# Star Sets in Regular Graphs

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ABSTRACT. Let $G$ be a finite graph and let $\mu$ be an eigenvalue of $G$ of multiplicity $k$. A star set for $\mu$ may be characterized as a set $X$ of $k$ vertices of $G$ such that $\mu$ is not an eigenvalue of $G - X$. It is shown that if $G$ is regular then $G$ is determined by $\mu$ and $G - X$ in some cases . The results include characterizations of the Clebsch graph and the Higman-Sims graph.

# 1 Background

Let $G$ be a finite simple graph with vertex set $V(G) = \{1, 2, \dots, n\}$, and let $\mu$ be an eigenvalue of $G$ (that is, an eigenvalue of the (0,1)-adjacency matrix of $G$). Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$; for example, $\mathbf{e}_1$ is the column $(1, 0, \dots, 0)^T$. Let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$, where $\mathcal{E}(\mu)$ is the eigenspace of $\mu$. The vectors $P\mathbf{e}_1, P\mathbf{e}_2, \dots, P\mathbf{e}_n$ span $\mathcal{E}(\mu)$, and so there exists a subset $X$ of $V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a subset is called a *star set* for $\mu$, and the corresponding basis is called a *star basis* for $\mathcal{E}(\mu)$. (The terminology reflects the fact that the vectors $P\mathbf{e}_1, P\mathbf{e}_2, \dots, P\mathbf{e}_n$ form a eutactic star as defined by Seidel [12].) The arguments of [5, Section 3] show that $X$ is a star set for $\mu$ if and only if $|X| =$ dim $\mathcal{E}(\mu)$ and $\mu$ is not an eigenvalue of $G - X$. Proofs of this and other results reviewed in this section may be found in [6, Chapter 7].

If $\mu_1, \mu_2, \dots, \mu_m$ are the distinct eigenvalues of $G$ then a *star partition* for $G$ is a partition $V(G) = X(\mu_1)\dot{\cup}X(\mu_2)\dot{\cup}\cdots\dot{\cup}X(\mu_m)$ such that $X(\mu_i)$ is a star set for $\mu_i$ $(i = 1, 2, \dots, m)$. Every graph has a star partition; indeed it was shown in [11] that if $X$ is a star set for $\mu_i$ then $G$ has a star partition in which $X(\mu_i) = X$. Given any star partition, a corresponding star basis for $\mathbb{R}^n$ is obtained by stringing together the star bases for each eigenspace.

Star partitions were introduced as part of an algebraic approach to the graph isomorphism problem: one can associate with a graph a star basis of $\mathbb{R}^n$ which is canonical in the sense that two cospectral graphs are isomorphic if and only if they determine the same canonical star basis

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(see [5, Section5], [3] and [6, Chapter 8]). Star partitions are however of interest in their own right because star sets are related directly to graph structure (see [5] and [9]). For example, let $X$ be a star set corresponding to the eigenvalue $\mu$, and let $\overline{X}$ be the complement of $X$ in $V(G)$. One can show that if $\mu \neq 0$ then $\overline{X}$ is a dominating set; thus if $G - X$ is connected then so is $G$. If $\mu \notin \{-1,0\}$ then $\overline{X}$ is even a location-dominating set, that is, a dominating set such that distinct vertices in $X$ have distinct neighbourhoods in $\overline{X}$. It follows that when $\mu \notin \{-1,0\}$, $|V(G)|$ is bounded in terms of $|\overline{X}|$, and hence that there are only finitely many graphs in which $\mathcal{E}(\mu)$ has prescribed co-dimension. The more we know about $\overline{X}$, the more we know about $G$; indeed, $G$ is determined uniquely if we know $\mu$, $G - X$ (the subgraph induced by $\overline{X}$) and the embedding of $\overline{X}$ in $G$. For subsequent reference, we provide further details. We label vertices of $X$ before those of $\overline{X}$ so that the adjacency matrix of $G$ has the form $A' = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$, where $A$ is the adjacency matrix of $G - \overline{X}$, $C$ is the adjacency matrix of $G - X$ and the non-zero entries of $B$ correspond to the edges between $X$ and $\overline{X}$. We have

$$\mu I - A' = \begin{bmatrix} \mu I - A & -B^T \\ -B & \mu I - C \end{bmatrix}.$$

Since $\mu$ is not an eigenvalue of $G - X$, $\mu I - C$ is invertible and accordingly the rows of $(-B \mid \mu I - C)$ form a basis for the row-space of the matrix $\mu I - A'$. It follows that there exists a matrix $L$ such that $\mu I - A = L(-B)$ and $-B^T = L(\mu I - C)$. We may eliminate $L$ to obtain

$$\mu I - A = B^T(\mu I - C)^{-1}B. \tag{1}$$

We can now see that $A$, and hence the adjacency matrix of $G$ itself, is determined by $\mu, B$ and $C$.

The foregoing remarks point to the possibility of characterizing graphs by properties of $\overline{X}$ which have implications for the set $E(X, \overline{X})$ of edges between $X$ and $\overline{X}$. Examples of properties which illustrate this principle are (i) the minimality of $\overline{X}$ as a dominating set (investigated in [10]), and (ii) the regularity of $G - X$ in a graph $G$ which is itself regular [11] . For regular graphs of prescribed degree the general principle applies if we simply specify the graph $G - X$, and the purpose of this paper is to demonstrate this in particular cases. For example, we investigate $k$-regular graphs ($k > 1$) in which $G - X$ is a $k$-star $K_{1,k}$ or a double $k$-star $S_{k,k}$. (Here $S_{k,k}$ denotes the tree with two adjacent vertices of degree $k$ and all other vertices of degree 1.) If $\mu \neq k$ then, since $\mathcal{E}(\mu) \perp \mathcal{E}(k)$ and $\mathcal{E}(k)$ contains the all-1 vector, we have $\sum_{j=1}^{n} P\mathbf{e}_j = \mathbf{0}$. We exploit this relation in conjunction with the linear independence of the vectors $P\mathbf{e}_j$ ($j \in X$) to show that $G - \overline{X}$ is regular.

It follows that if also $\mu \notin \{-1, 0\}$ then the $\overline{X}$-neighbourhoods of vertices in $X$ form a block design on $\overline{X}$, and its point-block incidence matrix is just the matrix $B$ of equation (1). In some cases (for example, $\mu = 1$ and $G - X \cong K_{1,5}$, or $\mu = 2$ and $G - X \cong K_{1,22}$) there is only one possibility for this block design, and so $G$ is then determined uniquely by $\mu$ and $G - X$. In this way we obtain characterizations of the Clebsch graph [1, p.35] and the Higman-Sims graph [1, p.107].

We use the following additional notation throughout. An all-1 matrix is denoted by $J$, and an all-1 column vector by $\mathbf{j}$. For any vertex $v$ of $G$ we write $\Delta(v)$ for the neighbourhood of $v$, that is, $\Delta(v) = \{u \in V(G) : u \sim v\}$. Also, $\Delta^*(v) = \Delta(v) \cup \{v\}$, $\Gamma(v) = \Delta(v) \cap X$ and $\overline{\Gamma}(v) = \Delta(v) \cap \overline{X}$. If $A'$ has spectral decomposition $\mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_m P_m$ then we have $A'P_i = \mu_i P_i = P_i A'$ $(i = 1, 2, \ldots, m)$. In particular, for $P$ and $\mu$ as above we have the basic relation

$$\mu P\mathbf{e}_j = \sum_{k \in \Delta(j)} P\mathbf{e}_k \quad (j \in V(G)). \tag{2}$$

## 2 Induced stars

**Lemma 2.1** *Let $G$ be a $k$-regular graph $(k > 0)$ with an eigenvalue $\mu$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $G - X \cong K_{1,k}$. Then the following hold:*

*(i)* $\mu \notin \{-1, 0\}$,

*(ii)* *if $\mu = k$ then $k = 2$, $m = 1$ and $G$ is a 4-cycle,*

*(iii)* *if $\mu \neq k$ then $G - \overline{X}$ is regular of degree $d$, where $d = \mu + \frac{(k-1)\mu}{\mu+1}$.*

**Proof:** Let $\overline{X} = \Delta^*(w)$, where $\deg(w) = k$. We deal first with the case $\mu = k$. Then $m$ is the number of components of $G$ [4, Theorem 3.23]; but $G$ is connected (because $\mu \neq 0$) and so $m = 1$. Hence $X$ consists of a single vertex adjacent to each vertex of $\Delta(w)$. Thus each vertex of $\Delta(w)$ has degree 2, and so $k = 2$. It follows that $G$ is a 4-cycle.

Now suppose that $\mu \neq k$. Since $\mathcal{E}(\mu)$ and $\mathcal{E}(k)$ are orthogonal we have, in the notation of §1,

$$\sum_{u \in X} P\mathbf{e}_u = - \sum_{v \in \Delta(w)} P\mathbf{e}_v - P\mathbf{e}_w. \tag{3}$$

From the basic relation (2) we have

$$\mu P\mathbf{e}_w = \sum_{v \in \Delta(w)} P\mathbf{e}_v\ , \tag{4}$$

and (for $u \in X$),

$$\mu P\mathbf{e}_u = \sum_{h\in\Gamma(u)} P\mathbf{e}_h + \sum_{j\in\overline{\Gamma}(u)} P\mathbf{e}_j.$$

Summing over $u \in X$, we obtain

$$\mu \sum_{u\in X} P\mathbf{e}_u = \sum_{u\in X} d_u P\mathbf{e}_u + (k-1) \sum_{v\in\Delta(w)} P\mathbf{e}_v \ , \tag{5}$$

where $d_u$ is the degree of $u$ in $G - \overline{X}$.

It follows from (3) and (4) that

$$\sum_{u\in X} P\mathbf{e}_u = -(\mu+1)P\mathbf{e}_w \ ,$$

and so $\mu \neq -1$ by linear independence of the vectors $P\mathbf{e}_u$ $(u \in X)$. If $\mu = 0$ then $\sum_{v\in\Delta(w)} P\mathbf{e}_v = \mathbf{0}$ and so $d_u = 0$ for all $u \in X$ by equation (5). In this case, $X$ is an independent set and each vertex of $X$ is adjacent to each vertex of $\Delta(w)$. Thus $|X| = k-1$ and $G \cong K_{k,k}$, a contradiction because the multiplicity of 0 as an eigenvalue of $K_{k,k}$ is $2(k-1)$. Hence $\mu \notin \{-1, 0\}$.

Now equations (3) and (4) yield

$$\sum_{u\in X} P\mathbf{e}_u = (-1 - \frac{1}{\mu}) \sum_{v\in\Delta(w)} P\mathbf{e}_v \ , \tag{6}$$

and on eliminating $\sum_{v\in\Delta(w)} P\mathbf{e}_v$ from equations (5) and (6) we have

$$\sum_{u\in X} \left(\mu - d_u + \frac{(k-1)\mu}{\mu+1}\right) P\mathbf{e}_u = \mathbf{0}.$$

It follows that $d_u = \mu + \frac{(k-1)\mu}{\mu+1}$ for all $u \in X$. □

**Theorem 2.2** *Let $G$ be a $k$-regular graph $(k > 0)$ with $n$ vertices and an eigenvalue $\mu \neq k$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $G - X \cong K_{1,k}$. Then the following hold:*

*(i)* $\mu \in \mathbb{N} \cup \{-2, \frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})\}$,

*(ii)* $G - \overline{X}$ *is regular of degree* $\mu^2(\mu+2)$,

*(iii)* $k = \mu(\mu^2+3\mu+1)$, $m = (\mu^2+3\mu+1)(\mu^2+2\mu-1)$ *and* $n = (\mu^2+3\mu)^2$,

*(iv) if* $\mu \in \mathbb{N}$ *then a clique in* $G$ *has at most* $\mu + 1$ *vertices.*

**Proof:** We retain the notation of Lemma 2.1 and make use of equation (1) in the form

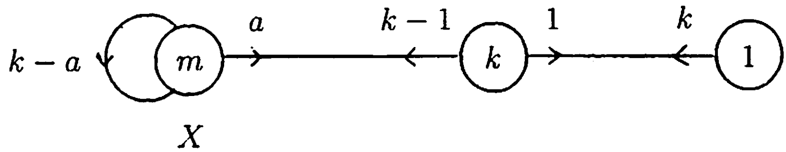


Fig.1

$$f(\mu)(\mu I - A) = B^T f(\mu)(\mu I - C)^{-1} B, \tag{7}$$

where $f$ is the minimal polynomial of $C$. Here $C = \begin{bmatrix} 0 & \mathbf{j}^T \\ \mathbf{j} & 0 \end{bmatrix}$, $f(x) = x(x^2 - k)$ and $f(\mu)\ (\mu I - C)^{-1} = (\mu^2 - k)I + \mu C + C^2$.

Each vertex of $X$ is adjacent to $a$ vertices of $\Delta(w)$, where $a = k - d$ and $d$ is given by Lemma 2.1. Thus $a = (k - \mu^2)/(\mu + 1)$ and $G$ has the form depicted in Fig.1. Each column of the matrix $B$ of equation (7) has weight $a$, and the first row of $B$ is zero. Accordingly a typical entry of the matrix $B^T \mu(\mu^2 - k)(\mu I - C)^{-1} B$ has the form

$$(0\ \ \mathbf{x}^T) \begin{bmatrix} \mu^2 & \mu \mathbf{j}^T \\ \mu \mathbf{j} & J \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix} + (0\ \ \mathbf{x}^T) \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & (\mu^2 - k)I \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix},$$

that is, $a^2 + (\mu^2 - k)\mathbf{x}^T\mathbf{y}$. On equating diagonal entries in (7) we find that $\mu^2(\mu^2 - k) = a^2 + (\mu^2 - k)a$, that is,

$$(\mu + 1)^2 \mu^2 (\mu^2 - k) = -\mu(\mu^2 - k)^2.$$

Since $\mu$ is not an eigenvalue of $C$, we may divide by $\mu(\mu^2 - k)$ to obtain $k = \mu(\mu^2 + 3\mu + 1)$. It follows that $d = \mu^2(\mu + 2)$ and $a = \mu(\mu + 1)$. Counting in two ways the edges between $X$ and $\overline{X}$, we have $ma = k(k-1)$, whence $m = (\mu^2 + 3\mu + 1)(\mu^2 + 2\mu - 1)$. Then $n = m + k + 1 = (\mu^2 + 3\mu)^2$.

On equating off-diagonal entries in equation (7) we find that $a^2 + (\mu^2 - k)\mathbf{x}^T\mathbf{y}$ is equal to $-\mu(\mu^2 - k)$ if the vertices corresponding to $\mathbf{x}$ and $\mathbf{y}$ are adjacent, and equal to zero otherwise. If we now express $a$ and $k$ in terms of $\mu$ we find that $\mathbf{x}^T\mathbf{y}$ is 0 or $\mu$, respectively. This tells us that for distinct vertices $u_1, u_2$ of $X$ we have:

$$|\overline{\Gamma}(u_1) \cap \overline{\Gamma}(u_2)| = \begin{cases} 0 \text{ if } u_1 \sim u_2 \\ \mu \text{ if } u_1 \not\sim u_2. \end{cases} \tag{8}$$

It follows that if $G-\overline{X}$ is not complete then $\mu \in \mathbb{N}$ (since $\mu \neq 0$ by Lemma 2.1). If $G-\overline{X}$ is complete then $d = m-1$, that is, $(\mu+2)(\mu+1)(\mu^2+\mu-1) = 0$. By Lemma 2.1, $\mu \neq -1$ and so $\mu$ is $-2$ or $\frac{1}{2}(-1 \pm \sqrt{5})$ in this case.

It remains to show that if $\mu \in \mathbb{N}$ and $H$ is a clique in $G$ with $t$ vertices then $t \leq \mu + 1$. We may suppose that $t \geq 3$, and in this case $H$ is contained in $G - \overline{X}$. To see this, note first that $w \notin V(H)$ and if $V(H) \cap \overline{X} \neq \emptyset$ then $V(H) \cap \overline{X}$ consists of a single vertex $v$ of $\Delta(w)$; but then $\Gamma(v)$ contains a pair $\{u_1, u_2\}$ of adjacent vertices, contradicting equation (8). Now the $t$ neighbourhoods $\overline{\Gamma}(u)$ $(u \in V(H))$ are pairwise disjoint subsets of $\Delta(w)$ of size $a$, and so $ta \leq k$, that is, $t\mu(\mu+1) \leq \mu(\mu^2+3\mu+1)$. Since $\mu \in \mathbb{N}$ it follows that $t(\mu+1) < \mu^2 + 3\mu + 2$, whence $t \leq \mu + 1$ as required. □

It is easy to see that, in the situation of Theorem 2.2, if $\mu = -2$ then $m = 1$ and $G$ is a 4-cycle, while if $\mu = \frac{1}{2}(-1 \pm \sqrt{5})$ then $m = 2$ and $G$ is a 5-cycle: in both cases, the hypotheses of the theorem are satisfied. If $\mu \in \mathbb{N}$ and $G$ is a strongly regular graph which satisfies the conclusions of Theorem 2.2 then $G$ is a negative Latin square graph of type $NL_\mu(\mu^2+3\mu)$; in other words, $G$ is a strongly regular graph with parameters $((\mu^2+3\mu)^2, \mu(\mu^2+3\mu+1), 0, \mu(\mu+1))$ (see [1, Chapter 2]). We give two examples which arise, and we shall see that our theorem enables us to characterize them among all regular graphs. The first is the Clebsch graph [1, p.35], the unique strongly regular graph with parameters (16, 5, 0, 2) : its eigenvalues are $5, 1, -3$ with multiplicities 1, 10, 5 respectively. Here we take $\mu = 1$ and $\overline{X} = \Delta^*(w)$, where $w$ is any vertex; then $G - \overline{X}$ is the Petersen graph, itself strongly regular, with eigenvalues $3, 1, -2$. It follows that a star partition of the Clebsch graph is given by $X(5) = \{w\}, X(1) = X$ and $X(-3) = \Delta(w)$. Another example is the Higman-Sims graph [1, p.107], the unique strongly regular graph with parameters (100, 22, 0, 6): its eigenvalues are $22, 2, -8$ with multiplicities 1, 77, 22 respectively. Here we take $\mu = 2$ and $\overline{X} = \Delta^*(w)$ where $w$ is any vertex; then $G - \overline{X}$ is the so-called 77-graph [1, p.109], itself strongly regular, with eigenvalues $16, 2, -6$. It follows that a star partition of the Higman-Sims graph is given by $X(22) = \{w\}$, $X(2) = X$, $X(-8) = \Delta(w)$.

We now use the proof of Theorem 2.2 to show that there are no further examples when $\mu \leq 2$; in particular we can characterize the Clebsch graph and the Higman-Sims graph in terms of the subgraph induced by the complement of a star set.

**Corollary 2.3** *Let $G$ be a $k$-regular graph $(k > 0)$ and let $\mu$ $(\neq k)$ be an eigenvalue of $G$ with a star set $X$ such that $G - X \cong K_{1,k}$. If $\mu = 1$ (or $k = 5$) then $G$ is the Clebsch graph; and if $\mu = 2$ (or $k = 22$) then $G$ is the Higman-Sims graph.*

**Proof:** If $\mu = 1$ then by Theorem 2.2, the sets $\overline{\Gamma}(u)$ $(u \in X)$ are ten distinct 2-element subsets of $\Delta(w)$. Since $|\Delta(w)| = 5$ these subsets are precisely all the 2-element subsets of $\Delta(w)$, and so $B$ is determined uniquely to within labelling of the vertices of $G$. Since $A$ is determined by $\mu, B, C$ the graph $G$ itself is unique.

If $\mu = 2$ then the sets $\overline{\Gamma}(u)$ $(u \in X)$ are 77 distinct 6-element subsets of the 22-element set $\Delta(w)$; moreover, by equation (8), any two of these subsets intersect in 0 or 2 elements. In particular, no triple lies in two of these sets which therefore account for $77 \times \binom{6}{3} = 1540$ triples from the set $\Delta(w)$. But the total number of such triples is $\binom{22}{3} = 1540$, and so each triple lies in exactly one of the sets $\overline{\Gamma}(u)$ $(u \in X)$.

Thus the non-zero rows of $B$ form the point-block incidence matrix of a (3, 6, 22)-design. By a theorem of Witt [13] there is only one such design; hence $B$ is unique (to within labelling of vertices), and so $G$ is unique. □

# 3 A generalization

Here we extend the techniques of §2 to an investigation of a $k$-regular graph $G$ with a star set $X$ such that $\overline{X} = \Delta^*(w)$ and $\Delta(w)$ induces a subgraph $hK_q$, where $k = hq$, $h \geq 1$ and $q > 1$. For example, if $q = 2$ then $G - X$ consists of $h$ triangles with a vertex in common; in other words, a *windmill* as defined in [1, p.31]. Recall that a *cocktail-party graph* is a graph of the form $\overline{hK_2}$.

**Lemma 3.1** *Let $G$ be a $k$-regular graph ($k > 0$) with $n$ vertices and an eigenvalue $\mu$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $\overline{X} = \Delta^*(w)$ where $\Delta(w)$ induces a regular subgraph of degree $r > 0$. Then the following hold:*

*(i)* $\mu \neq -1$,

*(ii) if $\mu = k$ then $m = 1, n = k+2, r = k-2$ and $G$ is a cocktail-party graph,*

*(iii) if $\mu \neq k$ then $G - \overline{X}$ is regular of degree $d$, where*

$$d = \mu + \frac{(k-1-r)\mu}{\mu+1}.$$

**Proof:** If $\mu = k$ then $m = 1$ because $G$ is connected; hence $n = k+2$, and so $G$ is a cocktail-party graph. Here the single vertex in $X$ is adjacent to each vertex in $\Delta(w)$. Thus if $v \in \Delta(w)$ then $\deg(v) = r+2$, and it follows that $r = k-2$.

When $\mu \neq k$ the remaining assertions are proved in similar fashion to Lemma 2.1, using the following three equations:

$$\sum_{u \in X} P\mathbf{e}_u = -\sum_{v \in \Delta(w)} P\mathbf{e}_v - P\mathbf{e}_w,$$

$$\mu P\mathbf{e}_w = \sum_{v\in\Delta(w)} P\mathbf{e}_v,$$

$$\mu \sum_{u\in X} P\mathbf{e}_u = \sum_{u\in X} d_u P\mathbf{e}_u + (k-1-r) \sum_{v\in\Delta(w)} P\mathbf{e}_v \, ,$$

where $d_u$ is the degree of $u$ in $G - \overline{X}$. □

We note that here, in contrast to Lemma 2.1, the possibility $\mu = 0$ cannot be excluded. Indeed if $\mu = 0$ then $|X| = k - r - 1$ and $X \cup \{w\}$ is an independent set of $k - r$ vertices adjacent to every vertex in $\Delta(w)$. The adjacency matrix of $G$ therefore has the form $A' = \begin{pmatrix} O & J^T \\ J & D \end{pmatrix}$, where $D$ is the adjacency matrix of the subgraph induced by $\Delta(w)$. Examples arise whenever this subgraph does not have 0 as an eigenvalue, for then 0 is not an eigenvalue of $G - X$, while the nullity of $A'$ is $k - r - 1$. To see this, it suffices to observe that $(\mathbf{0}^T|\mathbf{j}^T)$ does not lie in the row-space of $(J|D)$: indeed if $(\mathbf{0}^T|\mathbf{j}^T) = \mathbf{c}^T(J|D)$ then $\mathbf{0}^T = \mathbf{c}^T J$, $\mathbf{j}^T = \mathbf{c}^T D$ and so $\mathbf{j}^T\mathbf{j} = \mathbf{c}^T D\mathbf{j} = r\mathbf{c}^T\mathbf{j} = 0$, a contradicition.

The essential difference between the configurations considered in sections 2 and 3 is however the possible presence when $r \geq 1$ of a triangle with one vertex in $X$ and two vertices in $\overline{X}$. This will become apparent when we equate diagonal entries in equation (7), and it accounts for the condition $(\star)$ in the following theorem. We write $K_1\nabla H$ for the graph obtained from the graph $H$ by adding a vertex adjacent to every vertex in $H$.

**Theorem 3.2** *Let $G$ be a $k$-regular graph ($k > 0$) with $n$ vertices and an eigenvalue $\mu \neq k$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $G - X \cong K_1\nabla hK_q$, where $hq = k$ and $q > 1$. Suppose also that there exists a vertex $u$ of $X$ such that*

*$(\star)$ $G$ has no triangle with vertices $u, v_1, v_2$ where $v_1, v_2$ are adjacent vertices of $\overline{X}$.*

*Then*

*(i) $G - \overline{X}$ is regular of degree $\mu\{(\mu+1)^2 - q\}$,*

*(ii) $k = \mu(\mu^2 + 3\mu - q + 2)$, $m = (\mu^2 + 3\mu - q + 2)(\mu^2 + 2\mu - q)$,
$n = (\mu^2 + 3\mu - q + 1)^2$.*

**Proof:** The vertices of $\overline{X}$ may be labelled so that the adjacency matrix $C$ of equation (7) has the form $\begin{pmatrix} 0 & \mathbf{j}^T \\ \mathbf{j} & C' \end{pmatrix}$, where $C'$ is block-diagonal with $h$ blocks $J - I$ of size $q \times q$.
The characteristic polynomial of $C$ is

$$\{(x+1)^{q-1}(x-q+1)\}^h(x - \frac{k}{x-q+1})$$

[4, Theorem 2.8] and so its minimal polynomial is $f(x)$, where

$$f(x) = (x+1)(x-q+1)\{x^2 - (q-1)x - k\}.$$

It follows that

$$f(\mu)(\mu I - C)^{-1} = C^3 + \alpha C^2 + \beta C + \gamma$$

where

$$\begin{aligned} \alpha &= \mu - 2q + 3, \\ \beta &= \mu(\mu - 2q + 3) + (q-1)(q-3) - k, \\ \gamma &= \mu^2(\mu - 2q + 3) + \mu(q-1)(q-3) - k\mu + (q-1)^2 + (q-2)k. \end{aligned}$$

The matrix $f(\mu)(\mu I - C)^{-1}$ has the form

$$\alpha_0 I + \begin{bmatrix} * & * \\ * & (2q-2+\alpha)J \end{bmatrix} + \alpha_1 \begin{bmatrix} * & * \\ * & C' \end{bmatrix}$$

where $\alpha_0 = \gamma + (q-1)(q-2+\alpha)$ and $\alpha_1 = q^2 - 3q + 3 + \alpha(q-2) + \beta$.

Now we equate $(v, v)$-entries in equation (7), where $v$ is a vertex in $X$. Note that if $\mathbf{x}$ is the column of $B$ corresponding to $v$ then $\mathbf{x}$ has the form $(0, x_1, x_2, \ldots, x_k)^T$ and so we obtain

$$\mu f(\mu) = \alpha_0 a + (2q - 2 + \alpha)a^2 + \alpha_1 \sum_{i \sim j} x_i x_j, \tag{9}$$

where $a$ is the weight of $\mathbf{x}$. If now we take $v = u$ then the condition ($\star$) ensures that $\sum_{i \sim j} x_i x_j = 0$; while from Lemma 3.1 (with $r = q - 1$) we have

$$a = k - d = \frac{k + (q-1)\mu - \mu^2}{\mu + 1}.$$

We now substitute for $\alpha_0$ and $a$ in equation (9). Since $\mu$ is not an eigenvalue of $C$, we may divide by $\{\mu^2 - (q-1)\mu - k\}$ $(\mu - q + 1)$ to obtain

$$-\mu(\mu+1) = \frac{\mu^2-(q-1)\mu-k}{\mu+1}$$

It follows that $k = \mu(\mu^2+3\mu-q+2)$, hence that $d = \mu\{(\mu+1)^2-q\}$ and $a = \mu(\mu+1)$. Counting in two ways the edges between $X$ and $\overline{X}$, we find that $m = k(k-q)/a = (\mu^2+3\mu-q+2)(\mu^2+2\mu-q)$. Finally, $n = m+k+1 = (\mu^2+3\mu-q+1)^2$. □

The Paley graph $P(9)$ [1, p.34] provides an illustration of Theorem 3.1 with $\mu = -2, q = 2, k = 4$ and $G-\overline{X}$ a 4-cycle. Indeed, we have the following characterization.

**Corollary 3.3** *If $G$ is a graph which satisfies the hypotheses of Theorem 3.2 with $q = 2$ and $\mu = -2$ then $G \cong P(9)$.*

**Proof:** By Theorem 3.2, we have $k = 4, m = 4, n = 9$ and $G-\overline{X} \cong C_4$. Let $\Delta(v) = \{6,7,8,9\}$ where $6 \sim 7$ and $8 \sim 9$. By condition $(\star)$ the possible sets $\overline{\Gamma}(u)$ $(u \in X)$ are $\{6,8\}, \{6,9\}, \{7,8\}, \{7,9\}$, and each of these occurs exactly once because $\overline{X}$ is a location-dominating set. Only two graphs can now arise, according as $G-\overline{X}$ does or does not have adjacent vertices $u_1, u_2$ such that $\overline{\Gamma}(u_1) \cap \overline{\Gamma}(u_2) = \emptyset$. In the first case, the graph in question does not have $-2$ as an eigenvalue of multiplicity 4, and so $P(9)$ is the sole candidate. □

Any strongly regular graph which satisfies the hypotheses of Theorem 3.1 is of type $NL_\mu(\mu^2+3\mu-q+1)$. Indeed, two further examples arise as rank 3 graphs [1, p.36] associated with the group $O_4^-(I\!K)$ acting on a 4-dimensional vector space over a finite field $I\!K$ (see [2, Chapter 1]). The graphs in question are of type $C12^-$ in Hubaut's list [7] of strongly regular graphs. If $I\!K = GF(3)$ then we have an example with parameters $(81, 20, 1, 6)$ and eigenvalues $20, 2, -7$ of multiplicities $1, 60, 20$ respectively: here $\mu = 2$ and $q = 2$. If $I\!K = GF(4)$ then we have an example with parameters $(256, 51, 2, 12)$ and eigenvalues $51, 3, -13$ of multiplicities $1, 204, 51$ respectively: here $\mu = 3$ and $q = 3$. In both cases, $G - X$ has the required structure, and condition $(\star)$ holds for all vertices of $X$, because adjacent vertices are points of an isotropic line.

# 4 An alternative configuration

In the previous two sections, the star set $X$ was taken to be the set of non-neighbours of a single vertex. Here we explore a situation in which $X$ is the set of non-neighbours of two adjacent vertices.

**Lemma 4.1** *Let $G$ be a $k$-regular graph ($k > 1$) with an eigenvalue $\mu$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $G - X \cong S_{k,k}$. Then the following hold:*

*(i)* $\mu \neq 0$,

*(ii) if $\mu = k$ then $k = 2, m = 1$ and $G$ is a 5-cycle,*

*(iii) if $\mu \neq k$ then $G-\overline{X}$ is regular of degree $d$, where $d = \mu + \frac{(k-1)(\mu-1)}{\mu}$.*

**Proof:** Let $v, w$ be the adjacent vertices of degree $k$ in $G - X$, and let $\Delta = \Delta_1 \dot{\cup} \Delta_2$, where $\Delta_1 = \Delta(v)\backslash\{w\}$ and $\Delta_2 = \Delta(w)\backslash\{v\}$. Thus each vertex of $\Delta$ is adjacent to $k-1$ vertices of $X$. If $\mu = k$ then $m = 1$ because $G$ is connected, and so $k = 2, G$ is a 5-cycle. Accordingly, suppose that $\mu \neq k$. Then

$$Pe_v + Pe_w = -\sum_{u \in X} Pe_u - \sum_{j \in \Delta} Pe_j \tag{10}$$

and the basic relation (2) affords the following three equations:

$$\mu Pe_v = \sum_{j \in \Delta_1} Pe_j + Pe_w\ , \tag{11}$$

$$\mu Pe_w = \sum_{j \in \Delta_2} Pe_j + Pe_v\ , \tag{12}$$

$$\mu \sum_{u \in X} Pe_u = \sum_{u \in X} d_u Pe_u + (k-1) \sum_{j \in \Delta} Pe_j\ , \tag{13}$$

where $d_u$ is the degree of $u$ in $G - \overline{X}$. From equations (11) and (12) we have

$$(\mu - 1)(Pe_v + Pe_w) = \sum_{j \in \Delta} Pe_j. \tag{14}$$

From equations (10) and (14) we have

$$\mu \sum_{j \in \Delta} Pe_j = -(\mu - 1) \sum_{u \in X} Pe_u.$$

Since the vectors $Pe_u$ ($u \in X$) are linearly independent we have $\mu \neq 0$. We may now substitute for $\sum_{j \in \Delta} Pe_j$ in equation (13) to obtain

$$\sum_{u \in X} \left\{ d_u - \mu - \frac{(k-1)(\mu-1)}{\mu} \right\} Pe_u = \mathbf{0}\ ,$$

from which (iii) follows. □

Now suppose that, in the notation of Lemma 4.1, $G$ has an automorphism which interchanges $v$ and $w$. We shall go on to determine the graphs $G$ which arise when this symmetry condition is imposed. We shall need the following result, for which the author is indebted to F. K. Bell.

**Lemma 4.2** *If $x$ and $y$ are positive integers such that $x^2 - x = 3y^2 - 3y$ then there exists an integer $n \geq 0$ such that $(x, y) = (x_n, y_n)$, where $x_0 = 1, y_0 = 1$ and*

$$x_{n+1} = 2x_n + 3y_n - 2\ , \quad y_{n+1} = x_n + 2y_n - 1 \quad (n \geq 0).$$

**Proof:** If we write $p = 2x - 1$ and $q = 2y - 1$ then the original equation becomes: $p^2 - 3q^2 = -2$. If we now define non-negative integers $X, Y$ by $p = X + 3Y, q = X + Y$ then we obtain the Pell equation $X^2 - 3Y^2 = 1$, whose smallest solution in positive integers is $(X, Y) = (2, 1)$. Hence (by [8, Theorem 11.11]) the solutions with $X, Y \in \mathbb{N}$ are $(X, Y) = (X_n, Y_n)$ $(n \geq 0)$ where $X_n + \sqrt{3}Y_n = (2 + \sqrt{3})^n$. Thus (for $n \geq 0$)

$$X_n = \tfrac{1}{2}\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}\ , \quad Y_n = \tfrac{1}{2\sqrt{3}}\left\{(2+\sqrt{3})^n - (2-\sqrt{3})^n\right\},$$

from which it follows that $(p, q) = (p_n, q_n)$, where

$$p_n = X_n + 3Y_n = \tfrac{1}{2}\{(1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n \quad (n \geq 0),$$
$$q_n = X_n + Y_n = \tfrac{1}{2\sqrt{3}}\{(1+\sqrt{3})(2+\sqrt{3})^n - (1-\sqrt{3})(2-\sqrt{3})^n\} \quad (n \geq 0).$$

Thus $p_0 = q_0 = 1$ and $p_{n+1} + \sqrt{3}q_{n+1} = (1 + \sqrt{3})(2 + \sqrt{3})^{n+1} = (2 + \sqrt{3})(p_n + \sqrt{3}q_n)$ $(n \geq 0)$. We deduce that $p_{n+1} = 2p_n + 3q_n$ and $q_{n+1} = p_n + 2q_n$ $(n \geq 0)$. The recurrence relation for $(x_n, y_n)$ now follows. □

**Thoerem 4.3** *Let $G$ be a $k$-regular graph $(k > 1)$ with an eigenvalue $\mu$ of multiplicity $m$. Suppose that $G$ has a star set $X$ corresponding to $\mu$ such that $G - X \cong S_{k,k}$; and suppose also that $G$ has an automorphism which interchanges the central vertices of $G - X$. Then $k = 2$ and either (a) $\mu = 2, m = 1$ and $G$ is a 5-cycle, or (b) $\mu = \pm 1, m = 2$ and $G$ is a 6-cycle.*

**Proof:** We assume that conclusion (a) does not hold, so that by Lemma 4.1, $G - \overline{X}$ is regular of degree $d$, where $d = \mu + (k-1)(\mu-1)/\mu$ and $\mu \notin \{0, k\}$. Thus each vertex of $X$ is adjacent to $a$ vertices of $\overline{X}$, where $a = k - d = (k - \mu^2 + \mu - 1)/\mu$. Suppose, by way of contradiction, that $\mu$ is not an integer. Then $\mu$ has an algebraic conjugate which is also an eigenvalue of $G$ of multiplicity $m$. Hence $2m < |V(G)|$ and so $m \leq 2k - 1$. Counting in two ways the edges between $X$ and $\overline{X}$, we have $ma = 2(k-1)^2$, and so $m \neq 2k-1$. Moreover, if $m < 2k-2$ then $a > k-1$, whence $a = k$; but then $k = 2, m = 1, G$ is a 5-cycle and $\mu = 2$, a contradiction. It remains to consider the case $m = 2k-2$ : here we have $k-1 = a = (k-\mu^2+\mu-1)/\mu$, whence $(\mu-1)(\mu+k-1) = 0$ and $\mu \in \{1, 1-k\}$, a contradiction. Hence $\mu \in \mathbb{Z}$.

We label the vertices of $\overline{X}$ so that the adjacency matrix $C$ of $G - X$ has the form

$$\begin{bmatrix} O & \mathbf{j} & \mathbf{0} & O \\ \mathbf{j}^T & 0 & 1 & \mathbf{0}^T \\ \mathbf{0}^T & 1 & 0 & \mathbf{j}^T \\ O & \mathbf{0} & \mathbf{j} & O \end{bmatrix}. \text{ Then } C^2 = \begin{bmatrix} J & \mathbf{0} & \mathbf{j} & O \\ \mathbf{0}^T & k & 0 & \mathbf{j}^T \\ \mathbf{j}^T & 0 & k & \mathbf{0}^T \\ O & \mathbf{j} & \mathbf{0} & J \end{bmatrix},$$

$$C^3 = \begin{bmatrix} O & k\mathbf{j} & \mathbf{0} & J \\ k\mathbf{j}^T & 0 & 2k-1 & \mathbf{0}^T \\ \mathbf{0}^T & 2k-1 & 0 & k\mathbf{j}^T \\ J & \mathbf{0} & k\mathbf{j} & O \end{bmatrix}$$

and

$$C^4 = \begin{bmatrix} kJ & \mathbf{0} & (2k-1)\mathbf{j} & \mathbf{0} \\ \mathbf{0}^T & k^2+k-1 & 0 & (2k-1)\mathbf{j}^T \\ (2k-1)\mathbf{j}^T & 0 & k^2+k-1 & \mathbf{0}^T \\ O & (2k-1)\mathbf{j} & \mathbf{0} & kJ \end{bmatrix}.$$

By a formula of Heilbronner [4, Theorem 2.12], $C$ has characteristic polynomial $\{(x^2-k+1)x^{k-2}\}^2 - (x^{k-1})^2$. The minimal polynomial of $C$ is therefore $f(x)$, where

$$f(x) = xg(x)g(-x), \quad g(x) = x^2 + x - k + 1.$$

Accordingly, $(\mu I - C)^{-1}$ is a quartic in $C$, and we find that $f(\mu)(\mu I - C)^{-1} = C^4 + \alpha C^3 + \beta C^2 + \gamma C + \delta I$, where

$$\alpha = \mu, \quad \beta = \mu^2 - 2k + 1, \quad \gamma = \mu(\mu^2 - 2k + 1),$$
$$\delta = f(\mu)/\mu = \mu^2(\mu^2 - 2k + 1) + (k-1)^2.$$

Hence $f(\mu)(\mu I - C)^{-1} - \delta I =$

$$\begin{bmatrix} (k+\beta)J & (\alpha k+\gamma)\mathbf{j} & (2k-1+\beta)\mathbf{j} & \alpha J \\ (\alpha k+\gamma)\mathbf{j}^T & k^2+k-1+\beta k & \alpha(2k-1)+\gamma & (2k-1+\beta)\mathbf{j}^T \\ (2k-1+\beta)\mathbf{j}^T & \alpha(2k-1)+\gamma & k^2+k-1+\beta k & (\alpha k+\gamma)\mathbf{j}^T \\ \alpha J & (2k-1+\beta)\mathbf{j} & (\alpha k+\gamma)\mathbf{j} & (k+\beta)J \end{bmatrix}.$$

Now let $u \in X$ and (in the notation of Lemma 4.1) let $a_i = |\overline{\Gamma}(u) \cap \Delta_i|$ $(i = 1, 2)$, so that our graph has the form depicted in Fig.2. Then $a_1 + a_2 =$

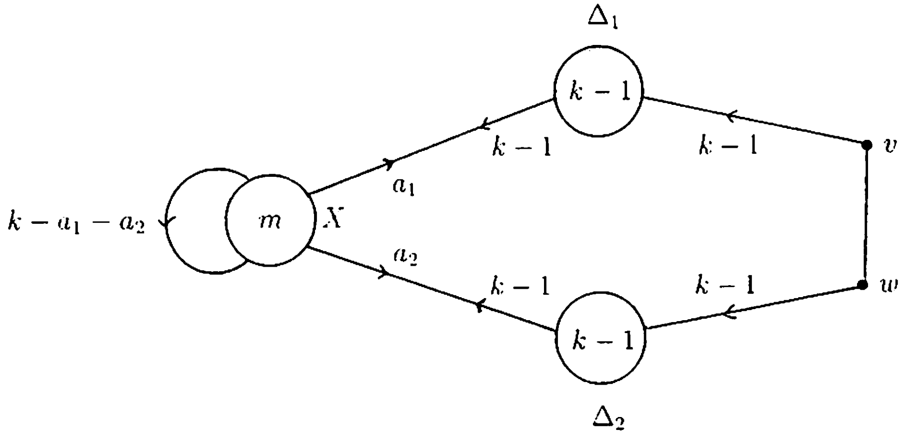


Fig.2

$a = -g(-\mu)/\mu > 0$ and the $u$-th row of $B^T$ has the form $(\mathbf{x}_1^T, 0, 0, \mathbf{x}_2^T)$ where $\mathbf{x}_i$ has weight $a_i$ $(i = 1, 2)$. On equating $(u, u)$ - entries in equation (7) we have

$$\mu f(\mu) = \delta(\mathbf{x}_1^T\mathbf{x}_1 + \mathbf{x}_2^T\mathbf{x}_2) + (k+\beta)(\mathbf{x}_1^T J\mathbf{x}_1 + \mathbf{x}_2^T J\mathbf{x}_2) + \alpha(\mathbf{x}_1^T J\mathbf{x}_2 + \mathbf{x}_2^T J\mathbf{x}_1),$$

equivalently,

$$\mu f(\mu) = \delta a + (k+\beta)(a_1^2 + a_2^2) + 2\alpha a_1 a_2. \tag{15}$$

We can now show that $a_1 \neq a_2$. For if $a_1 = a_2 = \frac{1}{2}a$ and we express $\alpha, \beta, \delta, a$ in terms of $k$ and $\mu$, we find that equation (15) becomes

$$2\mu^4 g(\mu)g(-\mu) = g(\mu)g(-\mu)^2(-2\mu + 1).$$

Since $\mu$ is not an eigenvalue of $G - X$ we may divide by $g(\mu)g(-\mu)$ to obtain

$$2\mu^4 = (\mu^2 - \mu - k + 1)(-2\mu + 1).$$

Since $2\mu^4$ and $2\mu - 1$ are coprime the only possibility is that $\mu = 1$ and $k = 3$; but this is a contradiction since $g(1) = 0$.

Next, let $t = \tau(u)$ where $\tau$ is an automorphism of $G$ which interchanges $v$ and $w$, and let $\mathbf{t}, \mathbf{u}$ be the columns of $B$ corresponding to the vertices $t, u$.

Since $\tau(X) = X$ while $\tau$ interchanges $\Delta_1$ and $\Delta_2$, we have $|\overline{\Gamma}(t)_\cap\Delta_1| = a_2$ and $|\overline{\Gamma}(t)_\cap\Delta_2| = a_1$. Since $a_1 \neq a_2$ we have $t \neq u$. Thus $\tau$ is fixed-point-free on $X$ and $|X|$ is even.

On equating $(t, u)$-entries in equation (7) we have

$$-f(\mu)a_{tu} = \delta \mathbf{t}^T\mathbf{u} + (k+\beta)2a_1a_2 + \alpha(a_1^2 + a_2^2), \tag{16}$$

where $a_{tu}$ is the $(t, u)$-entry of $A$. We can now show that if $a_1a_2 = 0$ then $G$ is a 6-cycle. For if $\{a_1, a_2\} = \{a, 0\}$ then $\overline{\Gamma}(t)$ and $\overline{\Gamma}(u)$ are disjoint; thus $\mathbf{t}^T\mathbf{u} = 0$ and from equation (16) we have $a_{tu}f(\mu) + \mu a^2 = 0$. Hence $t \sim u$ and $a = -\mu^2(a-2)$. It follows that $a = 1, \mu = \pm 1$ and hence that $k = 2, m = 2$. Then $G$ is a 6-cycle and we have case (b) of the theorem. Accordingly, we assume from now on that $a_1a_2 > 0$ for every choice of the vertex $u \in X$. We shall eliminate in turn the cases $\mu < 0, \mu > 1, \mu = 1$.

If we add equations (15) and (16) we obtain

$$(\mu - a_{tu})f(\mu) = \delta(a + \mathbf{t}^T\mathbf{u}) + (k + \beta + \alpha)a^2.$$

On expressing $\alpha, \beta, \delta$ and $a$ in terms of $k$ and $\mu$, this equation simplifies to

$$\mu^3(\mu - a_{tu}) = \mu^2\mathbf{t}^T\mathbf{u} + (1-\mu)(\mu^2 - \mu - k + 1). \tag{17}$$

Now suppose that $\mu$ is negative, say $\mu = -\lambda$, $\lambda > 0$. Equation (17) becomes

$$k = \frac{-\lambda^3(\lambda + a_{tu})}{\lambda+1} + \frac{\lambda^2}{\lambda+1}\,\mathbf{t}^T\mathbf{u} + \lambda^2 + \lambda + 1. \tag{18}$$

Since $0 \leq \lambda\mathbf{t}^T\mathbf{u} \leq \lambda a = \lambda^2 + \lambda - k + 1$, we have

$$k \;\leq\; -\lambda^4/(\lambda+1) - k + 2(\lambda^2 + \lambda + 1),$$

whence $\lambda^4 \leq 2(\lambda+1)(\lambda^2 + \lambda + 1 - k)$.

If $k \geq 3$ then $\lambda^4 \leq 2(\lambda+1)(\lambda+2)(\lambda-1)$, whence $\lambda = 2$ and $k \leq 4$. Equation (18) cannot be satisfied when $k = 4$ and $\lambda = 2$; while if $k = 3$ we have a contradiction because $f(-2) = 0$. If $k = 2$ then $G$ is a cycle and (since $|X|$ is even) the only possible value for $\lambda$ is 1; but then $a = 1$ and $a_1a_2 = 0$, contrary to assumption.

We now know that the integer $\mu$ cannot be negative. Next, suppose by way of contradiction that $\mu > 1$. From the expression for $\mu$ in Lemma 4.1, we know that $\mu$ divides $k-1$, say $k = s\mu + 1$, where $s \in I\!N$. Then $a = a_1 + a_2 = s - \mu + 1$. From equation (15) we have

$$\mu(\mu + 1 - s)(\mu^2 + \mu - 1 - s) = (\mu - s)(\mu - 1 - s) - 2a_1a_2,$$

whence

$$2a_1a_2 = -\mu^4 + (s-2)\mu^3 + (2s+1)\mu^2 - (2s+s^2)\mu + s^2 + s.$$

To simplify the arithmetic we write $b_i = a_i - 1$ $(i = 1,2)$ and $b = b_1 + b_2$: note that $b_1$ and $b_2$ are non-negative because $a_1, a_2 > 0$ ; and $b_1 \neq b_2$ because $a_1 \neq a_2$. We have $b = s - \mu - 1$ and we find that $2b_1b_2 = b(-s+\mu^2+2\mu)(\mu-1)$. Note that $b > 0$ for otherwise $k = \mu^2+\mu+1$ and $f(\mu) = 0$. Note also that since $\mu > 1$ and $b_1b_2 \geq 0$ we have $\mu^2+2\mu \geq s$, say $s = \mu^2+2\mu-\varepsilon$, where $\varepsilon \geq 0$. Now $b_1, b_2$ are roots of a quadratic with discriminant $\theta$, where $\theta = b^2 - 4b_1b_2$. We find that $\theta = b\{s(2\mu-1) - 2\mu^3 - 2\mu^2 + 3\mu - 1\}$, and since $\theta > 0$ we have

$$s > \frac{2\mu^3 + 2\mu^2 - 3\mu + 1}{2\mu - 1} \quad = \quad \mu^2 + 2\mu - \frac{\mu^2 + \mu - 1}{2\mu - 1}.$$

It follows that $\varepsilon < \frac{1}{2}\mu + 1$. For future reference we note here that if we express $\theta$ in terms of $\mu$ and $\varepsilon$ we find that $\theta = \alpha_1^2 - \alpha_2^2$, where

$$\alpha_1 = \mu^2 + \mu - 1 - \mu\varepsilon \quad \text{and} \quad \alpha_2 = (\mu - 1)\varepsilon. \tag{19}$$

Counting $X - \overline{X}$ edges in two ways, we find that $|X|a = 2(k-1)^2$. Since $|X|$ is even, this tells us that $a$ divides $\mu^2 s^2$. Since $a = s - \mu + 1$ we deduce that $a$ divides $\mu^2(\mu-1)^2$. Now consider the greatest common divisors $g = (a, \mu^2)$ and $h = (a, (\mu-1)^2)$. We have $(g,h) = 1$ and $a = gh$; moreover, $g = (a, \mu+1-\varepsilon)$ and $h = (a, 3\mu - \varepsilon)$ because $a = \mu^2 + \mu + 1 - \varepsilon$. Now $a > \frac{1}{3}(\mu+1-\varepsilon)(3\mu-\varepsilon)$ for otherwise $3(\mu^2+\mu+1-\varepsilon) \leq (\mu+1-\varepsilon)(3\mu-\varepsilon)$, whence $\varepsilon(\varepsilon - 4\mu + 2) \geq 3$, a contradiction.

It follows that $a = \rho(\mu+1-\varepsilon)(3\mu-\varepsilon)$ where $\rho \in \{1, \frac{1}{2}\}$. If $\rho = 1$ then $\mu^2+\mu+1-\varepsilon = 3\mu^2+3\mu-4\mu\varepsilon-\varepsilon+\varepsilon^2$, equivalently $(2\mu-\varepsilon)^2 = \mu^2+(\mu-1)^2$. Hence $\{\mu, \mu-1\} = \{e^2-f^2, 2ef\}$ and $2\mu - \varepsilon = e^2 + f^2$, where $e$ and $f$ are positive integers. If $\mu = e^2 - f^2$ then $2f^2 = \mu - \varepsilon > \frac{1}{2}\mu - 1 = ef - \frac{1}{2}$, whence $2f \geq e$ and $\mu = 2ef + 1 \geq e^2 + 1$, a contradiction. If $\mu = 2ef$ then $2f^2 = \mu - \varepsilon + 1 > \frac{1}{2}\mu$, whence $2f^2 > ef$ and $\mu > e^2$, a contradiction.

Now suppose that $\rho = \frac{1}{2}$. In this case we have $2(\mu^2 + \mu + 1 - \varepsilon) = 3\mu^2 + 3\mu - 4\mu\varepsilon - \varepsilon + \varepsilon^2$, which may be written:

$$(4\mu - 1 - 2\varepsilon)^2 = 2(2\mu)^2 + (2\mu - 3)^2.$$

Since $\varepsilon < \frac{1}{2}\mu + 1$, we have

$$4\mu - 1 - 2\varepsilon = \sqrt{12\mu^2 - 12\mu + 9}.$$

Hence $12\mu^2 - 12\mu + 9 = 9(2\nu - 1)^2$ for some positive integer $\nu$. Thus $\mu^2 - \mu = 3\nu^2 - 3\nu$ and $\varepsilon = 2\mu - 3\nu + 1$. We now use the fact that $\theta = \alpha_1^2 - \alpha_2^2$, where the integers $\alpha_1, \alpha_2$ are given by equation (19), while $\theta$ is itself a non-zero square. To exploit the form of the corresponding Pythagorean triple, we need some information on the greatest common divisor $\sigma$ of $\alpha_1$ and $\alpha_2$. Let $p$ be a prime which divides $\sigma$, and note that $3\nu^2 \equiv 2 \bmod p$ because $\alpha_1 + \alpha_2 = 3\nu^2 - 2$. We show first that $p$ does not divide $\varepsilon$; for otherwise we have $3\nu^2 \equiv 2, \mu^2 + \mu - 1 \equiv 0$ and $2\mu - 3\nu + 1 \equiv 0 \bmod p$, whence $0 \equiv (2\mu + 3\nu)(2\mu - 3\nu + 1) \equiv 4\mu^2 - 9\nu^2 + 2\mu + 3\nu \equiv 4\mu^2 - 9\nu^2 + 2\mu + 3\nu^2 - \mu^2 + \mu \equiv 3\mu^2 - 6\nu^2 + 3\mu \equiv 3(\mu^2 + \mu) - 6\nu^2 \equiv 3 - 4 \bmod p$, a contradiction. Hence $p$ does not divide $\varepsilon$; but $p$ divides $(\mu - 1)\varepsilon$ and so $\mu \equiv 1 \bmod p$. It follows that $0 \equiv \mu^2 - \mu \equiv 3\nu^2 - 3\nu$ and hence that $3\nu \equiv 2 \bmod p$. Now $0 \equiv 3\nu^2 - 3\nu - \nu(3\nu - 2)$, whence $\nu \equiv 0 \bmod p$ and necessarily $p = 2$. Thus $\sigma$ is a power of 2.

Now, since $\alpha_1^2 - \alpha_2^2$ is a non-zero square, there exist coprime positive integers $e, f$ of opposite parity such that either (i) $\alpha_1 = \sigma(e^2 + f^2)$ and $\alpha_2 = \sigma(e^2 - f^2)$ or (ii) $\alpha = \sigma(e^2 + f^2)$ and $\alpha_2 = 2\sigma ef$. In case (i) we have $3\nu^2 - 2 = \alpha_1 + \alpha_2 = 2\sigma e^2$ whence $\nu$ is even, say $\nu = 2\lambda$. But then $6\lambda^2 - 1 = \sigma e^2$ and so $\sigma = 1, e^2 \equiv -1 \bmod 3$, a contradiction. In case (ii) we have $3\nu^2 - 2 = \alpha_1 + \alpha_2 = \sigma(e + f)^2$, where $e + f$ is odd. If $\nu$ is even, say $\nu = 2\lambda$, we have $12\lambda^2 - 2 = \sigma(e + f)^2$, whence $\sigma = 2$ and $(e + f)^2 \equiv -1 \bmod 3$, a contradiction. If $\nu$ is odd then $\sigma = 1$; and if we write $\nu = 2\lambda - 1, e + f = 2\chi - 1$ then we obtain $3\lambda^2 - 3\lambda = \chi^2 - \chi$. In the notation of Lemma 4.2, we now have $\nu = y_m$ and $\lambda = y_n$ for some non-negative integers $m, n$; moreover $\nu = 2\lambda - 1$, and so $n > 0$ (for otherwise $y_m = y_n = 1$ and $\mu \in \{0, 1\}$). Hence $y_n < 2y_n - 1$; and since also $y_{n+1} = x_n + 2y_n - 1$ we have $y_n < y_m < y_{n+1}$. This contradiction eliminates the possibility that $\mu > 1$.

It remains to deal with the case $\mu = 1$. In this case $G - \overline{X} \cong (k-1)K_2$ by Lemma 4.1. Moreover the equation obtained by equating diagonal entries in (7) is

$$(k-1)(k-3) = (k-1)^2(k-3) + (2-k)(a_1^2 + a_2^2) + 2a_1a_2,$$

where $a_1 + a_2 = a = k - 1$. It follows that $a_1a_2 = k - 2$ and hence that $\{a_1, a_2\} = \{1, k-2\}$. Note that $k \geq 4$ because $a_1 \neq a_2$ and $a_1a_2 \neq 0$. Accordingly $X = T \dot{\cup} U$ where $\tau$ interchanges $T$ and $U$, and $a_1 = 1, a_2 = k - 2$ for each vertex in $T$, while $a_1 = k - 2, a_2 = 1$ for each vertex in $U$. For $t \in T$, $u \in U$ let $\mathbf{t}, \mathbf{u}$ denote the corresponding columns of $B$. On equating $(t, u)$-entries in equation (7) we find that $\mathbf{t}^T\mathbf{u} = 1 - a_{tu}$, i.e. $|\overline{\Gamma}(t) \cap \overline{\Gamma}(u)| = 1$ if $t \not\sim u$, 0 if $t \sim u$. If $u, u' \in U$ and we equate $(u, u')$-entries then we find that $\mathbf{u}^T\mathbf{u}' = k - 2 - a_{uu'}$, i.e.

$$|\overline{\Gamma}(u) \cap \overline{\Gamma}(u')| = \begin{cases} k-2 \text{ if } u \not\sim u' \\ k-3 \text{ if } u \sim u' \end{cases} . \tag{20}$$

Similarly, if $t, t' \in T$ then

$$|\overline{\Gamma}(t) \cap \overline{\Gamma}(t')| = \begin{cases} k-2 \text{ if } t \not\sim t' \\ k-3 \text{ if } t \sim t'. \end{cases} \tag{21}$$

For $y \in X$, we write $\Delta_i(y)$ for $\overline{\Gamma}(y) \cap \Delta_i$ $(i = 1, 2)$; thus for $t \in T$, $u \in U$ we have $|\Delta_1(t)| = |\Delta_2(u)| = 1$ and $|\Delta_2(t)| = |\Delta_1(u)| = k - 2$. We show first that the set of subsets $\Delta_1(u)$ $(u \in U)$ does not coincide with the set of all $(k-2)$-element subsets of $\Delta_1$. For otherwise by symmetry the set of subsets $\Delta_2(t)$ $(t \in T)$ coincides with the set of all $(k-2)$-element subsets of $\Delta_2$ and in this situation each vertex of $\Delta_1$ is adjacent to precisely one vertex of $T$, while each vertex of $\Delta_2$ is adjacent to precisely one vertex of $U$. It follows that the $U - \Delta_2$ edges and the $T - \Delta_1$ edges are independent. In particular $|\overline{\Gamma}(u) \cap \overline{\Gamma}(u')| = k - 3$ for every pair of distinct vertices $u, u'$ in $U$. It follows from equation (20) that $U$ induces a complete subgraph, a contradiction.

We now know that $U$ has at least two vertices $u_1, u_2$ with a common neighbourhood in $\Delta_1$, say $\Delta_1(u_1) = \Delta_1(u_2) = \Delta_1'$. Suppose that there is a third vertex $u_3 \in U$ such that $\Delta_1(u_3) = \Delta_1'$, and let $\Delta_2(u_i) = \{v_i\}$ $(i = 1, 2, 3)$. Note that $v_1, v_2, v_3$ are distinct because $\overline{X}$ is a location-dominating set. If $u$ is a vertex in $U$ such that $\Delta_1(u) \neq \Delta_1'$ then $u$ is adjacent to at most one of $v_1, v_2, v_3$; say $u \not\sim v_2$, $u \not\sim v_3$. Then $|\overline{\Gamma}(u) \cap \overline{\Gamma}(u_2)| = |\overline{\Gamma}(u) \cap \overline{\Gamma}(u_3)| = k - 3$, and so by equation (20) we have $u \sim u_2$ and $u \sim u_3$, a contradiction.

We conclude that either (1) $\Delta_1(u) = \Delta_1'$ for every $u \in U$, or (2) $\Delta_1(u) \neq \Delta_1'$ for all $u \in U \setminus \{u_1, u_2\}$. In case (1), $\Delta_1$ has a vertex adjacent to every vertex in $T$, and by symmetry, $\Delta_2$ has a vertex adjacent to every vertex in $U$. Thus $\overline{\Gamma}(t) \cap \overline{\Gamma}(u) = \emptyset$ for every $t \in T$ and every $u \in U$. By equation (17), every vertex in $T$ is adjacent to every vertex in $U$; but then $|T| = |U| = 1$ and $k = 2$, contrary to assumption. Thus case (1) does not arise. In case (2), consider a vertex $u \in U \setminus \{u_1, u_2\}$, and let $\Delta_2(u_i) = \{v_i\}$ $(i = 1, 2)$. At most one of $v_1, v_2$ is a neighbour of $u$ and so by equation (20), $u$ is adjacent to one of $u_1, u_2$. Since $u_1, u_2$ have degree 1 in $G - \overline{X}$, there are at most two such vertices $u$ and so $|X| \leq 4$, $k \in \{4, 5\}$. We eliminate the cases $k = 4$, $k = 5$ in turn.

In the case $k = 4$, let $U = \{u_1, u_2, u_3\}$, $\Delta_2(u_i) = \{v_i\}$ $(i = 1, 2)$, $\Delta_1(u_1) = \Delta_1(u_2) = \{w_1, w_2\}$, $\Delta_1(u_3) = \{w_2, w_3\}$, $\Delta_2 = \{v_1, v_2, v_3\}$. Since $u_3$ is not adjacent to both $u_1$ and $u_2$, it is adjacent to one of $v_1, v_2$, and there are two possibilities: (a) $u_1 \sim u_3 \sim v_2$, (b) $u_2 \sim u_3 \sim v_1$. Note that $v_3$ is the unique vertex in $\Delta_2$ adjacent to every vertex in $T$; hence $v_3 = \tau(w_2)$. Now consider case (a). Since $v_2$ is adjacent to two vertices in $U$, we also have $v_2 = \tau(w_3)$ and hence $v_1 = \tau(w_1)$. Let $\tau(u_i) = t_i$ $(i = 1, 2, 3)$. Then $t_1 \sim t_3$ and $u_2 \sim t_2$. By considering the images of $U - \Delta_1$ edges under $\tau$ we see that $t_1 \sim v_1$, $t_2 \sim v_1$, $t_3 \sim v_2$. Now since $t_2 \not\sim t_3$ we have $|\overline{\Gamma}(t_2) \cap \overline{\Gamma}(t_3)| = 2$

and so $\Delta_1(t_2) = \Delta_1(t_3)$. Necessarily $\Delta_1(t_2) = \Delta_1(t_3) = \{w_3\}$, and so the graph is completed by the edge $t_1w_1$. But now $\overline{\Gamma}(u_3) \cap \overline{\Gamma}(t_1) = \emptyset$, a contradiction because $t_1 \not\sim u_3$.

Now consider case (b), where $v_3 = \tau(w_2), v_2 = \tau(w_1), v_1 = \tau(w_3)$. If $t_i = \tau(u_i)$ $(i = 1, 2, 3)$ then $t_2 \sim t_3$, $t_1 \sim u_1$ and the images under $\tau$ of the $U - \Delta_1$ edges are $t_1v_2$, $t_1v_3$, $t_2v_2$, $t_2v_3$, $t_3v_1$, $t_3v_3$. Now by equation (21), $|\overline{\Gamma}(t_1) \cap \overline{\Gamma}(t_3)| = 2$ because $t_1 \not\sim t_3$, and so $t_1, t_3$ have a common neighbour in $\Delta_1$. This vertex is necessarily $w_3$ and the graph is completed by the edge $t_2w_1$. Now $|\overline{\Gamma}(u_2) \cap \overline{\Gamma}(t_2)| = 2$, a contradiction.

For the last step in the proof, suppose that $k = 5$, and let $U = \{u_1, u_2, u_3, u_4\}$, $\Delta_1(u_1) = \Delta_1(u_2) = \{w_1, w_2, w_3\}$, $\Delta_1(u_3) = \{w_2, w_3, w_4\}$, $\Delta_2(u_1) = \{v_1\}, \Delta_2(u_2) = \{v_2\}$. (Recall that $v_1 \neq v_2$ because $\overline{X}$ is a location-dominating set.) Again, there are two possibilities: (a) $v_1 \sim u_3 \sim u_2$, (b) $v_2 \sim u_3 \sim u_1$. Consider case (a). The vertex $u_4$, like $u_3$, is adjacent to exactly one of $v_1, v_2$; but since $u_2 \not\sim u_4$ we have $v_2 \sim u_4 \sim u_1$. Now $u_3 \not\sim u_4$ and so $|\overline{\Gamma}(u_3) \cap \overline{\Gamma}(u_4)| = 2$ whence $\Delta_1(u_4) = \Delta_1(u_3)$. In case (b), we have similarly $v_1 \sim u_4 \sim u_2$ and $\Delta_1(u_4) = \Delta_1(u_3)$. Accordingly we may interchange $u_3$ and $u_4$ if necessary and consider only case (a). Let $\Delta_2 = \{v_1, v_2, v_3, v_4\}$. By considering the number of $w_i - U$ edges $(i = 1, 2, 3, 4)$ we see that $\tau$ maps $\{w_2, w_3\}$ to $\{v_3, v_4\}$ and $v_3, v_4$ are adjacent to each vertex in $T$. Since $\{w_2, w_3\} = \Delta_1(u_2) \cap \Delta_1(u_3)$, $u_2 \sim u_3$ and $u_1 \sim u_4$, we may let $T = \{t_1, t_2, t_3, t_4\}$ where $\Delta_2(t_3) \cap \Delta_2(t_4) = \{v_3, v_4\}$, $t_3 \sim t_4$ and $t_1 \sim t_2$. The $\Delta_2$-neighbourhoods of vertices in $T$ are $\{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$ (each arising twice). Accordingly we may label $t_1, t_2$ so that $t_1 \sim v_1$ and $t_2 \sim v_2$. We can now obtain a contradiction : $t_2 \not\sim w_1$ since $v_2$ is already a common neighbour of $t_2$ and $u_2$, and $t_2 \not\sim w_4$ because $v_2$ is already a common neighbour of $t_2$ and $u_4$. Since $w_2$ and $w_4$ have no neighbours in $T$, it follows that $t_2$ has no neighbours in $\Delta_1$. This final contradiction completes the proof of the theorem.

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