Stratified Claw Domination in Prisms

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Dedicated to Ernie Cockayne, a Domination Trailblazer, on the occasion of his 60th birthday

Abstract

A graph G is 2-stratified if its vertex set is partitioned into two classes (each of which is a stratum or a color class), where the vertices in one class are colored red and those in the other class are colored blue. Let F be a 2-stratified graph rooted at some blue vertex v. An F-coloring of a graph is a red-blue coloring of the vertices of G in which every blue vertex v belongs to a copy of F rooted at v. The F-domination number $\gamma_F(G)$ is the minimum number of red vertices in an F-coloring of G. In this paper we determine the F-domination number of the prisms $C_n \times K_2$ for all 2-stratified claws F rooted at a blue vertex.

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1 Introduction

Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. The vertices of a graph can be divided into cut-vertices and non-cut-vertices. Equivalently, the vertices of a tree are divided into its leaves and non-leaves. The vertices of a graph is partitioned according to the degrees of its vertices. When studying distance, the vertices of a connected graph are partitioned according to their eccentricities. Also, in a connected rooted graph, the vertices are partitioned according to their distance from the root. Perhaps the best known example of this process, however, is graph coloring, where the vertex set of a graph is partitioned into classes each of which is independent in the graph.

In VLSI design, the design of computer chips often yields a division of the nodes into several layers each of which must induce a planar subgraph. So here too the vertex set of a graph is divided into classes. Motivated by these observations, Rashidi [7] defined a graph G to be a stratified graph if its vertex set is partitioned into classes.

Formally, then, a graph G whose vertex set has been partitioned is called a *stratified graph*. If the partition is $V(G) = \{V_1, V_2, \ldots, V_k\}$, then G is a k-stratified graph. The sets $V_1, V_2, \ldots, V_k$ are called the *strata* or *color classes* of G. If $k = 2$, we ordinarily color the vertices of V_1 red and the vertices of V_2 blue. In this paper, we will restrict our attention to 2-stratified graphs. In [7], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [1, 3].

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a *γ-set* of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. For a thorough study of domination in graphs, see Haynes, Hedetniemi and Slater [4, 5].

Let F be a 2-stratified graph rooted at some blue vertex v. By definition, F contains at least one red vertex. An *F-coloring* of a graph G is a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F rooted at v. The *F-domination number* $\gamma_F(G)$ of G was introduced in [2] as the minimum number of red vertices of G in an F-coloring of G. By a *minimum F-coloring* of G, we mean an F-coloring containing a minimum number of red vertices, that is, $\gamma_F(G)$ red vertices. The F-domination number was studied in [2] for 2-stratified graphs of order at most 3, where it was shown that F-domination generalizes not only ordinary domination but other types of domination that have been previously studied.

By a claw, we mean the graph $K_{1,3}$. There are eight possible choices for a 2-stratified claw rooted at a blue vertex v. These graphs are shown in Figure 1. In this paper we study "claw domination" for prisms, that is, graphs that are the Cartesian product $C_n \times K_2$ of an n-cycle and the complete graph of order 2.

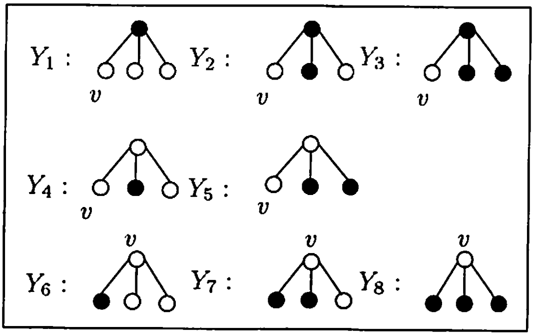

Figure 1: The distinct 2-stratified claws rooted at a blue vertex v

Claw domination is illustrated with the famed Petersen graph P in Figure 2, where for each 2-stratified claw Y_i $(1 \leq i \leq 8)$, a minimum Y_i-coloring of the Petersen graph is shown, thereby giving the following:

$$\gamma_{Y_1}(P) = 3, \ \gamma_{Y_2}(P) = 4, \ \gamma_{Y_3}(P) = 5, \ \gamma_{Y_4}(P) = 5$$
$$\gamma_{Y_5}(P) = 4, \ \gamma_{Y_6}(P) = 4, \ \gamma_{Y_7}(P) = 4, \ \gamma_{Y_8}(P) = 6$$

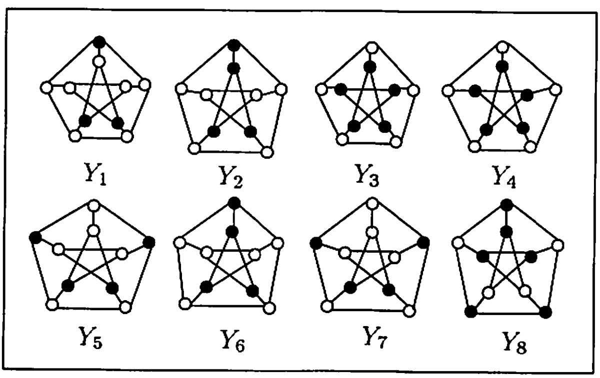

Figure 2: Y_i-Colorings of the Petersen graph

2 Stratified Claw Domination in Prisms

In this section we determine the Y_i-domination number of an important class of cubic graphs, namely the prisms $C_n \times K_2$, $n \geq 3$. The following lemma concerns the Y_1-domination number of cubic graphs in general.

Lemma 2.1 *If G is a cubic graph containing k pairwise disjoint copies of $K_{1,3}$, then $\gamma_{Y_1}(G) \geq k$.*

Proof. Let there be given a Y_1-coloring of G and let v be the central vertex in a copy of $K_{1,3}$. If v is blue, then v is adjacent to a red vertex in this copy. ■

In all proofs dealing with $C_n \times K_2$, we assume that $C_n \times K_2$ consists of two n-cycles $C : v_1, v_2, \cdots, v_n, v_1$ and $C' : v'_1, v'_2, \cdots, v'_n, v'_1$ with $v_i v'_i$ an edge for all i $(1 \leq i \leq n)$. We also assume that addition such as v_{a+b} is computed modulo n, and $a + b$ is one of the integers $1, 2, \cdots, n$.

Theorem 2.2 *For $n \geq 3$, $\gamma_{Y_1}(C_n \times K_2) = 2\lceil n/4 \rceil$.*

Proof. Let $G = C_n \times K_2$. Assume first that $n \equiv 0 \pmod 4$. So $2\lceil n/4 \rceil = n/2$. In this case, $V(G)$ can be partitioned into $n/2$ subsets, each of which induces a $K_{1,3}$. By Lemma 2.1, $\gamma_{Y_1}(G) \geq n/2$. By coloring the central vertex red in each such $K_{1,3}$ and all other vertices blue, we have a Y_1-coloring of G. Thus $\gamma_{Y_1}(G) \leq n/2$.

Next we consider the case where $n \equiv 1 \pmod 4$. Here $2\lceil n/4 \rceil = (n+3)/2$. Now let there be given a Y_1-coloring of G. Assume first that there are two adjacent red vertices. We consider two cases.

Case 1. *There exist adjacent red vertices v_i, v'_i for some i $(1 \leq i \leq n)$.* We may assume that $i = 1$. Since either v_2 or v'_2 is red, we may assume that v_2 is red. Then the set $V(G) - \{v_1, v'_1, v_2, v_{n-1}, v_n, v'_n\}$ can be partitioned into $(2n-6)/4$ subsets, each of which induces a copy of $K_{1,3}$. Therefore,

$$\gamma_{Y_1}(G) \geq 3 + \frac{2n-6}{4} = \frac{n+3}{2} = 2\left\lceil \frac{n}{4} \right\rceil.$$

Case 2. *For some i $(1 \leq i \leq n)$, v_i and v_{i+1} are red.* We can assume that $i = 1$. Then v'_1 and v'_2 are blue, otherwise we are in Case 1. Hence v'_3 and v'_n are red and the set $V(G) - \{v_1, v'_1, v_2, v'_2, v'_3, v'_n\}$ can be partitioned into $(2n-6)/4$ subsets, each of which induces a $K_{1,3}$. This implies that

$$\gamma_{Y_1}(G) \geq 4 + \frac{2n-6}{4} > 2\left\lceil \frac{n}{4} \right\rceil.$$

Hence we may assume that no two adjacent vertices are red. Without loss of generality, let v_1 be red. Then v'_1 and v_2 are blue. Let $j \geq 2$ be the smallest integer for which $\{v_j, v'_j\}$ is a pair of vertices of G containing a red vertex. Then $j \le 3$, for otherwise, the blue vertex v'_2 is not rooted at any copy of Y_1.

Next we claim that v_i and v'_{i+1} are red for some i $(1 \le i \le n)$. Otherwise, without loss of generality, we may assume that v_1 and v'_3 are both red and v_4 must consequently be blue. Also, since there are no adjacent red vertices, v'_1, v_2, v'_2, and v_3 are blue. Consequently, $v_5, v'_7, v_9, v'_{11}, \cdots, v_n$ are red. However, then, v_1 and v_n are adjacent red vertices, contrary to our assumption. Hence, as claimed, v_i and v'_{i+1} are both red for some i $(1 \le i \le n)$. Since $V(G) - \{v_i, v'_{i+1}\}$ can be partitioned into $(2n-2)/4$ subsets, each of which induces a copy of $K_{1,3}$, it follows that

$$\gamma_{Y_1}(G) \ge 2 + \frac{2n-2}{4} = 2\left\lceil \frac{n}{4} \right\rceil.$$

We have now proved that $\gamma_{Y_1}(G) \ge 2\lceil n/4 \rceil$. If we color the vertices v_1, v'_2, v_3, v'_4 red as well as the vertices v_6, $v'_8, v_{10}, \cdots, v'_{n-1}$ red with all other vertices colored blue, then we have a Y_1-coloring of G in which exactly $2\lceil n/4 \rceil$ vertices are colored red. Thus $\gamma_{Y_1}(G) = 2\lceil n/4 \rceil$.

Proofs of the cases $n \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ are similar to $n \equiv 1 \pmod 4$ and are therefore omitted. ∎

A set S of vertices in a graph G is called a *packing set* for G if the distance between every two vertices of S in G is at least 3. The *packing number* $\rho(G)$ is the cardinality of a maximum packing set (see [6]). The following result gives bounds for $\gamma_{Y_1}(G)$ of an arbitrary cubic graph G in terms of its order and packing number.

Theorem 2.3 *For every cubic graph G of order n,*

$$\rho(G) \le \gamma_{Y_1}(G) \le n - 3\rho(G).$$

Proof Since G contains at least $\rho(G)$ pairwise copies of $K_{1,3}$, the lower bound follows from Lemma 2.1. We now establish the upper bound. Let S be a maximum packing set for G, and let $N(S)$ be the neighborhood of S. Since S is a packing set and G is cubic, $|N(S)| = 3\rho(G)$. By coloring the vertices of $N(S)$ blue and other vertices red, we have a Y_1-coloring with $n - 3\rho(G)$ red vertices. ∎

We can use Theorem 2.2 to show that both bounds for $\gamma_{Y_1}(G)$ given in Theorem 2.3 are sharp. The lower bound in Theorem 2.3 is attained for the graph $C_n \times K_2$, where $n \equiv 0 \pmod 4$; while the upper bound is attained for the graph $C_n \times K_2$, where $n \equiv 1 \pmod 4$.

Now we turn our attention to Y_2. The following general lemma concerning Y_2-domination number of a cubic graph G is a consequence of the fact that in each Y_2-coloring of G, every $P_3 \times K_2$ contains at least two red vertices.

Lemma 2.4 *If G is a cubic graph containing k pairwise disjoint copies of $P_3 \times K_2$, then $\gamma_{Y_2}(G) \geq 2k$.*

Theorem 2.5 *For $n \geq 3$, $\gamma_{Y_2}(C_n \times K_2) = 2\lceil n/3 \rceil$.*

Proof Let $G = C_n \times K_2$. Assume first that $n \equiv 0 \pmod 3$. So $2\lceil n/3 \rceil = 2n/3$. In this case, $V(G)$ can be partitioned into $n/3$ subsets, each of which induces a $P_3 \times K_2$. By coloring the two central vertices in each such $P_3 \times K_2$ red and all other vertices blue, we have a Y_2-coloring of G. Thus, $\gamma_{Y_2}(G) \leq 2n/3$. By Lemma 2.4, $\gamma_{Y_1}(G) \geq 2n/3$.

Next we consider the case where $n \equiv 1 \pmod 3$. Then $2\lceil n/3 \rceil = (2n+4)/3$. Now let there be given a Y_2-coloring of G. Assume first that v_i and v_i' are colored red for some i with $1 \leq i \leq n$. Then $V(G) - \{v_i, v_i'\}$ can be partitioned into $(2n-2)/6$ subsets, each of which induces a copy of $P_3 \times K_2$. It follows by Lemma 2.4 that

$$\gamma_{Y_2}(G) \geq 2 + \frac{2n-2}{3} = \frac{2n+4}{3} = 2\left\lceil \frac{n}{3} \right\rceil .$$

Thus we can assume that for each i $(1 \leq i \leq n)$ at least one of v_i, v_i' is blue.

If there exists an integer i $(1 \leq i \leq n)$ such that exactly one vertex of each of the pairs $\{v_i, v_i'\}$, $\{v_{i+1}, v_{i+1}'\}$, $\{v_{i+2}, v_{i+2}'\}$, $\{v_{i+3}, v_{i+3}'\}$ is red, then the set $V(G) - \cup_{j=i}^{i+3}\{v_j, v_j'\}$ can be partitioned into $(2n-8)/6$ subsets, each of which induces a copy of $P_3 \times K_2$. Then

$$\gamma_{Y_2}(G) \geq 4 + \frac{2n-8}{3} = \frac{2n+4}{3} = 2\left\lceil \frac{n}{3} \right\rceil .$$

Otherwise, for every i $(1 \leq i \leq n)$ there is one pair in $\{v_i, v_i'\}$, $\{v_{i+1}, v_{i+1}'\}$, $\{v_{i+2}, v_{i+2}'\}$, and $\{v_{i+3}, v_{i+3}'\}$ containing only blue vertices.

Next we claim that there is no i $(1 \leq i \leq n)$ such that one vertex of each of three pairs $\{v_i, v_i'\}$, $\{v_{i+1}, v_{i+1}'\}$, and $\{v_{i+2}, v_{i+2}'\}$ is red. Without loss of generality, we consider $\{v_1, v_1'\}$, $\{v_2, v_2'\}$, $\{v_3, v_3'\}$, and $\{v_4, v_4'\}$, one of which contains only blue vertices. We now make several observations:

(1) Three consecutive vertices in C (or in C'), say v_1, v_2, v_3 (or v_1', v_2', v_3'), cannot all be colored red. Otherwise, the blue vertex v_2' (or v_2) is not rooted at any copy of Y_2.

(2) For three consecutive vertices of C, say v_1, v_2, v_3, none of the triples v_1, v_2, v_3'; v_1', v_2', v_3; v_1, v_2', v_3'; v_1', v_2, v_3 can all be colored red. If the first triple, say, is colored red, then v_4 and v_4' are both blue, implying that one of $\{v_5, v_5'\}$ is red. Then one of $\{v_4, v_4'\}$ is blue and is not rooted at any copy of Y_2.

(3) For each three consecutive vertices of C, say v_1, v_2, v_3, neither v_1, v_2', v_3 nor v_1', v_2, v_3' can all be colored red. If the first triple, say, is red, the blue vertex v_2 is not rooted at any copy of Y_2.

From these three observations, we conclude that at least one of every three pairs $\{v_i, v_i'\}$, $\{v_{i+1}, v_{i+1}'\}$, $\{v_{i+2}, v_{i+2}'\}$ contains only blue vertices.

We can now consider the three pairs $\{v_1, v_1'\}$, $\{v_2, v_2'\}$, $\{v_3, v_3'\}$, where we may assume that v_3 and v_3' are both blue. There are three cases.

Case 1. *v_1 and v_2 are red.* Then v_3, v_3', v_n, v_n' are all blue. This implies that v_4', v_5' are red and v_4, v_5, v_6, v_6' are blue. If $n = 7$, then the blue vertex v_7' is not rooted at a copy of Y_2. Hence, we may assume that $n \geq 10$. Next we have v_7, v_8 are red and v_7', v_8', v_9, v_9' are blue. Continuing this process, we conclude that v_{n-1} and v_{n-1}' are blue. However, then the blue vertex v_n' is not rooted at any copy of Y_2.

Case 2. *v_1 and v_2' are red.* Then $v_1', v_2, v_n, v_n', v_3, v_3'$ are blue, implying that the blue vertex v_1' is not rooted at any copy of Y_2.

Case 3. *v_2 is red and $v_1, v_1', v_2', v_3, v_3'$ are all blue.* Then the blue vertex v_2' is not rooted at any copy of Y_2.

Therefore, we are left with the only possibility that there exists an i ($1 \leq i \leq n$) such that exactly one vertex of each of the pairs $\{v_i, v_i'\}$, $\{v_{i+1}, v_{i+1}'\}$, $\{v_{i+2}, v_{i+2}'\}$, $\{v_{i+3}, v_{i+3}'\}$ is red. Thus, as we have seen, $\gamma_{Y_2}(G) \geq 2\lceil n/3 \rceil$.

By coloring v_1, v_1' red and v_{3i}, v_{3i}' red for all i with $1 \leq i \leq (n-1)/3$, and all other vertices blue, we obtain a Y_2-coloring of G with $2 + 2(n-1)/3 = (2n+4)/3$ red vertices. This implies that $\gamma_{Y_2}(G) \leq (2n+4)/3 = 2\lceil n/3 \rceil$.

Now let $n \equiv 2 \pmod 3$. Let there be given a Y_2-coloring of G. Since every blue vertex is adjacent to a red vertex having a red neighbor, we may assume that the pairs $\{v_1, v_1'\}$, $\{v_2, v_2'\}$ contain at least two red vertices. However, the remaining vertices of G can be partitioned into $(2n-4)/6$ subsets, each of which induces a copy of $P_3 \times K_2$ and so, by Lemma 2.4, $\gamma_{Y_2}(G) \geq 2 + 2(2n-4)/6 = (2n+2)/3 = 2\lceil n/3 \rceil$.

If we color the vertices v_1, v_1' red and v_{3i}, v_{3i}' red for all i with $1 \leq i \leq (n-2)/3$, and all other vertices blue, we obtain a Y_2-coloring of G with $(2n+2)/3$ red vertices. This implies that $\gamma_{Y_2}(G) \leq (2n+2)/3 = 2\lceil n/3 \rceil$. ■

We now turn our attention to $\gamma_{Y_3}(C_n \times K_2)$ and begin with a lemma.

Lemma 2.6 *Let G be a cubic graph containing $P_4 \times K_2$ as an induced subgraph. In every Y_3-coloring of G, at least four vertices of $P_4 \times K_2$ are red. Furthermore, if exactly four vertices of $P_4 \times K_2$ are colored red, then only the six 2-stratified graphs $P_4 \times K_2$ shown in Figure 3 can occur.*

We state two immediate consequences of Lemma 2.6.

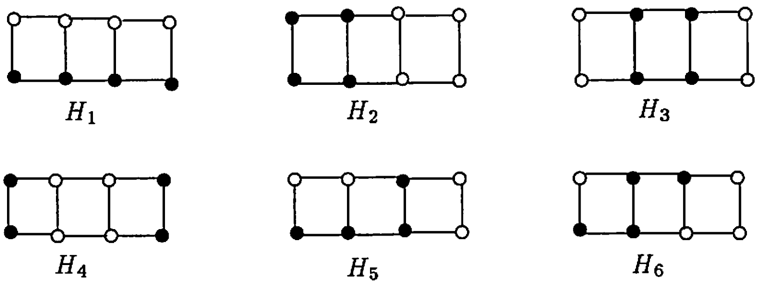

Figure 3: The six 2-stratified graphs $P_4 \times K_2$ containing exactly four red vertices in a cubic graph with a Y_3-coloring

Corollary 2.7 *If G is a cubic graph containing k pairwise disjoint copies of $P_4 \times K_2$, then $\gamma_{Y_3}(G) \geq 4k$.*

Corollary 2.8 *Let there be given a Y_3-coloring of the graph $G = C_n \times K_2$. If $v_i, v_i', v_{i+1}, v_{i+1}'$ are all blue for some i $(1 \leq i \leq n)$, then v_{i-2}, v_{i-2}', v_{i-1}, v_{i-1}', v_{i+2}, v_{i+2}', v_{i+3}, v_{i+3}' are all red.*

We now determine $\gamma_{Y_3}(G)$ for $G = C_n \times K_2$.

Theorem 2.9 *For $n \geq 3$, $\gamma_{Y_3}(C_n \times K_2) = n$.*

Proof. If we color all vertices of C red and all vertices of C' blue, then this is a Y_3-coloring with n red vertices. Hence $\gamma_{Y_3}(C_n \times K_2) \leq n$.

To establish the reverse inequality, we first assume that $n \equiv 0 \pmod 4$. Then $V(G)$ can be partitioned into $n/4$ subsets, each of which induces a $P_4 \times K_2$. It then follows by Lemma 2.6 that $\gamma_{Y_3}(C_n \times K_2) \geq 4(n/4) = n$.

Hence we may assume that $n \not\equiv 0 \pmod 4$. Now assume, to the contrary, that $\gamma_{Y_3}(C_n \times K_2) < n$. Let there be given a Y_3-coloring of G with $\gamma_{Y_3}(C_n \times K_2)$ red vertices. We consider three cases.

Case 1. $n \equiv 1 \pmod 4$. Then there exists some integer i $(1 \leq i \leq n)$ such that both vertices of $\{v_i, v_i'\}$ are blue, say $i = n$. Let $V_j = \{v_{4j-3}, v_{4j-3}', v_{4j-2}, v_{4j-2}', v_{4j-1}, v_{4j-1}', v_{4j}, v_{4j}'\}$ for $j = 1, 2, \cdots, (n-1)/4$. Then $\langle V_j \rangle = P_4 \times K_2$ for all such j. Necessarily, exactly four vertices in each subgraph $\langle V_j \rangle$ are colored red. Thus each induced subgraph $\langle V_j \rangle$ is one of the graphs $H_1, H_2, \cdots, H_6$ in Figure 3. If $\langle V_1 \rangle = H_1$, then either v_1 or v_1' is blue and is not rooted at a copy of Y_3. If $\langle V_1 \rangle = H_4$, then the blue vertex v_2 is not rooted at a copy of Y_3. The remaining possibilities are considered in four subcases.

Subcase 1.1. $\langle V_1 \rangle = H_2$. Necessarily, v_1, v_1', v_2, and v_2' are red by Corollary 2.8, which forces v_3, v_3', v_4, v_4' to be blue. Indeed, this forces $\langle V_j \rangle = H_2$

for $j = 2, 3, \cdots, (n-1)/4$, where $v_{4j-3}, v'_{4j-3}, v_{4j-2}, v'_{4j-2}$ are red and $v_{4j-1}, v'_{4j-1}, v_{4j}, v'_{4j}$ are blue. However, then $v_{n-2}, v'_{n-2}, v_{n-1}, v'_{n-1}$ are blue and so v_{n-1} is not rooted at any copy of Y_3.

Subcase 1.2. $\langle V_1 \rangle = H_3$. Then v_2, v'_2, v_3, v'_3 are red and v_1, v'_1, v_4, v'_4 are all blue. By Corollary 2.8, this forces $\langle V_j \rangle = H_2$ for $j = (n-1)/4, (n-5)/4, \cdots, 1$, which is a contradiction.

Subcase 1.3. $\langle V_1 \rangle = H_5$. Then, without loss of generality, either v_1 and v'_1 are blue or v_1 and v_2 are blue. In either case, v_1 is not rooted at any copy of Y_3, again a contradiction.

Subcase 1.4. $\langle V_1 \rangle = H_6$. If v_1 and v'_1 are blue, then v_1 is not rooted at a copy of Y_3. Hence, we may assume that v_1 is blue and v'_1 is red. But then $\langle V_j \rangle = H_2$ for all $j = 2, 3, \cdots, (n-1)/4$, where $v_{n-2}, v'_{n-2}, v_{n-1}, v'_{n-1}$ are blue, contradicting Corollary 2.8.

Hence if $n \equiv 1 \pmod 4$, then $\gamma_{Y_3}(C_n \times K_2) \geq n$.

Case 2. $n \equiv 2 \pmod 4$. There exists some integer i $(1 \leq i \leq n)$ such that there is at most one red vertex in $\{v_i, v'_i, v_{i+1}, v'_{i+1}\}$. Assume, without loss of generality, that $i = n-1$. So at most one of the vertices v_{n-1}, v'_{n-1}, v_n, v'_n is red. Let $V_j = \{v_{4j-3}, v'_{4j-3}, v_{4j-2}, v'_{4j-2}, v_{4j-1}, v'_{4j-1}, v_{4j}, v'_{4j}\}$ for $j = 1, 2, \cdots, (n-2)/4$.

Assume first that the vertices v_{n-1}, v'_{n-1}, v_n, v'_n are blue. Then all of v_1, v'_1, v_2, v'_2, v_{n-2}, v'_{n-2} v_{n-1}, v'_{n-1} are red. Now $\bigcup_{i=3}^{n-4}\{v_i, v'_i\}$ can be partitioned into $(n-6)/4$ subsets, each of which induces a $P_4 \times K_2$. It follows by Lemma 2.6 that $\gamma_{Y_3}(G) \geq 8 + 4\left(\frac{n-6}{4}\right) = n+2$, contradicting our assumption that $\gamma_{Y_3}(G) < n$.

Thus we may assume that exactly one vertex of v_{n-1}, v'_{n-1}, v_n, v'_n is red, say v_{n-1}. Then $\langle V_j \rangle = P_4 \times K_2$ for all j and exactly four vertices in each subgraph $\langle V_j \rangle$ are colored red. Thus each $\langle V_j \rangle$ is one of $H_1, H_2, \cdots, H_6$ by Lemma 2.6. Now $\langle V_1 \rangle \neq H_6$ since the blue vertex v_n is not rooted at a copy of Y_3. Indeed, by Lemma 2.6 then, $\langle V_1 \rangle = H_2$. In fact, this forces $\langle V_j \rangle = H_2$ for $j = 2, 3, \cdots, (n-2)/4$. Then $v_{n-3}, v'_{n-3}, v_{n-2}, v'_{n-2}$ are all blue and the blue vertex v'_{n-1} is not rooted at any copy of Y_3. Therefore, in this case, $\gamma_{Y_3}(C_n \times K_2) \geq n$.

Case 3. $n \equiv 3 \pmod 4$. Then there exists some integer i $(1 \leq i \leq n)$ such that at most two vertices of $\{v_i, v'_i, v_{i+1}, v'_{i+1}, v_{i+2}, v'_{i+2}\}$ are red, say $i = n-2$. Then there exists a pair v_j, v'_j of blue vertices for some $j = n-2, n-1, n$. If v_n and v'_n are both blue, then all of v_1, v'_1, v_2, v'_2 are red. Now $\bigcup_{i=3}^{n-1}\{v_i, v'_i\}$ can be partitioned into $(n-3)/4$ subsets, each of which induces a $P_4 \times K_2$. Hence, by Lemma 2.6, $\gamma_{Y_3}(G) \geq 4 + 4\left(\frac{n-3}{4}\right) = n+1$, a contradiction. Similarly, if both v_{n-2} and v'_{n-2} are blue, we have a contradiction. So we may assume that both v_{n-1} and v'_{n-1} are blue and that exactly one vertex in each of the pairs $\{v_{n-2}, v'_{n-2}\}$ and $\{v_n, v'_n\}$ is red. However, in any case, the blue vertex v_{n-1} is not rooted in any copy of Y_3. Therefore, $\gamma_{Y_3}(C_n \times K_2) \geq n$. ■

We now consider $\gamma_{Y_4}(C_n \times K_2)$ and begin with a lemma.

Lemma 2.10 *Let G be a cubic graph containing $P_5 \times K_2$ as an induced subgraph. In every Y_4-coloring of G, at least two vertices of $P_5 \times K_2$ are red.*

Proof. Assume, to the contrary, that there is a Y_4-coloring of G and that there is an induced subgraph $H = P_5 \times K_2$ in G containing at most one red vertex. We may assume that $V(H) = \{v_1, v'_1, v_2, v'_2, \cdots, v_5, v'_5\}$ in G, where $v_i v'_i \in E(H)$ for $1 \leq i \leq 5$. If H contains no red vertex, then the blue vertex v_3 is not rooted at any copy of Y_4. So H contains exactly one red vertex v. If $v = v_i$ (or $v = v'_i$), where $i = 1, 2$, then v_{i+1} (or v'_{i+1}) is not rooted in any copy of Y_4. Thus, $v = v_3$ (or $v = v'_3$). However, then, v'_3 (or v_3) is not rooted in any copy of Y_4. ■

Two consequences of Lemma 2.10 are given next.

Corollary 2.11 *If G is a cubic graph containing k pairwise disjoint copies of $P_5 \times K_2$, then $\gamma_{Y_4}(G) \geq 2k$.*

Corollary 2.12 *If n and k are integers with $n \geq 5k + 1 \geq 6$, then*

$$\gamma_{Y_4}(C_n \times K_2) \geq 2k + 1.$$

Proof. Let $G = C_n \times K_2$. Assume, to the contrary, that $\gamma_{Y_4}(G) \leq 2k$, where $n \geq 5k + 1 \geq 6$. Let there be given a Y_4-coloring of G with $\gamma_{Y_4}(G)$ red vertices. Let $S_i = \{ v_i, v'_i, v_{i+1}, v'_{i+1}, \cdots, v_{i+(n-5k-1)}, v'_{i+(n-5k-1)} \}$ for some i with $1 \leq i \leq n$ be a set of $n - 5k$ pairs of consecutive vertices in G. Then the set $V(G) - S_i$ can be partitioned into k subsets, each of which induces a $P_5 \times K_2$. These k copies of $P_5 \times K_2$ require at least $2k$ red vertices by Corollary 2.11. Hence, all vertices of S_i are blue. Since S_i is chosen arbitrarily, this implies that all vertices of G are blue, which is impossible. ■

The following theorem presents a formula for $\gamma_{Y_4}(C_n \times K_2)$. Since its proof involves extensive case consideration, we omit it.

Theorem 2.13 *For $n \geq 3$,*

$$\gamma_{Y_4}(C_n \times K_2) = \begin{cases} 2\lceil n/5 \rceil & \text{if } n \equiv 0, 3, 4 \ (mod\ 5) \\ & \text{or } n \equiv 2, 6 \ (mod\ 10), \\ 2\lceil n/5 \rceil - 1 & \text{if } n \equiv 1, 7 \ (mod\ 10). \end{cases}$$

We now consider $\gamma_{Y_5}(C_n \times K_2)$ and begin with a lemma.

Lemma 2.14 *Let G be a cubic graph containing $P_2 \times K_2$ as an induced subgraph. In every Y_5-coloring of G, at least two vertices of $P_2 \times K_2$ are red.*

Proof. Assume, to the contrary, that there is a Y_5-coloring of G and that there is an induced subgraph $H = P_2 \times K_2$ in G containing at most one red vertex. We may assume that $V(H) = \{v_1, v_1', v_2, v_2'\}$. Assume first that H contains no red vertices. Since v_1 is rooted at a copy of Y_5, it follows that v_1 is adjacent to a blue vertex w_1 that is adjacent to two red vertices. Necessarily, $w_1 \notin \{v_2, v_1', v_2'\}$. However, then, w_1 is not rooted at a copy of Y_5.

So H contains exactly one red vertex, say v_1. Since v_2 is rooted at a copy of Y_5, it follows that v_2 is adjacent to a blue vertex w_2 that is adjacent to two red vertices. Certainly, $w_2 \neq v_2'$. Then w_2 is not rooted at a copy of Y_5. So we have a contradiction in either case. ∎

Corollary 2.15 *If G is a cubic graph containing k pairwise disjoint copies of $P_2 \times K_2$, then $\gamma_{Y_5}(G) \geq 2k$.*

Theorem 2.16 *For $n \geq 3$, $\gamma_{Y_5}(C_n \times K_2) = 2\lceil n/2 \rceil$.*

Proof. Let $G = C_n \times K_2$. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* By Corollary 2.15, $\gamma_{Y_5}(G) \geq n$ for all even $n \geq 4$. If we color the vertices v_{2i+1}, v_{2i+1}' $(0 \leq i \leq n/2 - 1)$ red and the remaining vertices blue, we obtain a Y_5-coloring of G with n red vertices. Thus, $\gamma_{Y_5}(G) \leq n$. Therefore, $\gamma_{Y_5}(G) = 2\lceil n/2 \rceil$ for all even $n \geq 4$.

Case 2. *n is odd.* We first assume that $n \equiv 1 \pmod 4$. If we color v_n, v_n' red as well as $v_{4i+1}', v_{4i+2}', v_{4i+3}, v_{4i+4}$ red for all i with $0 \leq i \leq [(n-1)/4] - 1$ and the remaining vertices blue, we obtain a Y_5-coloring of G with $n+1$ red vertices. Thus, $\gamma_{Y_5}(G) \leq n+1$.

Next we show that $\gamma_{Y_5}(G) \geq n+1$. Assume, to the contrary, that $\gamma_{Y_5}(G) = n$. Let there be given a Y_5-coloring of G with n red vertices. First observe that for every i $(1 \leq i \leq n)$ the set $\{v_i, v_i'\}$ contains at most one red vertex, for otherwise, $V(G) - \{v_i, v_i'\}$ can be partitioned into $(n-1)/2$ subsets, each of which induces a $P_2 \times K_2$. By Corollary 2.15, $\gamma_{Y_5}(G) \geq [2(n-1)/2] + 2 = n+1$, a contradiction. Assume, without loss of generality, that v_n is blue and v_n' is red. Then the blue vertex v_n is rooted at a copy of Y_5. We may assume that v_1 is blue and v_1' and v_2 are both red. Then v_2' and v_3' are both blue and v_3 is red. Hence v_4' is red and v_4 is blue. Continuing this procedure, we have that $v_{4i+1}, v_{4i+2}', v_{4i+3}', v_{4i+4}$ are blue and $v'_{4i+1}, v_{4i+2}, v_{4i+3}, v'_{4i+4}$ are red for all $1 \leq i \leq [(n-1)/4]-1$. However, then the blue vertex v_{n-1} is not rooted at any copy of Y_5, a contradiction. Hence, $\gamma_{Y_5}(G) \geq n+1$. Therefore, $\gamma_{Y_5}(G) = n+1$ if $n \equiv 1 \pmod 4$.

Since the proof of the case when $n \equiv 3 \pmod 4$ is similar, we omit it. ■

We now turn to the 2-stratified claw Y_6. First we make the following observation concerning packing numbers and cubic graphs.

Proposition 2.17 *If G is a cubic graph, then $\gamma_{Y_6}(G) \geq \rho(G)$.*

Proof. Let there be given a Y_6-coloring of G with $\gamma_{Y_6}(G)$ red vertices. Also, let S be a maximum packing set. For each $v \in S$, at least one vertex of the closed neighborhood $N[v]$ of v is red, and so $\gamma_{Y_6}(G) \geq \rho(G)$. ■

In order to compute $\gamma_{Y_6}(C_n \times K_2)$, we begin with a lemma.

Lemma 2.18 *Let G be a cubic graph containing $P_4 \times K_2$ as an induced subgraph. In every Y_6-coloring of G, at least two vertices of $P_4 \times K_2$ are red. Furthermore, if exactly two vertices of $P_4 \times K_2$ are colored red, then only the three 2-stratified graphs $P_4 \times K_2$ shown in Figure 5 can occur.*

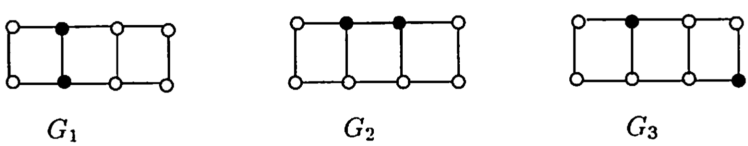

Figure 4: The three 2-stratified graphs $P_4 \times K_2$ containing exactly two red vertices and in which every blue vertex is rooted at a copy of Y_6

Proof. Assume, to the contrary, that there is a Y_6-coloring of G and that there is an induced subgraph $H = P_4 \times K_2$ in G containing at most one red vertex. We may assume that $V(H) = \{v_1, v'_1, v_2, v'_2, \cdots, v_4, v'_4\}$ and that any red vertex is one of v_1 and v_2. Then v'_2 is not rooted in any copy of Y_6, a contradiction. ■

Corollary 2.19 *If G is a cubic graph containing k pairwise disjoint copies of $P_4 \times K_2$, then $\gamma_{Y_6}(G) \geq 2k$.*

Corollary 2.20 *If n and k are positive integers with $n \geq 4k+1$, then*

$$\gamma_{Y_6}(C_n \times K_2) \geq 2k+1.$$

Proof. We may assume that v_1 is red. Then $V(G) - \{v_1, v'_1\}$ can be partitioned into k subsets, each of which induces a graph containing $P_4 \times K_2$ as a subgraph. Hence, by Lemma 2.19, $\gamma_{Y_6}(C_n \times K_2) \geq 2k+1$. ■

It is easy to see that $\gamma_{Y_6}(C_3 \times K_2) = 2$. For $n \geq 4$, $\gamma_{Y_6}(C_n \times K_2)$ is given below.

Theorem 2.21 *Let $n \geq 4$. If $n \equiv i$ (mod 4), where $0 \leq i \leq 3$, then*

$$\gamma_{Y_6}(C_n \times K_2) = 2\lfloor n/4 \rfloor + i.$$

Proof. Let $G = C_n \times K_2$. We first assume that $n \equiv 0$ (mod 4). Then $n = 4k$ for some integer $k \geq 1$. By Corollary 2.19, $\gamma_{Y_6}(G) \geq 2k$. On the other hand, if we color the vertices v_{4i+1}, v'_{4i+3} red for all i with $0 \leq i \leq k-1$ and the remaining vertices blue, then this is a Y_6-coloring with $2k$ red vertices. Hence, $\gamma_{Y_6}(G) \leq 2k$. Therefore, if $n \equiv 0$ (mod 4), then $\gamma_{Y_6}(G) = 2\lfloor n/4 \rfloor$. Hence we may assume that $n \not\equiv 0$ (mod 4). We consider three cases.

Case 1. $n \equiv 1$ (mod 4). Then $n = 4k+1$ for some integer $k \geq 1$. By Corollary 2.20, $\gamma_{Y_6}(G) \geq 2k+1$. We now color v_n red as well as v_{4i+1}, v'_{4i+3} red for all i with $0 \leq i \leq k-1$ and the remaining vertices blue. This is a Y_6-coloring with $2k+1$ red vertices. Hence, $\gamma_{Y_6}(G) \leq 2k+1$.

Case 2. $n \equiv 2$ (mod 4). Then $n = 4k+2$ for some integer $k \geq 1$. We now color the vertices v_{n-1}, v_n red as well as v_{4i+1}, v'_{4i+3} red for all i with $0 \leq i \leq k-1$ and the remaining vertices blue. This is a Y_6-coloring with $2k+2$ red vertices. Hence, $\gamma_{Y_6}(G) \leq 2k+2$.

Next, we show that $\gamma_{Y_6}(G) \geq 2k+2$. By Corollary 2.20, $\gamma_{Y_6}(G) \geq 2k+1$. Assume, to the contrary, that $\gamma_{Y_6}(G) = 2k+1$. Then there exists some integer i $(1 \leq i \leq n)$ such that $\{v_i, v'_i\}$ contains exactly one red vertex, say v_n is red and v'_n is blue.

Observe that there do not exist two consecutive pairs $\{v_i, v'_i\}$, $\{v_{i+1}, v'_{i+1}\}$ for some i $(1 \leq i \leq n)$ that contains two red vertices, for otherwise, the set $V(G) - \{v_i, v'_i, v_{i+1}, v'_{i+1}\}$ can be can be partitioned into k subsets, each of which induces a $P_4 \times K_2$, which implies that $\gamma_{Y_6}(G) \geq 2k+2$, a contradiction. Hence every 2-stratified graph $P_4 \times K_2$ in this Y_6-coloring is the graph G_3. Let $V_j = \{v_{4j+1}, v'_{4j+1}, v_{4j+2}, v'_{4j+2}, v_{4j+3}, v'_{4j+3}, v_{4j+4}, v'_{4j+4}\}$ with $0 \leq j \leq k-1$. Then $\langle V_j \rangle = G_3$ for all j. There are only two possible positions for the two red vertices in $\langle V_0 \rangle$: (1) v'_2 and v_4 are red and the remaining vertices are blue; (2) v_1 and v'_3 are red and the remaining vertices are blue. If (1) occurs, then each $\langle V_j \rangle$ has v'_{4j+2} and v_{4j+4} red and the remaining vertices blue; while if (2) occurs, then each $\langle V_j \rangle$ has v_{4j+1} and v'_{4j+3} red and the remaining vertices blue. In (1), the blue vertex v_{n-1} is not rooted at any copy of Y_6, while in (2), the blue vertex v'_{n-1} is not rooted at any copy of Y_6. These both produce contradictions.

Case 3. $n \equiv 3$ (mod 4). Then $n = 4k+3$ for some integer $k \geq 1$. We now color the vertices v_{n-2}, v_{n-1}, v_n red as well as v_{4i+1}, v'_{4i+3} red for all i with

$0 \leq i \leq k-1$ and the remaining vertices blue. This is a Y_6-coloring with $2k+3$ red vertices. Hence, $\gamma_{Y_6}(G) \leq 2k+3$.

We next show that $\gamma_{Y_6}(G) \geq 2k+3$. For $1 \leq i \leq n$, let $S_i = \{v_i, v_i', v_{i+1}, v_{i+1}', v_{i+2}, v_{i+2}'\}$. If some set S_i contains at least three red vertices, then $\gamma_{Y_6}(G) \geq 2k+3$ since $V(G) - S_i$ contains at least $2k$ red vertices by Corollary 2.19. Hence, we may assume that each set S_i contains at most two red vertices. We now make another observation. Suppose that there exists i $(1 \leq i \leq n)$ such that v_i, v_i' are red, say v_1, v_1' are red. Then v_j and v_j' are red for all $j \equiv 1 \pmod 3$. But then $\gamma_{Y_6}(G) > 2k+3$, a contradiction. Hence, at most one of v_i and v_i' is red for all i $(1 \leq i \leq n)$.

For $1 \leq i \leq n$, let $T_i = \{v_i, v_i', v_{i+1}, v_{i+1}'\}$. If some T_i contains only blue vertices, then v_{i+2}, v_{i+2}' are red, which, as we have seen, is impossible. Hence every set T_i contains at least one red vertex.

Now, $\gamma_{Y_6}(G) \geq 2k+1$ by Corollary 2.20. However, $\gamma_{Y_6}(G) \neq 2k+1$, for otherwise, let there be given a Y_6-coloring of G with exactly $2k+1$ red vertices. Then there exists i with $1 \leq i \leq n$ such that $\{v_i, v_i'\}$ contains exactly one red vertex, say v_n is red and v_n' is blue. Applying Lemma 2.18 to the vertices v_i, v_i', $1 \leq i \leq n-3$, we see that $v_{n-1}, v_{n-1}', v_{n-2}, v_{n-2}'$ are blue. On the other hand, applying Lemma 2.18 to the vertices v_i, v_i', $3 \leq i \leq n-1$, we see that v_1, v_1', v_2, v_2' are blue. But this implies that v_1 is not rooted at a copy of Y_6, a contradiction.

Assume, to the contrary, that $\gamma_{Y_6}(G) = 2k+2$ and let there be given a Y_6-coloring of G with exactly $2k+2$ red vertices. We show that for each i $(1 \leq i \leq n)$, the set $\{v_i, v_{i+1}\}$ (and $\{v_i', v_{i+1}'\}$) contains at most one red vertex, for suppose that v_{n-1} and v_n are red, say. Then v_{n-1}' and v_n' are blue. For each j with $0 \leq j \leq k-1$, let $V_j = \{v_{4j+1}, v_{4j+1}', v_{4j+2}, v_{4j+2}', \cdots, v_{4j+4}, v_{4j+4}'\}$. Since the blue vertex v_{n-2} is rooted at a copy of Y_6, it follows that v_{n-3}' and v_{n-5} are red, that is, $\langle V_{k-1} \rangle = G_3$. Continuing in this manner, we see that v_{4j+2} and v_{4j+4}' are red and so $\langle V_j \rangle = G_3$ for all j $(0 \leq j \leq k-1)$. However, then, the blue vertex v_1 is not rooted at a copy of Y_6, a contradiction. Hence, for each i, the set $\{v_i, v_{i+1}\}$ (or $\{v_i', v_{i+1}'\}$) contains at most one red vertex.

We now show that each set T_i contains exactly one red vertex. We have already seen that each set T_i contains at least one and at most two red vertices. Assume, to the contrary, that there exists some i $(1 \leq i \leq n)$ such that T_i contains exactly two red vertices, say T_{n-1} contains exactly two red vertices. Necessarily, then either v_{n-1}, v_n' are red or v_{n-1}', v_n are red, say the former. However, then, the blue vertex v_{n-1}' is not rooted at a copy of Y_6, a contradiction. Therefore, each set T_i contains exactly one red vertex, as claimed.

We now claim that there exists i $(1 \leq i \leq n)$ such that the set S_i contains exactly two red vertices, for if every set S_i has exactly one red vertex, then this red vertex must be v_{i+1} or v_{i+1}', but then v_{i+3} or v_{i+3}'

is red as well since S_{i+1} contains exactly one red vertex. We may assume that S_{n-2} contains exactly two red vertices. Then either v_{n-2} and v'_n are red or v'_{n-2} and v_n are red, say the former. This forces v_{4j+2} and v'_{4j+4} to be red for all j $(0 \leq j \leq k-1)$ and the remaining vertices to be blue and so $\langle V_j \rangle = G_3$. However, then, the blue vertex v_{n-3} is not rooted at a copy of Y_6, a contradiction. ∎

Next we consider Y_7, again beginning with a lemma.

Lemma 2.22 *Let G be a cubic graph containing $P_2 \times K_2$ as an induced subgraph. In every Y_7-coloring of G, at least two vertices of $P_2 \times K_2$ are red.*

Proof. Assume, to the contrary, that there is a Y_7-coloring of G for which some induced subgraph $H = P_2 \times K_2$ in G contains at most one red vertex. We may assume that $V(H) = \{v_1, v'_1, v_2, v'_2\}$ and that the vertices v'_1, v_2, v'_2 are blue. Then v'_2 is not rooted at a copy of Y_7, a contradiction. ∎

An immediate corollary now follows.

Corollary 2.23 *If G is a cubic graph containing k pairwise disjoint copies of $P_2 \times K_2$, then $\gamma_{Y_7}(G) \geq 2k$.*

We now present a formula for $\gamma_{Y_7}(C_n \times K_2)$.

Theorem 2.24 *For $n \geq 3$, $\gamma_{Y_7}(C_n \times K_2) = 2\lceil n/2 \rceil$.*

Proof. Let $G = C_n \times K_2$. We consider two cases.

Case 1. n is even. By Corollary 2.23, $\gamma_{Y_7}(C_n \times K_2) \geq n$. If we color the vertices v_{2i+1}, v'_{2i+1} $(0 \leq i \leq (n-2)/2)$ red and the remaining vertices blue, then we obtain a Y_7-coloring of G with n red vertices. Thus, $\gamma_{Y_7}(G) \leq n$. Therefore, $\gamma_{Y_7}(G) = n$.

Case 2. n is odd. If we color v_n, v'_n red as well as v_{2i+1}, v'_{2i+1} $(0 \leq i \leq (n-1)/2)$ red and the remaining vertices blue, then we obtain a Y_7-coloring of G with $n+1$ red vertices. Thus, $\gamma_{Y_7}(G) \leq n+1$.

We next show that $\gamma_{Y_7}(G) \geq n+1$. Let there be given a Y_7-coloring of G with $\gamma_{Y_7}(G)$ red vertices. If some the set $\{v_i, v'_i\}$, $1 \leq i \leq n$, contains two red vertices, then $\gamma_{Y_7}(G) \geq n+1$ since $V(G) - \{v_i, v'_i\}$ can be partitioned into $(n-1)/2$ subsets, each of which induces a $P_2 \times K_2$. By Corollary 2.23, this implies that $\gamma_{Y_7}(G) \geq [2(n-1)/2] + 2 = n+1$. Hence, we may assume that every set $\{v_i, v'_i\}$ contains at most one red vertex. Let $\{v_i, v'_i\}$ be a set with one red vertex. Then again by Corollary 2.23, we have that $\gamma_{Y_7}(G) \geq n$. Assume that $\gamma_{Y_7}(G) = n$. Therefore, $\{v_i, v'_i\}$ contains exactly one red vertex for each i with $1 \leq i \leq n$. Since n is odd, there must be two vertices v_i, v_{i+1} (or v'_i, v'_{i+1}) that are both red. However, there cannot be three red vertices v_i, v_{i+1}, v_{i+2} (or v'_i, v'_{i+1}, v'_{i+2}). We may assume that v_1, v_2 are red. This, in fact, forces $v'_3, v'_4, v'_n, v'_{n-1}$ to be red. Since v'_3, v'_4 are red, we must also have v_5, v_6, v'_7, v'_8, etc. red. Since v'_{n-1} is red, so is v'_{n-2}. However, then, the blue vertex v_{n-1} is not rooted at a copy of Y_7, a contradiction. ■

Finally, we turn to Y_8. Since the result (and proof) is similar to that of Y_7, we state only the final result.

Theorem 2.25 *For $n \geq 3$, $\gamma_{Y_8}(C_n \times K_2) = 2\lceil n/2 \rceil$.*

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