Sensitivity of the upper irredundance number to edge addition

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January 5, 2000

Abstract

In this study, we consider the effect on the upper irredundance number $IR(G)$ of a graph G when an edge is added joining a pair of non-adjacent vertices of G. We say that G is *IR-insensitive* if $IR(G+e) = IR(G)$ for every edge $e \in \overline{E}$. We characterize IR-insensitive bipartite graphs and give a constructive characterization of graphs G for which the addition of any edge decreases $IR(G)$. We also demonstrate the existence of a wide class of graphs G containing a pair of non-adjacent vertices u, v such that $IR(G+uv) > IR(G)$.

Dedicated to Ernie Cockayne on the occasion of his 60th birthday.

1 Introduction

In this paper, all graphs considered will be finite and without multiple edges or loops. Let G be a graph with vertex set V and edge set E and let $\overline{E}$ denote the edge set of the complementary graph $\bar{G}$ of G. For any vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v in G and let $N[v] = N(v) \cup \{v\}$. For any subset $U \subseteq V$, let $N(U) = \bigcup_{v \in U} N(v)$ and $N[U] = N(U) \cup U$.

A non-empty subset $S \subseteq V$ is said to be an *irredundant set* in G if, for every vertex $x \in S$, there exists a vertex $v \in V$ such that $v \in N[x]$ and $v \notin N[S - \{x\}]$. Such a vertex v is called a *private neighbour of* x (with respect to S). The set of all private neighbours of a vertex x with respect to a set S is denoted by $I_G(x, S)$

JCMCC 33 (2000), pp. 65-79

(or just by $I(x,S)$ if the underlying graph is clear). If for a given set S and a given vertex $x \in S$, $I(x,S) \neq \emptyset$, then the vertex x is said to be *irredundant* with respect to S; otherwise, x is said to be *redundant*. A set containing a redundant vertex is called *redundant*. The maximum cardinality of an irredundant set in G is called the *upper irredundance number* of G and denoted by $IR(G)$. An irredundant set in G of cardinality $IR(G)$ is called an IR-set.

In this study, we consider the effect on $IR(G)$ of the addition of an edge joining two non-adjacent vertices of G. We show that, in contrast to the situation with the domination and independence numbers, if the addition of such an edge changes $IR(G)$, then in general it may either decrease or increase it. We say that G is *IR-sensitive* if there exists an edge $e \in \overline{E}$ such that $IR(G+e) \neq IR(G)$ and *IR-insensitive* otherwise. In section 3, we obtain necessary and sufficient conditions for the existence of an edge $e \in \overline{E}$ for which $IR(G+e) > IR(G)$ and also for which $IR(G+e) < IR(G)$. We demonstrate the existance of a wide class of graphs G for which $IR(G)$ may be increased by the addition of a suitably chosen missing edge.

The situation in the case of bipartite graphs is simpler to describe. By a result of Cockayne, Favaron, Payan and Thompson [2], $IR(G) = \beta(G)$ when G is bipartite. In section 4, we show that in this case, $IR(G)$ cannot be *increased* by the addition of an edge and give a characterization of bipartite graphs that are insensitive to edge addition.

A domination critical graph was defined by Sumner and Blitch in [8] as a graph G for which the domination number decreases when any edge joining two non-adjacent vertices is added to G. Following their notation, we call a graph *k-IR-critical* if $IR(G) = k \geq 2$ and $IR(G+e) < k$, for all $e \in \overline{E}$. In section 5, we obtain a constructive characterization of k-IR-critical graphs for all $k \geq 2$. This characterization has been obtained independently by Grobler and Mynhardt [5] in a study of the effect on the upper domination parameters caused by adding or removing an edge.

Related results on the effect on the domination number of a graph caused by adding an edge have been obtained by Carrington, Harary and Haynes [1]. A bound on the irredundance number after removal of a vertex has been obtained by Favaron [3]. Finally, there is an extensive literature on the effect on the domination number caused by removing a vertex and by adding or removing an edge; the interested reader is referred to [6] and [7] for an excellent survey.

We use the following notation and terminology. Let S be an irredundant set in G and let $x \in S$. If $N(x) \subseteq V - S$, then we call x an *isolate* in S. Thus x is an isolate in S if and only if $x \in I(x,S)$. We define two subsets $S_0, S_1 \subseteq S$, where possibly one of S_0 and S_1 is empty, by $s \in S_0$ if $I(s,S) = \{s\}$, and $s \in S_1$ otherwise. Then every vertex of S_0 is an isolate in S, while every vertex that is in a non-trivial component of the induced subgraph $\langle S \rangle$ is in S_1 (but note that S_1 also contains any isolate in S that has a private neighbour in $V - S$).

If $S_1 \neq \emptyset$, every vertex of S_1 has at least one private neighbour in $V - S$. For each vertex $x \in S_1$, make an arbitrary selection of just one vertex of $I(x,S) \cap (V - S)$ and label it $f(x)$. Then $f : S_1 \rightarrow V - S$, defined by $x \mapsto f(x)$, is an injection. We shall call f a *private neighbour function* (or *pnf*) for S_1. For

$U \subseteq S_1$, denote the set $\{f(u) : u \in U\}$ by $f(U)$.

We shall suppose throughout this paper that G is a graph of order $n \geq 2$ and that $G \not\cong K_n$.

2 Preliminary Results

The results in this section establish some structural properties of an IR-set for G. In particular, we obtain in the first three results, information about vertices that are contained in *every* IR-set in G.

Lemma 2.1 *Let S be an irredundant set in G, with $S_1 \neq \emptyset$ and let $f : S_1 \to V - S$ be a pnf for S_1. Then the set $S' = S_0 \cup f(S_1)$ is an irredundant set in G, with $|S'| = |S|$. In particular, when S is an IR-set, then so is S'.*

Proof. Consider the set $S' = S_0 \cup f(S_1)$. Since there are no edges in G between the sets S_0 and $f(S_1)$, by the definition of f, we have $x \in I(x, S')$, for all $x \in S_0$. Further, since there are no edges between S_0 and S_1, we have $f^{-1}(y) \in I(y, S')$, for all $y \in f(S_1)$. Thus S' is an irredundant set in G. Clearly $|f(S_1)| = |S_1|$, and hence $|S'| = |S|$. ■

Corollary 2.2 *Let $u \in V$ have the property that it is contained in every IR-set in G. Then for any IR-set S, $u \in S_0$.*

We shall say that a subset X of the vertices of a graph G is *minimal* with respect to a given property P if no proper subset of X possesses property P. With this definition of minimality, we establish the following result.

Lemma 2.3 *Let $X \subset V$ be an independent set that is minimal with respect to the property that $|X| > |N(X)|$. Then X is contained in every IR-set in G.*

Proof. Let X satisfy the conditions of the lemma and let S be any IR-set in G. If $N(X) = \emptyset$, then X is an isolated vertex in G and the result is clearly true. So suppose that $|N(X)| \geq 1$, and hence $|X| \geq 2$. Let $X_0 = X \cap S_0$, $X_1 = X \cap S_1$ and $X_2 = X \cap (V - S)$, where one or more of the sets X_0, X_1, X_2 may be empty. If $X_2 = \emptyset$, there is nothing to prove, so suppose otherwise. We note that

$$|X| = |X_0| + |X_1| + |X_2|.$$

We shall define four subsets Y_i of $N(X)$, where $i = 0, 1, 2, 3$. Firstly, if $X_0 \neq \emptyset$, let $Y_0 = N(X_0)$. Since the vertices of X_0 are isolates in S, we have $Y_0 \subset V - S$. Further, X_0 is a proper subset of X and hence $|Y_0| \geq |X_0|$, by the minimality of X. When $X_0 = \emptyset$, let $Y_0 = \emptyset$.

When $S_1 \neq \emptyset$, let $f : S_1 \to V - S$ denote a pnf for S_1. If $X_1 \neq \emptyset$, let $Y_1 = f(X_1)$. Then since f is injective, we have $|Y_1| = |X_1|$. When $X_1 = \emptyset$, let $Y_1 = \emptyset$. Let $Y_2 = N(X) \cap S$. Then since X is an independent set, $Y_2 \subseteq S - X$. Finally, let $Z = \{s \in S_1 - N[X] : f(s) \in N(X_2)\}$ and if $Z \neq \emptyset$, let $Y_3 = f(Z)$. Then $|Y_3| = |Z|$. When $Z = \emptyset$, let $Y_3 = \emptyset$.

We note that the sets Y_i, $i = 0, 1, 2, 3$, are pairwise disjoint. Thus, in all cases, we have the following inequality:

$$|N(X)| \geq \sum |Y_i| \geq |X_0| + |X_1| + |Y_2| + |Z|.$$

Hence the condition $|X| > |N(X)|$ gives $|X_2| > |Y_2| + |Z|$.

Let $S' = (S \cup X_2) - (Z \cup Y_2)$. Then X is contained in S' as a set of isolates and hence $x \in I(x, S')$, for all $x \in X$. Further, for each $s \in S_1' - X$, $f(s) \in V - N[X]$, and hence $I(s, S') \neq \emptyset$ for all $s \in S' - X$. Thus S' is an irredundant set. However, $|S'| > |S|$, contradicting the choice of S. Hence $X_2 = \emptyset$, and $X \subseteq S$. ■

The final result in this section establishes a useful upper bound on the degree of a vertex contained in an irredundant set.

Lemma 2.4 *Let S be any irredundant set in G. Then* $\deg x \leq n - |S|$, *for all* $x \in S$.

Proof. If $x \in S_0$, then x is not adjacent to any vertex of S and the result follows. Otherwise, $x \in S_1$. Let $f : S_1 \to V - S$ be a pnf for S_1. Then x is not adjacent to any vertex of $S_0 \cup f(S_1 - \{x\})$. But $|S_0 \cup f(S_1 - \{x\})| = |S_0| + |S_1| - 1 = |S| - 1$, so that in either case, $\deg x \leq n - |S|$. ■

In the case when S is an IR-set, Lemma 2.4 gives the following theorem of Favaron [4], which we use in the proof of Lemma 5.5 and Lemma 5.6.

Theorem 2.5 (Favaron, 1988) *In any graph G with minimum degree δ, $IR(G) \leq n - \delta$.*

3 Effect on IR of edge addition

In general, the addition of an edge joining two non-adjacent vertices of G may do any one of three things: it may decrease $IR(G)$, increase $IR(G)$ or leave it unchanged.

In order to *decrease* $IR(G)$, the addition of $e \in \overline{E}$ must make at least one vertex of every IR-set S in G redundant. It can only do this by depriving some vertex of each IR-set S for G of all of its private neighbours with respect to S. Thus each IR-set S must contain at least one vertex with the property that it has a unique private neighbour with respect to S and e must join this unique private neighbour to another vertex of S. On the other hand, the addition of the edge e *increases* $IR(G)$ if it enables a vertex to be adjoined to some IR-set W of G by providing it with a private neighbour in $V - W$.

The graphs shown in Figure 1 illustrate various situations that can occur. The graph G_1 has just two IR-sets: $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. It is easily seen that for any edge $e \in \overline{E}$, $IR(G_1 + e) = 2$ and hence G_1 is an example of a 3-IR-critical graph. However, for the graph G_2, we have $IR(G_2) = 2$, whereas $IR(G_2 + x_1y_1) = 3$: the addition of any edge other than x_1y_1 leaves $IR(G_2)$ unchanged.

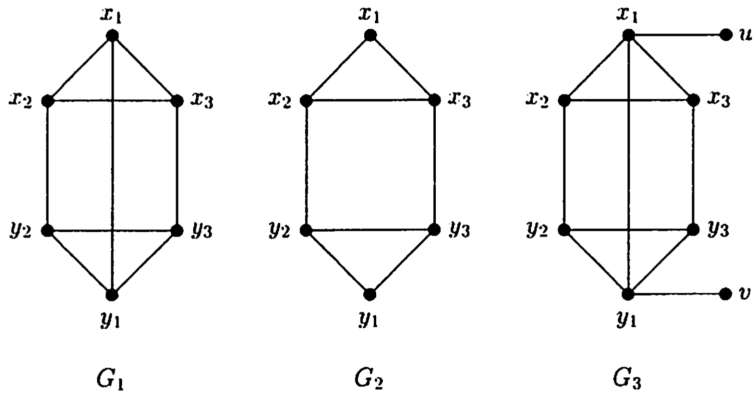

Figure 1.

The situation is further complicated by the fact that the addition of an edge may cause both an increasing and a decreasing effect simultaneously, as the example of the graph G_3 shown in Figure 1 illustrates. This graph has just the six IR-sets $\{u, v, x_2, y_3\}$, $\{u, v, y_2, x_3\}$, $\{u, v, x_2, x_3\}$, $\{u, v, y_2, y_3\}$, $\{u, y_1, y_2, y_3\}$ and $\{v, x_1, x_2, x_3\}$. Then in $G_3 + uv$, u or v is redundant in each of the first four of these sets, y_1 is redundant in the fifth and x_1 is redundant in the sixth. However, G_3 also contains the irredundant set $W = \{x_1, x_2, x_3\}$, and $\{u\} \cup W$ is irredundant in $G_3 + uv$. Hence $IR(G_3 + uv) = 4 = IR(G_3)$, so that the net effect of adding the edge uv has been to leave the irredundance number unchanged. Thus, in general, we cannot isolate decreasing effects from increasing effects.

In this section, we obtain conditions on a pair of non-adjacent vertices u, v so that $IR(G + uv) \neq IR(G)$. We also show that there exists a wide class of graphs containing an edge $e \in \overline{E}$ such that $IR(G + e) > IR(G)$.

We first obtain necessary and sufficient conditions for the existence of an edge $e \in \overline{E}$ that has an increasing effect on $IR(G)$.

Lemma 3.1 *Let u, v be a pair of non-adjacent vertices in G and suppose there exists a set $T \subset V$, where $|T| \geq 2$, with the property that T is irredundant in $G + uv$, but redundant in G. Then one of u and v, say u, is in a non-trivial component of T and $W = T - u$ is irredundant in G; further, $v \in V - N[W]$, $u \in N(W) - W$ and $I_G(w, W) \not\subseteq N[u]$, for any $w \in W$.*

Proof. Since T is redundant in G but irredundant in $G + uv$, the addition of the edge uv must provide one of u or v with a private neighbour with respect to T in $G + uv$. Without loss of generality, we may assume that $u \in T$ and $\{v\} = I_{G+uv}(u, T)$. It follows that u is in a non-trivial component of T and $v \in V - T$. Hence $u \in N(W) - W$ and $v \in V - N[W]$. Further, since T is irredundant in $G + uv$, we have $I_G(w, W) \not\subseteq N[u]$ for any $w \in W$. Hence for all $x \in W$, $I_G(x, W) = I_{G+uv}(x, T)$ and so W is irredundant in G. ■

It is useful to note that for the irredundant set W described in Lemma 3.1, $W_1 \neq \emptyset$. This is because u has a neighbour x, say, in T. Then since T is irredundant in $G + uv$, x has a private neighbour $x' \in V - T$. Thus $x \in W_1$.

Corollary 3.2 *Let $\{u, v\}$ be a pair of non-adjacent vertices. Then $IR(G + uv) > IR(G)$ if and only if there exists an IR-set W in G such that one of u and v, say v, is in $V - N[W]$, the other vertex, u, is in $N(W) - W$ and, in this case, $I_G(w, W) \not\subseteq N[u]$, for any $w \in W$.*

Proof. If $IR(G + uv) > IR(G)$, then the existence of W follows immediately from the lemma, by choosing T as an IR-set for $G + uv$. Conversely, if W exists satisfying the conditions of the corollary, then $W \cup \{u\}$ is irredundant in $G + uv$, establishing the result. ■

The proof of Lemma 3.1 also establishes the following.

Corollary 3.3 *Let $e \in \overline{E}$ be such that $IR(G + e) > IR(G)$. Then $IR(G + e) = IR(G) + 1$. Moreover, if T is an IR-set for $G + e$, then some subset $S \subset T$ is an IR-set for G.*

Note that although the relationship between u and v in the statement of Lemma 3.1 and Corollary 3.2 appears to be asymmetrical, this is not in fact the case when W is maximal irredundant in G. Then, $W \cup \{v\}$ contains a redundant vertex. Thus there exists a vertex $y \in W_1$ such that $I_G(y, W) \subseteq N(v)$ and we can define a pnf $f : W_1 \to V - W$ for W_1. By Lemma 2.1, $W' = f(W_1) \cup W_0$ is also an irredundant set in G, with $|W'| = |W|$. Then $f(y) \in W'$, and the roles of u and v with respect to W are interchanged with respect to W'.

Lemma 3.4 *Let $IR(G) = k \geq 2$, and let $\{u, v\}$ be a pair of non-adjacent vertices in G. Then $IR(G + uv) < IR(G)$, if and only if both the following conditions are satisfied.*

(a) *For any IR-set S in G, at least one vertex of $\{u, v\}$ is in S, say $u \in S$, and $\{v\} = I_G(x, S)$, for some $x \in S$ (where possibly $x = v$).*

(b) *There exists no irredundant set W of size $k - 1$ in G such that one of $\{u, v\}$, say v, is in $V - N[W]$; the other vertex $u \in N(W) - W$ and $I_G(w, W) \not\subseteq N[u]$, for any $w \in W$.*

Proof. Suppose that there exists a pair of non-adjacent vertices $\{u, v\} \subseteq V$ satisfying the conditions of the lemma. Suppose further that there exists an irredundant set T of size k in $G + uv$. If T is also irredundant in G, then $\{u, v\}$ and T satisfy condition (a), and x is redundant with respect to T in $G + uv$.

Thus we may assume that T contains a redundant vertex in G. Since T is irredundant in $G + uv$, the conditions of Lemma 3.1 are satisfied. Hence we may assume that u is in T and G contains the irredundant set $W = T - \{u\}$, where $v \in V - N[W]$, $u \in N(W) - W$ and $I(w, W) \not\subseteq N[u]$ for any $w \in W$. But this contradicts condition (b) of the Lemma. Thus every set of k vertices contains a redundant vertex in $G + uv$. Hence $IR(G + uv) < IR(G)$.

Now suppose conversely that there exists a pair of non-adjacent vertices $u, v \in V$ such that $IR(G + uv) < IR(G)$. Let S be an IR-set in G. Then S contains a redundant vertex, x say, in $G + uv$. Thus the addition of the edge uv deprives x of every private neighbour with respect to S. Hence at least one of u and v, say u, is in S. Two possibilities now arise, according to whether $u = x$ or $u \neq x$. If $u = x$, then u is redundant in $G + uv$. But this occurs only if $I_G(u, S) = \{u\}$. Hence $u \in S_0$ and $v \in S$. Otherwise, $u \neq x$. This occurs only if $I_G(x, S) = \{v\}$. Thus in either case, the sets $\{u, v\}$ and S satisfy condition (a).

Suppose that condition (b) is not also satisfied. Then there exists an irredundant set W of size $k - 1$ in G such that $u \in N(W) - W$, $I(w, W) \not\subseteq N[u]$ for any $w \in W$, and $v \in V - N[W]$. Then $W \cup \{u\}$ is an irredundant set of size k in $G + uv$, contrary to hypothesis. Thus condition (b) is also satisfied. ■

From the proof of Lemma 3.4, we deduce an obvious parallel to Corollary 3.3.

Corollary 3.5 *Let $e \in \overline{E}$ be such that $IR(G + e) < IR(G)$. Then $IR(G + e) = IR(G) - 1$. Moreover, if S is an IR-set for G, then some proper subset S' of S is an IR-set for $G + e$.*

Finally in this section, we demonstrate the existence of a wide class of graphs G for which there is an edge $e \in \overline{E}$ such that $IR(G + e) > IR(G)$.

Theorem 3.6 *Let G be a graph of order $2k + 2$, where $k \geq 2$. Then there is an edge $e \in \overline{E}$ such that $IR(G + e) = IR(G) + 1$ if the following conditions are satisfied.*

(a) *V can be partitioned into two subsets $X = \{u, x_1, ..., x_k\}$, $Y = \{v, y_1, ..., y_k\}$, such that the only edges between the sets X and Y are $\{x_i y_i : 1 \leq i \leq k\}$;*

(b) *$\{ux_i : 1 \leq i \leq k\} \cup \{vy_i : 1 \leq i \leq k\} \subset E$;*

(c) *the induced subgraphs $\langle X - \{u\}\rangle$ and $\langle Y - \{v\}\rangle$ each contain at least one edge.*

Proof. It is easily seen that X is irredundant in $G + uv$ and $X - \{u\}$ is irredundant in G. Hence $IR(G + uv) \geq k + 1$ and $IR(G) \geq k$. We shall show that $IR(G) = k$.

If $k = 2$, then G is isomorphic to the graph G_2 shown in Figure 1. Since $IR(G_2) = 2$, the result holds in this case. We shall therefore suppose that $k \geq 3$ and that G contains an irredundant set S of cardinality $k + 1$. We first note that $\beta(G) \leq k$ and hence $S_1 \neq \emptyset$. Let $f : S_1 \to V - S$ be a pnf for S_1.

Suppose that $u \in S_1$. Then $f(S_1 - \{u\}) \cap X = \emptyset$. Thus if also $x_i \in S$, then $f(x_i) = y_i$. Hence $y_i \notin S$ and further $v \notin S$, since v is adjacent to y_i. Let $f(u) = x_j$, where $j \neq i$. Then neither x_j nor y_j is in S. Hence including both u and x_i in S excludes at least one other vertex of X and three vertices of Y. For each additional vertex of X included in S, there is at least one further vertex of Y excluded from S. Suppose $y_p \in S$, where $p \notin \{i, j\}$. Then $f(y_p)$ is excluded from S if $y_p \in S_1$ and x_p is excluded from S when $y_p \in S_0$. But this gives $|S| \leq k$, contrary to assumption.

Thus we may assume that when $u \in S$, then S contains no other vertex of X. However, if $y_j \in S$, then $I(y_j, S) \subseteq Y$. But $\langle Y \rangle$ contains an induced K_3 and hence S contains at most $k-1$ vertices of Y, again implying that $|S| \leq k$.

Lastly, assume $u \notin S$, $v \notin S$. Then we have $\{x_i, y_i\} \subset S_1$, for some i, $1 \leq i \leq k$. Then $f(x_i) \in X$ and $f(y_i) \in Y$. However, if $f(x_i) = u$, then no other vertex of X is in S. In this case, we cannot also have $f(y_i) = v$, because this would give $|S| = 2$. Thus $f(y_i) \in Y - \{v, y_i\}$. Then, since $f(y_i)$ is excluded from S, we again have $|S| \leq k$. But if $f(x_i) \neq u$, then $f(x_i) \in X - \{u, x_i\}$. Then since both $f(x_i)$ and $f(y_i)$ are excluded from S, we have $|S| \leq k$ in this case too. Thus $IR(G) = k$ and the result follows using Corollary 3.3. ■

We remark here that not all graphs G for which there is an edge $e \in \overline{E}$ such that $IR(G+e) > IR(G)$ are induced subgraphs of a graph obtained by this construction. It is possible to have $\deg u < k$ and $\deg v < k$ when there are sufficient edges in $\langle X - \{u\} \rangle$ and $\langle Y - \{v\} \rangle$. For example, a graph G with V satisfying condition (a) of Theorem 3.6, and with $\langle X - \{u\} \rangle \cong K_k \cong \langle Y - \{v\} \rangle$, $N(u) = \{x_1, x_2\}$, $N(v) = \{y_1\}$, satisfies $IR(G) = k$ and $IR(G + uv) = k + 1$ when $k > 3$.

4 IR-insensitive bipartite graphs

It is well known that the upper domination parameters of G are related by the inequalities

$$\beta(G) \leq \Gamma(G) \leq IR(G),$$

where $\beta(G)$ denotes the independence number of G and $\Gamma(G)$ is the cardinality of a largest minimal dominating set for G. In the case of bipartite graphs, however, $IR(G) = \beta(G)$ by a theorem of Cockayne, Favaron, Payan and Thomason [2]. It follows that every bipartite graph G contains an IR-set consisting only of isolates.

In this section, we characterize bipartite graphs that are IR-insensitive. We show first that when G is bipartite, $IR(G)$ cannot be increased by the addition of an edge.

Lemma 4.1 *Let G be a bipartite graph with bipartition $V(A, B)$ and let $\{u, v\} \subseteq A$. Suppose there exists $T \subset V$, with $|T| = t \geq 2$, and such that T is irredundant in $G + uv$ but redundant in G. Then G contains an independent set of size $t + 1$.*

Proof. By Lemma 3.1, we may assume that u is in a non-trivial component of T and $W = T - \{u\}$ is irredundant in G, satisfying the conditions $u \in N(W) - W$, $v \in V - N[W]$ and $I(w, W) \not\subseteq N[u]$ for any $w \in W$. Then $W_1 \neq \emptyset$ and hence we can define a pnf $f : W_1 \to V - W$ for W_1. Let $R_A = (W_1 \cup f(W_1)) \cap A$. Then $W' = R_A \cup W_0$ is an independent set in G of cardinality $t - 1$ and hence $W' \cup \{u, v\}$ is an independent set in G of cardinality $t + 1$. ■

Lemma 4.2 *If G is a bipartite graph, then $IR(G+uv) \leq IR(G)$ for all $uv \in \overline{E}$.*

Proof. Let G be a bipartite graph with bipartition $V(A, B)$, and let $uv \in \overline{E}$. If $G + uv$ is a bipartite graph, then $IR(G + uv) = \beta(G + uv) \leq \beta(G) = IR(G)$. If $G + uv$ is not a bipartite graph, then u and v lie in the same partite set of $V(G)$. Let T be an IR-set of $G + uv$ and assume that $IR(G + uv) > IR(G)$. Then the set T is redundant in G and by Lemma 4.1 G has an independent set of vertices of order $|T| + 1 > IR(G) = \beta(G)$, a contradiction. ■

Lemma 4.3 *Let G be a bipartite graph and let $u, v \in V$ be a pair of non-adjacent vertices such that $IR(G + uv) < IR(G)$. Then $\{u, v\}$ is contained in every IR-set in G.*

Proof. Let S be an arbitrary IR-set S in G. Then by Lemma 3.4, at least one of the vertices u and v is in S, say $u \in S$. Suppose that $v \notin S$. Then again by Lemma 3.4, $\{v\} = I(x, S)$, for some $x \in S - \{u\}$. Thus x is in a non-trivial component of S, so that $S_1 \neq \emptyset$. Let G have bipartition $V(A, B)$ and suppose that $x \in A$; then $v \in B$. Let $f : S_1 \to V - S$ be a pnf for S_1 and let $R_A = (S_1 \cup f(S_1)) \cap A$. Then the set $S' = S_0 \cup R_A$ is an independent IR-set in G. Since S' is also irredundant in $G + uv$, we obtain $IR(G + uv) = IR(G)$, contrary to hypothesis. Hence $u, v \in S$. ■

Theorem 4.4 *Let G be a bipartite graph containing no isolated vertex. Then there exists an edge $e \in \overline{E}$ such that $IR(G + e) < IR(G)$ if and only if there is an independent set $X \subset V$ such that $|X| > |N(X)|$.*

Proof. Let $IR(G) = k$ and let G have bipartition $V(A, B)$. Suppose that G contains an independent set $X \subset V$ such that $|X| > |N(X)|$. Let X' be a subset of X that is minimal with respect to the property that $|X'| > |N(X')|$. Then $|X'| \geq 2$, since $|N(X')| \geq 1$. It follows from the minimality of X', that X' is contained in just one set of the bipartition of V. Assume that $X' \subseteq A$. Further, X' is contained in every IR-set in G, by Lemma 2.3. Let $u, v \in X'$ and let S be an arbitrary IR-set in G. Then by Corollary 2.2, $u, v \in S_0$ and hence $\{u, v\}$ satisfies condition (a) of Lemma 3.4.

Suppose that $\{u, v\}$ does not also satisfy condition (b) of Lemma 3.4. Then we can find an irredundant set W in G with $|W| = k - 1$ and such that $u \in N(W) - W$, $v \in V - N[W]$, and $I(w, W) \nsubseteq N[u]$ for any $w \in W$. Then $T = W \cup \{u\}$ is irredundant in $G + uv$, with $|T| = k$, and hence T satisfies the conditions of Lemma 4.1. But then G contains an irredundant set of cardinality $k + 1$, contradicting $IR(G) = k$. Thus $\{u, v\}$ satisfies condition (b) of Lemma 3.4 and hence $IR(G + uv) < IR(G)$.

Now suppose, conversely, that G contains an edge $e \in \overline{E}$ such that $IR(G + e) < IR(G)$. Let X be the set of all vertices that are contained in every IR-set in G. By Lemma 4.3, $|X| \geq 2$. Without loss of generality, we may suppose that $X \cap A \neq \emptyset$. Since G is bipartite, there is an independent IR-set S in G and by assumption, $X \subseteq S$. Let $S_A = S \cap A$, $S_B = S \cap B$. If $|S_A| > |N(S_A)|$, then the theorem is proved. So, suppose that $|N(S_A)| \geq |S_A|$. Since $X \cap A \subseteq S_A$, the set

$S_A \neq \emptyset$. Consider the set $S' = S_B \cup N(S_A)$. Since G has no isolated vertex, $S' \neq \emptyset$. Further, since S is an independent set, $|S'| = |S_B| + |N(S_A)| \geq |S|$. Hence S' is an independent set with at least $|S|$ vertices. Thus S' is an IR-set that does not contain $X \cap A$, contradicting the choice of X. Hence $|S_A| > |N(S_A)|$, and the theorem is proved. ■

Thus we have the following characterisation of IR-insensitive bipartite graphs containing no isolated vertex.

Corollary 4.5 *A bipartite graph G containing no isolated vertex is IR-insensitive if and only if every independent set $X \subset V$ satisfies the condition that $|X| \leq |N(X)|$.*

It is interesting to note that in such a graph G with bipartition $V(A, B)$, then $|A| \leq |N(A)| = |B|$ and $|B| \leq |N(B)| = |A|$; hence $|A| = |B| = IR(G)$.

5 IR-critical graphs

Let G be a graph of order n with $IR(G) = k \geq 2$. We shall say that G is *k-IR-critical* if $IR(G + e) < k$, for all $e \in \overline{E}$. It follows from Corollary 3.5 that if G is k-IR-critical, then $IR(G + e) = k - 1$, for all $e \in \overline{E}$. In the following lemmas, we establish the structure of such graphs. This characterization has also been established by Grobler and Mynhardt, using a different proof, in a paper [5] which appears elsewhere in this issue.

Throughout this section, we shall assume that G is k-IR-critical and S is an IR-set in G.

Lemma 5.1 *G, S satisfy the following conditions:*

(a) *$G - S$ is a clique;.*

(b) *$\langle S_1 \rangle$ has at most one component and, if $|S_1| \geq 2$, then $\langle S_1 \rangle$ is a clique;*

(c) *if $v \in V - S$ and $\{v\} \neq I(x, S)$, for any $x \in S$, then* $\deg v = n - 1$.

Proof. Let u, v be a pair of non-adjacent vertices in G. Then S is irredundant in $G + uv$ in each of the following cases: (i) $u, v \in V - S$; (ii) $u, v \in S_1$; (iii) $u \in S$, $v \in V - S$ and $\{v\} \neq I(x, S)$ for any vertex $x \in S - \{u\}$. But this contradicts the definition of G, so that none of these cases can arise. ■

Lemma 5.2 *If $S_1 \neq \emptyset$, then $|S_1| \geq 3$.*

Proof. Let $f : S_1 \to V - S$ be a pnf for S_1. Suppose first that $S_1 = \{u\}$, for some vertex $u \in S$. Then since $|S| \geq 2$, there exists $w \in S_0$. Then w and $f(u)$ are non-adjacent. But then S is irredundant in $G + wf(u)$, contradicting the definition of G. Hence $|S_1| \geq 2$.

Now suppose that $S_1 = \{u, v\}$, where u, v are distinct vertices in S. Then the set $S' = (S - \{v\}) \cup \{f(v)\}$ is an independent set of k vertices in $G + vf(u)$, again contradicting the definition of G. ■

It is worth noting that the argument used in the proof of Lemma 5.2 fails if $|S_1| \geq 3$. In this case, S_1 contains a K_3, with vertex set $\{x, y, z\}$, say. Then $\langle f(x), f(y), f(z) \rangle$ is also a K_3, by Lemma 5.1. Hence the vertices x, y, z, $f(x), f(y), f(z)$, induce a subgraph of G isomorphic to the graph G_1, shown in Figure 1. It is easily seen that $IR(G_1 + xf(y)) = 2$.

Lemma 5.3 *If $u \in S_1$, then u has exactly one private neighbour in $V - S$.*

Proof. Suppose to the contrary that v, w are distinct vertices in $I(u, S\}$. Let $x \in S - \{u\}$. Then S is an irredundant set in $G + xv$. ■

The preceding results place restrictions on the structure of a k-IR-critical graph G and any IR-set in G. These are summarized in the following lemma.

Lemma 5.4 *Let G be a k-IR-critical graph of order n and let S be an IR-set in G. Then $G - S$ is a clique, each vertex of S has a unique private neighbour and, if $v \in V - S$ is such that $\{v\} \neq I(s, S)$ for any $s \in S$, then* $\deg v = n - 1$. *Further, just one of the following three possibilities arise for the structure of S.*

(a) $\langle S \rangle \simeq K_k$ *and* $|V - S| \geq k$;

(b) $\langle S \rangle \simeq \bar{K}_k$;

(c) $\langle S \rangle \simeq K_t \cup \bar{K}_{k-t}$, *and* $|V - S| \geq t$, *where* $k \geq t \geq 3$.

Proof. Let $|S_1| = t$, where $0 \leq t \leq k$. When $t > 0$, we can define a pnf f for S_1. Then, in this case, since $f(S_1) \subseteq V - S$, we have $|V - S| \geq t$. If $t = k$, we have $S = S_1$, and hence $\langle S \rangle \simeq K_k$, by Lemma 5.1, giving case (a). When $t = 0$, we have $S = S_0$, giving case (b). Otherwise, Lemma 5.1 and Lemma 5.2 give case (c). ■

We now describe three families of graphs using the following notation. Let h, k be integers such that $n = k + h$ and let $V = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, ..., u_k\}$ and $V_2 = \{v_1, v_2, ..., v_h\}$. Denote the set of edges between V_1 and V_2 by E^*.

Then the graph G belongs to $\mathcal{F}_1$ if and only if the following conditions are satisfied:

1. $h \geq k \geq 3$;
2. $\langle V_1 \rangle \cong K_k$ and $\langle V_2 \rangle \cong K_h$;
3. $E^* = \{u_i v_i : 1 \leq i \leq k\} \cup \{u_i v_j : 1 \leq i \leq k,\ k + 1 \leq j \leq h\}$.

The graph G belongs to $\mathcal{F}_2$ if and only if the following conditions are satisfied:

1. $h \geq 0$, $k \geq 2$;
2. $\langle V_1 \rangle \cong \bar{K}_k$ and $\langle V_2 \rangle \cong K_h$;
3. $E^* = \{u_i v_j : 1 \leq i \leq k,\ 1 \leq j \leq h\}$.

The graph G belongs to $\mathcal{F}_3$ if and only if the following conditions are satisfied::

1. $h \geq t \geq 3$, $k \geq t+1$, for some integer t;

2. $\langle\{u_1, u_2, ..., u_t\}\rangle \cong K_t$, $\langle\{u_{t+1}, u_{t+2}, ..., u_k\}\rangle \cong \bar{K}_{k-t}$ and $\langle V_2\rangle \cong K_h$;

3. $E^* = \{u_iv_i : 1 \leq i \leq t\} \cup \{u_iv_j : 1 \leq i \leq k, t+1 \leq j \leq h\}$.

Remark 1 *If G is in any one of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, then V_1 is an irredundant set in G and hence $IR(G) \geq k$.*

Lemma 5.5 *If $G \in \mathcal{F}_1$, then G is k-IR-critical.*

Proof. Let $G \in \mathcal{F}_1$. Since $\delta(G) = \deg u_1 = n - k$, it follows from Remark 1 and Theorem 2.5 that $IR(G) = k$. Thus to show that G is k-IR-critical, it suffices to show that $IR(G+e) < k$, for all $e \in \overline{E}$. Without loss of generality, we may take $e = u_1v_2$.

Suppose that $G+u_1v_2$ contains an irredundant set S with $|S| = k$. By Lemma 2.4, no vertex of $\{v_{k+1}, v_{k+2}, ..., v_h\}$ is in S. Also, since u_2 is redundant in V_1 and v_1 is redundant in V_2, S is neither V_1 nor the set $W = \{v_1, v_2, ..., v_k\}$ and hence S contains at least one element from each of V_1 and W. Assume, without loss of generality, that $u_p, u_q, v_r \in S$, with $p \neq r$, $r \leq k$. Then since $u_pu_q \in E$, $S_1 \neq \emptyset$. Let $f : S_1 \to V - S$ be a pnf for S_1. We show that u_p is redundant in S. This follows by noting that $f(u_p) \notin V_1$, since every vertex of V_1 is adjacent to u_q, and $f(u_p) \neq v_p$, since v_p is adjacent to v_r. Hence $IR(G+u_1v_2) < k$ and G is k-IR-critical. ■

Lemma 5.6 *If $G \in \mathcal{F}_2$, then G is k-IR-critical.*

Proof. Let $G \in \mathcal{F}_2$. (Note that the restriction on k arises from the assumption that $G \not\cong K_n$.) As in the case when $G \in \mathcal{F}_1$, we have $\delta(G) = \deg u_1 = n - k$, giving $IR(G) = k$. Thus again it suffices to show that $IR(G+e) < k$, for any edge $e \in \overline{E}$. Without loss of generality, we may take $e = u_1u_2$.

Suppose that $G + u_1u_2$ contains an irredundant set S with $|S| = k$. When $h = 0$, we have $G \cong \bar{K}_k$ and $IR(G+u_1u_2) = k-1$, so that the result holds. Assume that $h > 0$. Then no vertex of V_2 is in S, by Lemma 2.4. Hence $S = V_1$. But u_2 is redundant with respect to V_1 in $G+u_1u_2$. This contradiction establishes that $IR(G+u_1u_2) < k$ and G is k-IR-critical. ■

Lemma 5.7 *If $G \in \mathcal{F}_3$, then G is k-IR-critical.*

Proof. Suppose first that G contains an irredundant set T with $|T| = k+1$. Then T contains no vertex of the set V_2, by Lemma 2.4. Hence $T \subseteq V_1$. But $|V_1| = k$. This contradiction establishes that $IR(G) \leq k$, and hence by Remark 1, we have $IR(G) = k$.

Suppose that S is an irredundant set with $|S| = k$ in $G + e$, where we may assume that e is one of the edges $e_1 = u_1u_{t+1}$, $e_2 = u_1v_2$, $e_3 = u_{t+1}v_1$ and, when $k \geq t+2$, $e_4 = u_{t+1}u_{t+2}$. By Lemma 2.4, S contains no vertex of the set $\{v_{t+1}, v_{t+2}, \dots, v_h\}$. Also, V_1 is redundant in each of the graphs $G + e_i$, $i = 1, 2, 3, 4$. Thus, without loss of generality, we may assume that S contains u_p, u_q, v_r, where $1 \leq p, q, r \leq t$ and $r \neq p$. Then as in the proof of Lemma 5.5, it is easily verified that u_p is redundant with respect to S. Thus G is k-IR-critical. ■

The last four lemmas give the following characterization of k-IR-critical graphs.

Theorem 5.8 *Let G be a graph with $IR(G) = k \geq 2$. Then G is k-IR-critical if and only if G belongs to one of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.*

The previous theorem characterizes graphs with the property that the addition of any missing edge *decreases* $IR(G)$. In the context of irredundance, there are two other senses in which a graph might be said to be edge-critical: (i) if the addition of any missing edge *increases* $IR(G)$; (ii) if the addition of any missing edge *changes* $IR(G)$. We call an edge uv joining a non-adjacent pair of vertices $\{u, v\}$ an *increasing* edge if $IR(G+uv) > IR(G)$; a *decreasing* edge if $IR(G+uv) < IR(G)$; and a *neutral* edge if $IR(G+uv) = IR(G)$.

We consider first the existence of graphs with $IR \geq 2$ for which every missing edge is an increasing edge. Suppose this class is non-empty and, among all such graphs, let G be one of least order n. Then G contains no vertex of degree $n-1$. This follows by noting that a vertex x with $\deg x = n-1$ cannot be in any IR-set S for G, by Lemma 2.4; further, x cannot be the unique private neighbour of any vertex of S. Thus $IR(G-x) = IR(G)$ and G has the property that every missing edge is an increasing edge if and only if $G - x$ has that property, contradicting the choice of G. Thus we may assume that every vertex of G is incident with at least one missing edge.

Let u be any vertex of G. Then there is a vertex $v \notin N[u]$, and uv is an increasing edge. It follows from Corollary 3.2 that there exists an IR-set W for G with the property that $v \in V - N[W]$, $u \in N(W) - W$ and $I_G(w, W) \not\subseteq N[u]$, for any $w \in W$.

The existence of these IR-sets would appear to be quite a restrictive condition on G. In particular, we show in the following lemma that no graph satisfying the conditions of Theorem 3.6 has the property that every missing edge is an increasing edge.

Lemma 5.9 *Suppose that G is a graph satisfying the conditions of Theorem 3.6. Then not every missing edge of G is an increasing edge.*

Proof. With the notation of Theorem 3.6, we may suppose that y_1, y_2 are adjacent. Now x_1y_2 is a missing edge. We shall show that there is no IR-set W for G satisfying the conditions that $y_2 \in V - N[W]$, $x_1 \in N(W) - W$ and $I_G(w, W) \not\subseteq N[x_1]$, for any $w \in W$.

Suppose to the contrary that such an IR-set W exists. It follows from the discussion immediately preceding Corollary 3.2 that $W_1 \neq \emptyset$. Let $f : W_1 \to$

$V - W$ be a pnf for W_1. Now W does not contain x_1 or any vertex of $N[y_2]$, and hence W contains no vertex of the set $\{x_1, x_2, y_1, y_2, v\}$. However, some vertex of $N[x_1]$ is in W_1. Suppose that $u \in W_1$. Then $f(u)$ is the only vertex of $f(W_1)$ in X. Hence no pair of vertices $\{x_r, y_r\}$ is in W, for $3 \leq r \leq k$. But then $|W| < k$, a contradiction.

We may thus assume that $u \notin W$. Then $W \subset \{x_3, \ldots, x_k\} \cup \{y_3, \ldots, y_k\}$ and hence W contains a pair of vertices $\{x_r, y_r\}$ for at least two distinct values of r, with $3 \leq r \leq k$, say $x_4, x_5, y_4, y_5 \in W$. Then for $r = 4, 5$, $f(y_r) \neq v$; further, $I_G(y_r, W) \neq \{y_1\}$, because $y_1 \in N[x_1]$. Thus, without loss of generality, we may take $f(y_4) = y_s$ and $f(y_5) = y_t$, where $6 \leq s, t \leq k$ and $s \neq t$. But then the vertices x_s, y_s, x_t, y_t are excluded from W, and similarly, for each additional pair $\{x_p, y_p\}$ that is included in W, there is a pair $\{x_{p'}, y_{p'}\}$ which is excluded. Thus we again have $|W| < k$, establishing the Lemma. ■

It is not difficult to find examples of graphs containing both increasing and decreasing edges. For example, suppose G is a graph satisfying the conditions of Theorem 3.6 with $k \geq 3$ and $\langle X \rangle \cong \langle Y \rangle \cong K_{k+1}$. Then uv is an increasing edge and, by an argument similar to that used in Lemma 5.5, every edge of the form $x_i y_j$, $i \neq j$, is a decreasing edge. However, it can also be shown that the edges of the set $\{uy_j : 1 \leq j \leq k\} \cup \{vx_i : 1 \leq i \leq k\}$ are neutral edges. If we add these edges to G, then neither u nor v can be in any IR-set for G by Lemma 2.4. Hence uv is now a neutral edge. Adding uv to G gives a graph in the family $\mathcal{F}_1$.

We conclude with the following conjecture.

Conjecture 1 . *The only graphs with $IR(G) \geq 2$ that satisfy the property that $IR(G + e) \neq IR(G)$, for all $e \in \overline{E}$, are the graphs of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, for which every edge is a decreasing edge.*

6 Acknowledgements

We would like to thank the (unknown) referee for several helpful and constructive comments.

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