has at least two pns in her column.

(c) Suppose q has exactly two pns. Obviously, q and q' have at most five common neighbours on L and, by Proposition 4(a), at most six common neighbours not on L. The only possibility is that they have 11 common neighbours and q covers exactly 13 squares, *i.e.*, q lies on the edge of Q_5. By considering all these configurations, it is apparent that there is (up to symmetry) only one such configuration. See Figure 2 for an example, where the queens are indicated by black dots and the pns by open circles. Then, by (ii), q can be annihilated from at most four squares on the same line l as the pns (the squares in column 1, but not row 3, in Figure 2) and, by (iii), from at most three squares not on l (squares $(2, 3)$, $(3, 2)$ and $(3, 4)$ in Figure 2). Let the queen q have three or more pns. If there are at least three pns on the same line l as q, then q can be annihilated from at most four squares on l and from at most two squares not on l (Proposition 4(b)). If at least three pns are on the line l', where q is not on l', then q lies on one of the squares mentioned in Proposition 4(b), and so q can be annihilated from at most five squares on l' and from at most one square not on l'.

Suppose no line contains at least three pns of q. If q lies in column 1 or 5, then by (ii), (iii) and the fact that q has at least two pns in her column, q can be annihilated from at most seven squares. Suppose q lies in column

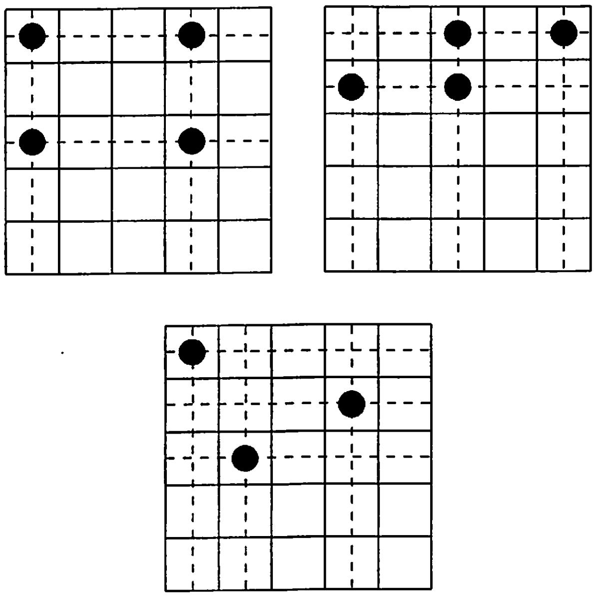

Figure 1: If there are two queens on Q_5, then there are at most seven pn-free squares.

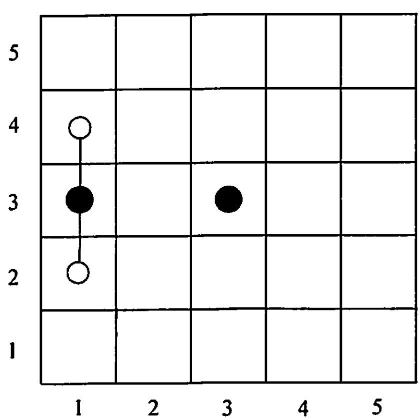

Figure 2: Each queen can be annihilated from at most seven squares.

2. If q' lies in row 1 or 5, then only two lines of q' intersect column 1. Since one of these must be L, it follows that q has three pns in her column, a contradiction. Hence we assume that q' lies in rows 2 to 4. Now, if q is in row 1 or 5, then q' is not in the same row as q, and hence, by (i), q has two pns in her row. As above, q can be annihilated from at most seven squares. Suppose q is in row 2, *i.e.*, q has coordinates $(2,2)$. The only position for q' that does not leave three pns for q in the same line, which we shall call a *valid* position, is $(3,3)$. But then $(2,1)$ and $(2,5)$ are pns of q and hence, by (ii) and (iv), q can be annihilated from at most six squares. Now say q lies on $(2,3)$. The only valid positions for q' are $(3,2)$, $(3,3)$, $(3,4)$ and $(4,3)$. If q' is on $(3,2)$ or $(3,4)$, then q has pns $(1,3)$ and $(5,3)$, and if q' is on $(3,3)$, then $(2,1)$ and $(2,5)$ are pns of q; hence by (ii) and (iv), q can be annihilated from at most six squares. If q' is on $(4,3)$, then q has two pns in column 1 and so can be annihilated from at most seven squares. By symmetry we have shown that the result holds for q in column 2 and thus also for q in column 4.

Suppose q is in column 3. As above we may assume that q and q' lie in rows 2 to 4. If q lies on $(3,2)$, the only valid positions for q' are $(2,3)$ and $(4,3)$. In each case $(3,1)$ and $(3,5)$ are pns of q and we are done. Finally, suppose q is on $(3,3)$. Up to symmetry, the only valid position for q' is $(2,2)$, in which case q has two pns in column 5 and we are done. ■

Of course, the result in Lemma 6(a) can be improved by considering squares covered by the diagonals of the queens, but we don't need a stronger result here.

Theorem 7 $ir(Q_5) = 3$.

Proof. Suppose $ir\,(Q_5) = 2$ and consider a maximal irredundant set of Q_5 consisting of two queens. There are 23 unoccupied squares. All of them must be either annihilators or pn-free. This is impossible, because by Lemma 6 each queen can be annihilated from at most seven squares, and the number of pn-free squares is at most seven, *i.e.*, the total number of annihilators or pn-free squares is at most $7 + 7 + 7 < 23$. ■

In the final two results of this paper we show that $ir\,(Q_6) = 3$ $(= \gamma\,(Q_6)$ – see [13]).

Lemma 8 *Suppose X is a maximal irredundant set of two queens on Q_6.*

(a) *There are at most* 14 *pn-free squares.*

(b) *Each queen has at least four pns.*

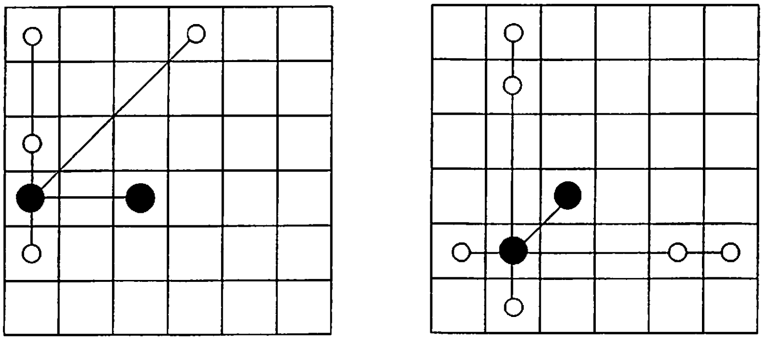

Figure 3: If $ir(Q_6) = 2$, then each queen has at least four private neighbours.

(c) *Each queen can be annihilated from at most four squares.*

Proof. As in the proof of Lemma 6, the two queens have a common line L, which we may assume is not a column.

(a) Again, the pn-free squares relative to X are the dominated but unoccupied squares that are not in line with the open squares, which are arranged as stated in Proposition 5. There are at least two rows and two columns containing these open squares. This leaves at most 16 squares, *i.e.*, 14 unoccupied squares.

(b) Since L is not a column, it follows from (i) in the proof of Lemma 6 that each queen has at least three pns in her column. If L is a row, each queen covers at least five other squares in her diagonals, and so at least one diagonal contains three or more squares. Again by (i) each queen has at least one additional pn in a diagonal. Similarly, if L is a diagonal, each queen has at least three additional pns in her row (see Figure 3).

(c) For any queen q, there are at least three pns in the same column x as q plus at least one other pn, say r. The lines of r intersect x in at most three squares, one of these being the square containing q. Thus there are at most two squares in x from which q can be annihilated. Further, for any given three squares in the same column, there are at most two squares not in the column whose lines intersect all three given squares. Thus there are at most two squares not on x from which q can be annihilated. ■

Theorem 9 $ir(Q_6) = 3$.

Proof. Suppose $ir\,(Q_6) = 2$ and consider a maximal irredundant set X of Q_6 consisting of two queens. There are 34 unoccupied squares. By the maximality of X, all of them must be either annihilators or pn-free. This is impossible, because each queen can be annihilated from at most four squares, and the number of pn-free squares is at most 14, that is, the total number of annihilators or pn-free squares is at most $14 + 4 + 4 < 34$. ■

Using the same method as in the proofs of the above theorems, it can be shown that $ir(Q_7) = 4$. However, the proof in this case is much more technical and is not given here. It is also not clear how this method can be extended to determine $ir(Q_n)$ for large n, as the application of the inclusion-exclusion principle (to determine the numbers of pns and annihilators) is bound to become too complicated.

The existence of a maximal irredundant set X of queens on Q_n with $|X| < \gamma(Q_n)$ for some n seems unlikely, as the (average) number of pns per queen seems to increase rapidly as n increases, as does the cardinality of R, the set of open squares, and hence the cardinality of $N[R]$. For every square in $N[R]$ to annihilate a queen in X (see Theorem 1) is a tall order!

(Note: Harborth [11] recently reported that Jens-P. Bode had verified by computer that $ir(Q_n) = \gamma(Q_n)$ for $n \leq 10$.)

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References

[1] B. Bollobás and E. J. Cockayne. Graph theoretic parameters concerning domination, independence and irredundance. *J. Graph Theory,* 3:241-250,1979.

[2] A. P. Burger, E. J. Cockayne, and C. M. Mynhardt. Domination and irredundance in the queen's graph. *Discrete Math.*, 163:47-66, 1997.

[3] A. P. Burger and C. M. Mynhardt, Symmetry and domination in queens graphs, *Bulletin of the ICA*, to appear.

[4] A. P. Burger and C. M. Mynhardt, An upper bound for the minimum number of queens covering the $n \times n$ chessboard, submitted.

[5] A. P. Burger and C. M. Mynhardt, Queens on hexagonal boards, *J. Combin. Math. Combin. Comput.* 31, 1999.

[6] A. P. Burger and C. M. Mynhardt, Properties of dominating sets of the queens graph Q_{4k+3}, *Utilitas Math.*, to appear.

[7] E. J. Cockayne. Chessboard domination problems. *Discrete Math.*, 86:13-20, 1990.

[8] E. J. Cockayne, P. J. P. Grobler, S. T. Hedetniemi and A. A. McRae. What makes an irredundant set maximal? *J. Combin. Math. Combin. Comput.*, 25:213-223, 1997.

[9] O. Favaron, G. Fricke, D. Pritikin and J. Puech, Domination and irredundance in the kings graph. Preprint.

[10] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, C. K. Wallis, M. S. Jacobson, H. W. Martin and W. D. Weakley. Combinatorial Problems on Chessboards: A Brief Survey. In Y. Alavi and A. J. Schwenk (Eds.), *Graph Theory, Combinatorics, and Algorithms, Proceedings of the Seventh Quadrennial International Conference on the Theory and Application of Graphs*, (Kalamazoo, MI 1992), Volume 1, Wiley-Interscience, New York, 1995, pages 507-528.

[11] H. Harborth, personal communication, January 2000.

[12] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[13] S. M. Hedetniemi, S. T. Hedetniemi and R. Reynolds. Combinatorial problems on chessboards: II. In T. W. Haynes, S. T. Hedetniemi and P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998, 133-162.

[14] M. D. Kearse, A lower bound on irredundance in the queens' graph, preprint.

[15] W. D. Weakley. Domination in the queen's graph. In Y. Alavi and A. J. Schwenk (Eds.), *Graph Theory, Combinatorics, Algorithms, and Applications, Proc. Seventh Quad. Internat. Conf. on the Theory and Application of Graphs*, Volume 2, pp. 1223-1232, (Kalamazoo, MI 1992), 1995. Wiley-Interscience, New York, 1995.

[16] W. D. Weakley, Upper bounds for domination numbers of the queen's graph, preprint.