The lower irredundance and domination parameters are equal for inflated trees

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Abstract

The inflated graph G_I of a graph G is obtained by replacing every vertex of degree d by a clique K_d. We pursue the investigation of domination related parameters of inflated graphs initialized by Dunbar and Haynes. They conjectured that the lower irredundance and domination parameters are equal for inflated graphs. Favaron showed that in general the difference between them can be as large as desired. In this article we prove that the two parameters are equal for inflated trees.

Dedicated to Ernie Cockayne on the occasion of his 60th birthday

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of finite order $|V(G)| = n(G) \geq 2$ and size $|E(G)| = m(G)$. For a set A of vertices of the graph G and a vertex v of G, the degree of v (that is the number $|N(v)|$ of its neighbors) is denoted by $d(v)$, the number of the vertices of A which are adjacent to the vertex v is denoted by $d_A(v)$, and the subgraph induced by the set A is denoted by $G[A]$. If $A \subseteq V$, $N[A]$ denotes the union of closed neighborhoods of elements of A and for subsets A and B of V we say that B is *dominated by* A (or that A *dominates* B) if $B \subseteq N[A]$. An *independent* set S is a set of nonadjacent vertices. The minimum (resp. maximum) cardinality of a maximal (under inclusion) independent set is denoted by $i(G)$ (resp. $\beta(G)$). A set D of vertices of G is *dominating* if every vertex of $V - D$ has at least one neighbor in D. The minimum (resp. maximum) cardinality of a minimal dominating set is denoted by $\gamma(G)$ (resp. $\Gamma(G)$). For a set W of vertices of the graph G and a vertex w of W, the *W-private neighborhood of* w, is the set $pn_W(w) = N[w] - N[W - \{w\}]$. The elements of $pn_W(w)$ are called *W-private neighbors of* w. A W-private neighbor of w is either w itself, in which case w is an isolated vertex of $G[W]$, or is a neighbor of w in $V - W$ which is not adjacent to any vertex of $W - \{w\}$. This latter type will be called an *external* W-private neighbor (abbreviated W-EPN) of w. In what follows if w is not an isolated vertex of $G[W]$ and has a unique external W-private neighbor we denote the latter by $\tilde{w}$. The set W is *irredundant* if for all $w \in W$, $pn_W(w) \neq \emptyset$. The minimum (resp. maximum) cardinality of a maximal irredundant set is denoted by $ir(G)$ (resp. $IR(G)$). An *ir-set* is a maximal irredundant set of cardinality $ir(G)$. We mention the well-known chain of inequalities among these parameters :

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Throughout this paper, when W is a set of vertices of G, we can partition the set of vertices of G into the disjoint union $Y_W \sqcup Z_W \sqcup B_W \sqcup M_W \sqcup R_W$ (if there is no ambiguity we omit the letter W) where

- the set $Z_W = \{w \in W \mid d_W(w) = 0\}$ is the set of isolated vertices of W,
- the set $Y_W = W - Z_W$ is the set of nonisolated vertices of W,
- the set $B_W = \{v \in V - W \mid d_W(v) = 1\}$ is the set of W-EPN of some vertex in W,
- the set $M_W = \{v \in V - W \mid d_W(v) \geq 2\}$ is the set of vertices of $V - W$ with at least two neighbors in W,
- the set $U_W = \{v \in V - W \mid d_W(v) = 0\}$ is the set of vertices of $V - W$ which are undominated by W.

We need one additional concept about private neighborhoods. A vertex v *annihilates* (or W-annihilates if any confusion occurs) a vertex w of an irredundant set W, if v dominates the W-private neighborhood $pn_W(w)$. We can now state a necessary and sufficient condition for an irredundant set to be maximal.

Theorem 1.1 (Cockayne et al., [2]) *An irredundant set W is maximal, if and only if, for every $t \in N[U_W]$ there exists $w \in W$ such that t annihilates w.*

Haynes and Schmidt [5] defined a graph operation called the *inflated graph* G_I of a simple graph G without isolates and which is obtained as follows. Each vertex x of degree $d(x)$ of G is replaced by a clique $C(x) \simeq K_{d(x)}$ and each edge (x, y) of G is replaced by an edge (u, v) in such a way that $u \in V(C(x))$ and $v \in V(C(y))$. Moreover different edges of G are replaced by nonadjacent edges of G_I. Note that there are two different kinds of edges in G_I. The edges of the cliques $C(x)$ are colored red and these cliques are called *red cliques*. The other edges, which correspond to the edges of G, are colored blue and form a perfect matching of G_I. So every vertex of G belongs to exactly one red clique and is incident to exactly one blue edge (Note that vertices with degree 1 in G lie in a red clique K_1 and a blue edge). Two adjacent vertices of G_I are said to be *red-adjacent* (and each is a *red-neighbor* of the other) if they belong to a same red clique and *blue-adjacent* (and each is the *blue-neighbor* of the other) otherwise. Observe that, since G is simple, two red cliques are joined by at most one blue edge. In this paper we adopt the convenient following notation : if x is a vertex of G_I we denote by x' the other endvertex of the blue edge containing x, by $C(x)$ the red clique containing x, and if S is a subset of $V(G)$ we denote by S_I the inflated graph of $G[S]$. If S is a subset of $V(G_I)$, a red clique $C = \{x_1, x_2, \cdots, x_p\}$ is dominated by S if and only if either C is *occupied* by S that is $C \cap S \neq \emptyset$, or C is *besieged* by S that is $\{x'_1, x'_2, \cdots, x'_p\} \subseteq S$ (indeed, the neighborhood of x_i located out of the red clique C is only x'_i for $i \in \{1, 2, \cdots, p\}$).

We point out the following results of common use when W is an irredundant set of an inflated graph G_I.

Result $\mathbf{R_1}$ If the set W is maximal irredundant, then for every $u \in U_W$ there exists at least one vertex y in Y_W which is annihilated by u (see Theorem 1.1).

Result $\mathbf{R_2}$ If $y \in Y_W$ is annihilated by some $u \in U_W$, then the external W-private neighborhood of y is exactly one vertex denoted by $\tilde{y}$.

Proof of $\mathbf{R_2}$. Suppose that the external W-private neighborhood of y contains at least two vertices. If the blue-neighbor y' of y is one of them, these two W-private neighbors have no common neighbor different from y since the graph G is simple. Otherwise these two W-private neighbors belong to $C(y)$, their common neighbors are in $C(y)$ and hence are dominated by y. In either case, $pn_W(y)$ cannot be dominated by any vertex of U_W, a contradiction. ∎

Dunbar and Haynes initialized in [3] the study of the six previous parameters concerning independance, domination and irredundance in the case of inflated graphs. In particular, they pointed out that $\gamma(G_I) = i(G_I)$ for every graph G since every inflated graph is claw-free (without an induced subgraph isomorphic to $K_{1,3}$) and since Allan and Laskar presented in [1] a sufficient condition for $\gamma = i$ in terms of this forbidden subgraph. We now give some results Dunbar and Haynes proved for inflated trees.

Theorem 1.2 (Dunbar,Haynes, [3]) *Every inflated tree T_I satisfies*
1) $\gamma(T_I) = i(T_I) \leq n(T) - 1$ and equality holds if and only if the tree T is isomorphic to the star $K_{1,p}$.
2) $\beta(T_I) = \Gamma(T_I) = IR(T_I) = n(T) - 1$.

In the same paper, they conjectured that $ir(G_I) = \gamma(G_I)$ for every graph G. This was later shown not to be true by Favaron.

Theorem 1.3 (Favaron, [4])
For every integer k there exist 2-connected graphs G such that $\gamma(G_I) - ir(G_I) \geq k$.

But in this article we show that the two parameters are equal for inflated trees.

Theorem 1.4 *For every tree T, we have $ir(T_I) = \gamma(T_I)$.*

2 Irredundance and domination in inflated graphs

We begin by establishing three results concerning maximal irredundant sets of inflated graphs in order to prove Theorem 1.4.

Definition 2.1 *If W is an irredundant set of an inflated graph we consider*
- *the set Y_1 which is the set of vertices y of Y which are annihilated by some vertex u of U and such that the unique (by Result R_2) W-EPN $\tilde{y}$ of y is its blue neighbor y'.*
- *the set Y_2 which is the set of vertices y of Y which are annihilated by some vertex u of U, and in such a way that the unique (by Result R_2) W-EPN $\tilde{y}$ of y belongs to $C(y)$ and that the vertex u annihilates no vertex of Y_1.*
- *the set B_1 which is the subset $\{y' \mid y \in Y_1\}$ of B.*
- *the set B_2 which is the set of vertices of B which have a blue-neighbor in U.*
- *the set J which is the set of vertices of W admitting a vertex b of B_2 as an W-EPN.*

Remark 2.2 *Observe that from these definitions we have $Y_1 \cap Y_2 = \emptyset$, $|B_1| = |Y_1|$, $Y_2 \subseteq J$, $|B_2| \geq |Y_2|$, $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 \subseteq B$ (may be strict). Moreover, every vertex u of U annihilates at least one vertex y of $Y_1 \cup Y_2$. If $y \in Y_1$ then $N_W(y) \in C(y)$. If $y \in Y_2$ then $N_W(y) = \{y'\}$ and thus a vertex of Y_1 and a vertex of Y_2 are never adjacent.*

Lemma 2.3 *If W is an irredundant set of an inflated graph, then no vertex of M is adjacent to two vertices of Y_2.*

Proof. Suppose that two vertices y and z of Y_2 have a common neighbor t in M. Since y' and z' are in W, the edges ty and tz should be red, which contradicts the fact that y and z are not in the same red clique. ∎

Lemma 2.4 *Let W be a maximal irredundant set of an inflated graph, y a vertex of Y_2 and $\tilde{y}$ the only W-private neighbor of y. Then the vertex y is annihilated by only one vertex in U which is the blue-neighbor $\tilde{y}'$ of $\tilde{y}$. Moreover, we have $C(\tilde{y}') \cap W = \emptyset$ and for every vertex t in $C(\tilde{y}') \cap U$, there exists some vertex z in Y_2 such that the vertex t is the blue-neighbor $\tilde{z}'$ of the unique W-EPN $\tilde{z}$ of z.*

Proof. By definition of Y_2, the vertex y is annihilated by some vertex u of U which is necessarily blue-adjacent to the unique W-EPN $\tilde{y}$ of y (indeed, $C(y) \cap U = \emptyset$ since $y \in W$). Hence the vertex u is the blue-neighbor $\tilde{y}'$ of $\tilde{y}$. The red clique $C(\tilde{y}')$ has no vertex in W since the vertex $\tilde{y}'$ is not dominated by the set W. Let $t \in C(\tilde{y}') \cap U$. The vertex t annihilates some vertex $z \in Y$ by R_1, and z has a unique W-EPN $\tilde{z}$ by R_2. If $z \in Y_1$, then $z\tilde{z}$ is a blue edge, $t\tilde{z}$ a red edge belonging to $C(\tilde{y}')$, and thus $\tilde{y}'$ also annihilates the vertex $z \in Y_1$. Thus the vertex $\tilde{y}'$ annihilates $y \in Y_2$ and $z \in Y_1$ which contradicts the definition of Y_2. Hence $z \in Y - Y_1$, the edge $z\tilde{z}$ is red, and the edge $t\tilde{z}$ is blue. Finally, if t annihilates any vertex of Y_1, then $\tilde{y}'$ annihilates the same vertex, leading again to a contradiction. So t annihilates no vertex of Y_1, and thus $z \in Y_2$. ∎

Proposition 2.5 *In every inflated graph G_I there exists an ir-set W such that its associated set Y_2 (see Definition 2.1) is empty.*

Proof. We consider an arbitrary ir-set X of G_I and from this set X we will construct an *ir*-set W as required. Let ψ be the function from X into $V(G_I)$ which is the identity on $X - Y_2(X)$ and which is defined for every $y \in Y_2(X)$ by $\psi(y) = \tilde{y}$ (where the vertex $\tilde{y}$ is the unique W-EPN of y). Observe that the vertex $\psi(y)$ is in the red clique $C(y)$. Let W be $\psi(X)$. We claim that W is an *ir*-set as required.

By the private neighbor property the function ψ is an injection and we therefore have $|W| = |\psi(X)| = |X| = ir$. Let us first prove that W is a maximal irredundant set. If $y \in Y_2(X)$, then the vertex $\psi(y) = \tilde{y}$ is red-adjacent to y, blue-adjacent to some vertex $u \in U$, and thus cannot be adjacent to any other X-EPN. Hence the vertices $\psi(y)$ for $y \in Y_2$ are isolated in W, and $pn_W(x) \supseteq pn_X(x) \neq \emptyset$ for every other vertex x of W (in this case $x \in X - Y_2(X)$). Therefore W is an irredundant set of cardinality ir. From the definition of W and the fact that no vertex of M_X is adjacent to two vertices of $Y_2(X)$ (see Lemma 2.3), it follows that $N[W] \supseteq N[X]$ and thus $U_W \subseteq U_X$. Moreover, if $pn_W(x) \neq pn_X(x)$ for some $x \in X - Y_2(X)$, that is if there exists some $b \in pn_W(x) - pn_X(x)$, then b is adjacent to a vertex $v \in Y_2(X)$, and the edge bv is red since by Remark 2.2 the blue-neighbor v' of v is in X. Hence b is adjacent to the vertex $\tilde{v} = \psi(v)$ of $C(v)$.

Then b is adjacent to the two vertices x and $\psi(v)$ of W, in contradiction with $b \in pn_W(x)$. Therefore $pn_W(x) = pn_X(x)$ for all $x \in X - Y_2(X)$. Now let t be any vertex of $N[U_W]$. Since $U_W \subseteq U_X$, it follows that $t \in N[U_X]$. Since the irredundant set X is maximal and by Theorem 1.1, the vertex t dominates $pn_X(v)$ for some $v \in X$. Suppose that $v \in Y_2(X)$ and that $pn_X(v) = \{\tilde{v}\} = \{\psi(v)\}$. If $t \in C(v) = C(\tilde{v})$, then, since $t \notin pn_X(v)$, the vertex t is adjacent (necessarily by a blue edge) to another vertex of X. Thus the vertex t and all its neighbors are dominated by X, and therefore $t \notin N[U_X]$. Hence thr vertex t is blue-adjacent to $\tilde{v}$, that is, $t = \tilde{v}'$. By Lemma 2.4, each of the other neighbors of t, that is each vertex of $C(t)$, is blue-adjacent to a vertex of W, which contradicts $t \in N[U_W]$. Therefore $v \in X - Y_2(X)$. In this case we saw that $pn_W(v) = pn_X(v)$, and thus t dominates $pn_W(v)$. Hence by Theorem 1.1 W is a maximal irredundant set of order ir.

Finally we have to prove that $Y_2(W) = \emptyset$. Since the vertices $\psi(v)$ for $v \in Y_2(X)$, are isolated in W, we have $Y_W \subseteq Y_X - Y_2(X)$. Let y be a vertex of $Y_1(X)$ and u a vertex of U_X dominating $pn_X(y)$. Since u does not X-annihilate any vertex of $Y_2(X)$, by the definition of $Y_2(X)$, the vertex u is also in U_W, and since $pn_W(y) = pn_X(y)$, the vertex u W-annihilates y. Hence $y \in Y_1(W)$. The vertices of $Y_2(W)$ belong thus to $Y(X) - [Y_1(X) \cup Y_2(X)]$. Let $v \in Y(X) - [Y_1(X) \cup Y_2(X)]$. Then either v is X-annihilated by no vertex of U_X, or $pn_X(v)$ is a red-neighbor $\tilde{v}$ of v, the vertex v is X-annihilated by the vertex $u = \tilde{v}'$ of U_X, and u also X-annihilates a vertex y of $Y_1(X)$. In the first case, from $pn_W(v) = pn_X(v)$ and $U_W \subseteq U_X$, it follows that v is W-annihilated by no vertex of U_W and thus $v \notin Y_2(W)$. In the second case, u belongs to U_W since any edge between a vertex $\psi(t)$ with $t \in Y_2(X)$ should be blue, and u is already blue-adjacent to $\tilde{v}$. Since $pn_W(y) = pn_X(y)$, the vertex u W-annihilates the vertex y of $Y_1(W)$ and thus $v \notin Y_2(W)$. Therefore $Y_2(W) = \emptyset$.

Thus W is a maximal irredundant set as required and the proposition holds. ∎

We are now able to prove the main result which states that the lower irredundance and domination parameters are equal for inflated trees.

Proof of Theorem 1.4 : By Proposition 2.5, there exists an ir-set W of T_I whose associated set Y_2 is empty. From W we will construct a dominating set D of T_I such that $|D| \leq |W|$, from which it will follow that $\gamma(T_I) \leq |D| \leq |W| = ir(T_I)$ and thus the theorem will hold since $ir(G) \leq \gamma(G)$ for any graph G. Moreover our dominating set D will satisfy the following Property $\mathcal{P}$: for every red clique C of T_I, $C \cap W \neq \emptyset$ implies

$C \cap D \neq \emptyset$.

In order to construct the dominating set D of T_I, we consider the following skeleton $\mathcal{O}_{\mathcal{I}}$ of the inflated tree T_I. Let O be the subset $U \cup Y_1 \cup B_1 \cup B_2$ of T_I and $\mathcal{O}$ the subgraph of T generated by the edges of T, whose corresponding edges in T_I are the blue edges of $T_I[O]$. The graph $\mathcal{O}$ is a forest and the graph $\mathcal{O}_{\mathcal{I}}$ is the subgraph of T_I induced by the vertices of O whose blue-neighbors are also in O.

Claim 1 One can partition the set of the red cliques in $\mathcal{O}_I$ into three disjoint sets $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ defined as follows. We say that the red clique C in $\mathcal{O}_I$ belongs to

- the class $\mathcal{T}_1$ if $V(C) \subseteq (U \cup B_1)$,
- the class $\mathcal{T}_2$ if $V(C) \subseteq Y_1$,
- the class $\mathcal{T}_3$ if $V(C) \subseteq B_2$.

Proof of Claim 1 : Since by the definition of these different sets, there is no edge between Y_1 and B_2, Y_1 and U, B_1 and B_2, and all the edges between Y_1 and B_1 and between B_2 and U are blue, a red clique cannot intersect two sets among the three sets $U \cup B_1$, Y_1 and B_2.

Claim 2 Every vertex $u \in U$ is blue-adjacent to a red clique in $\mathcal{O}_I$ which belongs either to the class $\mathcal{T}_1$ or to the class $\mathcal{T}_3$.

Proof of Claim 2 : Suppose that the blue neighbor u' of a neighbor u of U does not belong to O, and thus belongs to M by the definition of B_2. The vertex u' is red-adjacent to two vertices x_1 and x_2 of Y, which form a red clique with u'. Hence u' cannot be adjacent to any vertex of B and $W \cup \{u'\}$ is irredundant, a contradiction with the maximality of W. Therefore every vertex u of U has its blue neighbor u' in $U \cup B_2$. We denote by C the red clique of T_I containing u', by c its corresponding vertex in T (by what precedes, $c \in \mathcal{O}$), and by $C/\mathcal{O}_I$ the red clique of $\mathcal{O}_I$ corresponding to the vertex c of $\mathcal{O}$. Observe that $C/\mathcal{O}_I = T_I[C \cap V(\mathcal{O}_I)]$. The clique $C/\mathcal{O}_I$ contains u' and $u' \in U \cup B_2$. By Claim 1, if $u' \in U$ then $C/\mathcal{O}_I \in \mathcal{T}_1$, while if $u' \in B_2$ then $C/\mathcal{O}_I \in \mathcal{T}_3$.

Let us denote by D_0 the subset $W - (Y_1 \cup J)$ of W.

Claim 3 We have $V(\mathcal{O}_I) = O$ and the set $D_0 \cup J \cup Y_1$ dominates $V(T_I) - U$.

Proof of Claim 3 : By definition of $\mathcal{O}_I$, we have $V(\mathcal{O}_I) \subseteq O$. On the other hand, there is a blue perfect matching between Y_1 and B_1 and every vertex of B_2 is blue-adjacent to some vertex of U. Therefore by Claim 2, the blue-neighbor of every vertex in O is in O. Hence $O \subseteq V(\mathcal{O}_I)$ and consequently $V(\mathcal{O}_I) = O$. Finally the set $D_0 \cup J \cup Y_1 = W$ obviously dominates $V(T_I) - U$ by the definition of U.

Claim 4 If the red clique C belongs to the class $\mathcal{T}_1$, then $C \cap U \neq \emptyset$ and $C \cap B_1 \neq \emptyset$.

Proof of Claim 4 : Suppose first that the red clique C contains some vertex $u \in U$. Then, by result R_1 and since $Y_2 = \emptyset$, the vertex u annihilates some vertex in Y_1 and therefore we have $C \cap B_1 \neq \emptyset$. On the other hand, suppose that the red clique C contains some vertex $y \in B_1$. Then it immediately follows from the definition of Y_1 that $C \cap U \neq \emptyset$.

Claim 5 A red clique C in $\mathcal{O}_I$ which corresponds to a leaf c in $\mathcal{O}$ does not belong to the class $\mathcal{T}_1$.

Proof of Claim 5 : If Claim 5 is false, then by Claim 4, we have $|C| \geq 2$, which contradicts the fact that the vertex c is a leaf in $\mathcal{O}$.

Claim 6 Denote the elements of the set J by $x_1, x_2, \cdots, x_q$. Then, for every $i \in \{1, 2, \cdots, q\}$ and for any choice of $b_i \in B_2$ such that b_i is an W-EPN of x_i, the set $D_0 \cup \{b_1, b_2, \cdots, b_q\} \cup Y_1$ dominates $V(T_I) - U$, and satisfies Property $\mathcal{P}$, that is for every red clique C of T_I, $C \cap W \neq \emptyset$ implies that $C \cap [D_0 \cup \{b_1, b_2, \cdots, b_q\} \cup Y_1] \neq \emptyset$.

Proof of Claim 6 : Let Λ be the set $D_0 \cup \{b_1, b_2, \cdots, b_q\} \cup Y_1 = (W - J) \cup \{b_1, b_2, \cdots, b_q\}$. By the choice of vertices b_i, that is, since $b_i \in V(\mathcal{C}(x_i))$, it is clear that the set Λ satisfies Property $\mathcal{P}$. Let i be any integer in $\{1, 2, \cdots, q\}$. Observe that $V(\mathcal{C}(x_i)) \cap W = \{x_i\}$ since x_i and its W-EPN b_i are in the same red clique (see Definition 2.1). By Claim 3 and by the choice of vertices b_i (each vertex b_i dominates the red clique $\mathcal{C}(x_i)$), in order to prove that the set Λ dominates $V(T_I) - U$, it is sufficient to check that the set Λ dominates x_i' for $1 \leq i \leq q$. We consider two cases. Suppose first that $x_i \in Z$. Then the vertex x_i' is red-dominated in $\mathcal{C}(x_i')$ by W, for otherwise x_i' (resp. b_i') would be a $(W \cup \{b_i\})$-PN of x_i (resp. b_i) and $W \cup \{b_i\}$ would be an irredundant set, contradicting the maximality of W. In the second case, since $x_i \notin Z$ then $x_i' \in W$. Observe that in both cases $\mathcal{C}(x_i') \cap W \neq \emptyset$ and since the set Λ satisfies Property $\mathcal{P}$, the set Λ dominates x_i'.

We are now ready to construct the dominating set D of T_I as required. We first put the vertices of D_0 in D, and then we put recursively vertices in D by adding vertices from $Y_1 \cup J$ as follows.

Let R be a tree of the forest $\mathcal{O}$, let c_0 be some leaf of the tree R, C_0 be the red clique in R_I corresponding to c_0 and $D \cap C_0$ be the set C_0 (which is one vertex). Observe that by Claim 5 the red clique C_0 belongs either to the class $\mathcal{T}_2$ or to the class $\mathcal{T}_3$. Fix c_0 in R as its root and consider it as a rooted tree. We apply to the rooted tree R the depth-first search algorithm beginning at the root c_0 and we construct the dominating set D while visiting once each vertex of R.

Suppose that we are visiting a red clique C in R_I corresponding to a vertex c in R. We denote by C^- the red clique visited just before C in the process and by $\alpha(C)$ the vertex of C which is blue-adjacent to C^-.

- If the red clique C is C_0 (which belongs to $\mathcal{T}_2$ or $\mathcal{T}_3$), let the set $D \cap C_0$ be C_0 (which is one vertex).
- If the red clique C belongs to the class $\mathcal{T}_1$, then we do nothing.
- If the red clique C belongs to the class $\mathcal{T}_2$ and is not C_0, then the red clique C^- belongs to the class $\mathcal{T}_1$. By Claim 5, the red clique C^- is not C_0 and the vertex $\alpha(C^-)$ is therefore well-defined. Let the set $D \cap C$ be $C - \{\alpha(C)\}$ and the set $D \cap C^-$ be $\{\alpha(C^-)\}$ (the set $D \cap C^-$ corresponding to the red clique C^- has possibly already been defined while visiting another red clique E in the class $\mathcal{T}_2$, but it is straitforward to see that in this case $D \cap C^- = D \cap E^-$).
- If the red clique C belongs to the class $\mathcal{T}_3$ and is not C_0, let the set $D \cap C$ be $\{\alpha(C)\}$.

This process stops when all the vertices of the tree R have been visited once. We repeat the process for each tree of the forest $\mathcal{O}$.

Claim 7 The set D satisfies Property $\mathcal{P}$ and we have $|D| \leq |W|$.

Proof of Claim 7 : It follows from the construction of D that $|D| \leq |W|$, since each clique C of $\mathcal{T}_2$ provides at most $|C|$ vertices in D, and each clique C of $\mathcal{T}_3$ provides one (the red clique $\mathcal{C}(c) \cap J$ in T_I is one vertex x and in the process we associate uniquely x to its W-EPN $\alpha(C) \in B_2$). Moreover, in the construction of D we move the vertices of $Y_1 \cup J$ without leaving the red clique containing them except if the red clique C considered in the process belongs to the class $\mathcal{T}_2$. However in this case, the vertices of $C - \{\alpha(C)\}$ are in D, so that $C \cap D \neq \emptyset$ except if the red clique C is exactly one vertex c. But since $c \in Y_1$, the vertex c has then at least one red-neighbor in $W - (Y_1 \cup J) = D_0 \subseteq D$ and the set D therefore satisfies Property $\mathcal{P}$.

Claim 8 If the red clique C in $\mathcal{O}_I$ belongs to the class $\mathcal{T}_1$ and if the vertex $\alpha(C)$ is in the set U, then the vertex $\alpha(C)$ is in the set D.

Proof of Claim 8 : Since by Claim 5 the red clique C in $\mathcal{T}_1$ does not correspond to a leaf in $\mathcal{O}$, there exists at least one red clique C^+ visited in the process just after C. By Claim 4 and by the definition of B_1, we can suppose without loss of generality that the red clique C^+ belongs to the class $\mathcal{T}_2$. Then, by construction of D we get that the vertex $\alpha(C)$ belongs to the set D.

Claim 9 If the red clique C belongs to the class $\mathcal{T}_2$, then the set D dominates not only the red clique C but also the blue-neighbor of $\alpha(C)$.

Proof of Claim 9 : Since by Claim 7 the set D satisfies Property $\mathcal{P}$, then the set D dominates the red clique C. By construction of D the red clique C^- belongs to the class $\mathcal{T}_1$ and the vertex $\alpha(C^-)$ is in D. Then the blue-neighbor of $\alpha(C)$, which belongs to C^-, is dominated by the vertex $\alpha(C^-)$ of D.

Claim 10 The set D dominates the set $V(T_I) - U$.

Proof of Claim 10 : Let C be a red clique which belongs to the class $\mathcal{T}_3\}$ and c its corresponding vertex in T. The red clique $\mathcal{C}(c) \cap J$ in T_I is one vertex x and in the process we associate uniquely x to its W-EPN $\alpha(C) \in B_2$. By Claim 6 the set $V(T_I) - U$ is dominated by the set $D' = D_0 \cup \{\alpha(C) \mid \text{the red clique } C \text{ belongs to the class } \mathcal{T}_3\} \cup Y_1$. We obtain the set D from D' by moving some vertices from Y_1 to U as described in the process of construction of D (see the case where the red clique C belongs to the class $\mathcal{T}_2$). By Claim 9, the blue neighbors of these vertices of Y_1 and the red cliques containing these vertices of Y_1 are still dominated by D and thus the set D dominates $V(T_I) - U$.

Claim 11 The set D is a dominating set of T_I.

Proof of Claim 11 : By Claim 10, it is sufficient to prove that every $u \in U$ is dominated by the set D. Let u be any vertex in U and C be the red clique $\mathcal{C}(u)$ in $\mathcal{O}_I$ containing the vertex u. Observe that the red clique C belongs to the class $\mathcal{T}_1$, so that by Claim 5 the red clique C^- visited just before C in the construction of D exists. We consider two cases. In the first case, we suppose that the red clique C^- belongs to the class $\mathcal{T}_1$ or to the class $\mathcal{T}_3$. Then the vertex $\alpha(C)$ is in U and by Claim 8 the vertex $\alpha(C)$ is in the set D. In the second case, the red clique C^- belongs to the class $\mathcal{T}_2$ and $\alpha(C)$ is in B_1 . Suppose that the vertex $\alpha(C)$ is not in D, which means that the red clique C is not occupied by D. Then, by the construction of D, the set $C \cap B_1$ is exactly the vertex $\alpha(C)$. By Claim 2, the cliques C^+ visited just after C in the process (at least one exists by Claim 5) belong either to the class $\mathcal{T}_1$ or to the class $\mathcal{T}_3$. If such a red clique C^+ belongs to the class $\mathcal{T}_1$, then the vertex $\alpha(C^+)$ is in U, so that by Claim 8 the vertex $\alpha(C^+)$ is in D. If such a red clique C^+ belongs to the class $\mathcal{T}_3$, it immediately follows from the construction of D that the vertex $\alpha(C^+)$ is in D. Thus the vertices α associated to all the cliques visited just after C are in D. Since the blue-neighbor of $\alpha(C)$ is also in D, the red clique C is therefore besieged by the set D. Hence, the vertex u is dominated by the set D whether the red clique C^- belongs to the class $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Thus by Claims 7 and 11, the set D is a dominating set of T_I as required and the theorem holds. ∎

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