The Characteristic Polynomials of Interpolations between Coxeter Arrangements

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Abstract

We apply a lattice point counting method due to Blass and Sagan [2] to compute the characteristic polynomials for the subspace arrangements interpolated between the Coxeter hyperplane arrangements. Our proofs provide combinatorial interpretations for the characteristic polynomials of such subspace arrangements. In the process of doing this, we explore some interesting properties of these polynomials.

1 Introduction

A *central subspace arrangement*

$$\mathcal{A} = \{K_1, K_2, \ldots, K_m\}$$

in the Euclidean space $\mathbf{R}^n$ is a finite collection of linear subspaces K_i of $\mathbf{R}^n$. We assume that, for simplicity, there are no containments among the K_i. Then $\mathcal{A}$ is a *hyperplane arrangement* if $\operatorname{codim} K_i = 1$ for all i. Also, we write $\bigcup \mathcal{A}$ for the set-theoretic union of the subspaces in $\mathcal{A}$, i.e., $\bigcup_{i=1}^m K_i$.

Let $x_1, x_2, \ldots, x_n$ be the coordinate functions in $\mathbf{R}^n$. The *coordinate hyperplane arrangement* $\mathcal{Q}_n$ is defined by

$$\mathcal{Q}_n = \{x_i = 0 : 1 \le i \le n\}.$$

The *Coxeter arrangements of type* B, D and A are defined, respectively, by

$$\begin{aligned} \mathcal{B}_n &= \{x_i = \pm x_j : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}, \\ \mathcal{D}_n &= \{x_i = \pm x_j : 1 \le i < j \le n\}, \\ \mathcal{A}_{n-1} &= \{x_i = x_j : 1 \le i < j \le n\}. \end{aligned}$$

It is clear that $\mathcal{Q}_n \subset \mathcal{B}_n$ and $\mathcal{A}_{n-1} \subset \mathcal{D}_n \subset \mathcal{B}_n$.

For a fixed integer k, $1 \le k \le n$, let $I = \{1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and

$$K_I = \{\mathbf{x} \in \mathbf{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}.$$

Let $\mathcal{Q}_{n:k}$ be the set of all such subspaces K_I. Then $\mathcal{Q}_{n:k}$ was defined in [7] as the *k-equal subspace arrangement* and studied further in [8].

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The *intersection lattice* $\mathcal{L}(\mathcal{A})$ (or $\mathcal{L}$) of a subspace arrangement $\mathcal{A}$ is the poset of nonempty intersections of these subspaces ordered by reverse inclusion. Thus in $\mathcal{L}$, a unique minimal element $\hat{0}$ corresponds to $\mathbf{R}^n$ and a unique maximal element $\hat{1}$ corresponds to $\bigcap_{K\in\mathcal{A}} K$. If $\mathcal{A}$ and $\mathcal{B}$ are subspace arrangements such that $\mathcal{A} \subseteq \mathcal{L}(\mathcal{B})$, then we say that $\mathcal{A}$ is *embedded* in $\mathcal{B}$. Since $\mathcal{Q}_{n:k} \subset \mathcal{L}(\mathcal{Q}_n)$, the arrangement $\mathcal{Q}_{n:k}$ is embedded in $\mathcal{Q}_n$ and $\mathcal{B}_n$. Given an arrangement $\mathcal{A}$, let $\mu(\mathbf{x}) = \mu(\hat{0}, \mathbf{x})$ denote the Möbius function of $\mathcal{L}(\mathcal{A})$, i.e.,

$$\mu(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \hat{0}, \\ -\sum_{\mathbf{y}<\mathbf{x}} \mu(\mathbf{y}) & \text{if } \mathbf{x} > \hat{0}. \end{cases} \tag{1}$$

Then the *characteristic polynomial* of $\mathcal{L}$ (or of $\mathcal{A}$) is defined by

$$\chi(\mathcal{L}, t) = \sum_{\mathbf{x}\in\mathcal{L}} \mu(\mathbf{x})\, t^{\dim \mathbf{x}}. \tag{2}$$

The characteristic polynomial of an arrangement is one of its most important combinatorial invariants. Zaslavsky [5, 6] has related the characteristic polynomials of certain arrangements to the chromatic polynomials of signed graphs. Blass and Sagan [2] have generalized one of Zaslavsky's results by showing that these two polynomials both count a certain set of lattice points in $\mathbf{Z}^n$, where $\mathbf{Z}$ represents the integers.

Theorem 1.1 (Blass-Sagan) *Let $\mathcal{A}$ be a subspace arrangement such that $\mathcal{A} \subseteq \mathcal{L}(\mathcal{B}_n)$, and let $t = 2s+1$ be a positive odd integer. Define*

$$[-s, s] = \{-s, -(s-1), \ldots, -1, 0, 1, \ldots, s\}$$

and $C_t = [-s, s]^n$, the cube of side t in $\mathbf{Z}^n$ centered at 0. Then

$$\chi(\mathcal{A}, t) = |C_t \setminus \bigcup \mathcal{A}|.$$

The significance of Theorem 1.1 is that it provides an efficient way to determine certain characteristic polynomials without even computing any Möbius functions. It is the only known purely combinatorial technique to obtain such a result. In an earlier work [8], we applied Theorem 1.1 to the k-equal subspace arrangements and obtained the following theorem.

Theorem 1.2 *The characteristic polynomial of $\mathcal{Q}_{n:k}$ has the following two forms:*

$$\chi(\mathcal{Q}_{n:k}, t) = \sum_{i=0}^{k-1} \binom{n}{i} (t-1)^{n-i}, \tag{3}$$

$$\chi(\mathcal{Q}_{n:k}, t) = (t-1)^{n-k+1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} t^{k-i-1}. \tag{4}$$

Our aim in this paper is to apply Theorem 1.1 to compute the characteristic polynomials of the subarrangements which are interpolated between the three families of the Coxeter hyperplane arrangements of type $\mathcal{B}_n$, $\mathcal{D}_n$ and $\mathcal{A}_{n-1}$. Józefiak and Sagan [3] introduced these subarrangements. Also, they used the Saito-Terao method to obtain these characteristic polynomials. Their proofs, however, employ a purely algebraic method which gives no combinatorial insight into these characteristic polynomials. By Theorem 1.1, we are able to obtain the same characteristic polynomials in a combinatorial way, i.e., by counting the lattice points in certain sets. As a result, not only are our proofs much easier in most cases but they also provide combinatorial interpretations for the characteristic polynomials of these interpolations.

The following results are well-known about the characteristic polynomials of the coordinate hyperplane arrangements and the Coxeter hyperplane arrangements (also, they can be easily proved by Theorem 1.1):

$$\begin{aligned}
\chi(\mathcal{Q}_n, t) &= (t-1)^n, \\
\chi(\mathcal{B}_n, t) &= (t-1)(t-3)\cdots(t-2n+1), \\
\chi(\mathcal{D}_n, t) &= (t-1)(t-3)\cdots(t-2n+3)(t-n+1), \\
\chi(\mathcal{A}_{n-1}, t) &= t(t-1)(t-2)\cdots(t-n+1).
\end{aligned}$$

2 Unordered Interpolations

In this section, we consider so-called (linear) interpolations between three pairs of Coxeter arrangements, namely, $\{\mathcal{A}_{n-1}, \mathcal{A}_n\}$, $\{\mathcal{B}_{n-1}, \mathcal{B}_n\}$ and $\{\mathcal{D}_n, \mathcal{B}_n\}$. All of these interpolations have been described in detailed in [3].

We begin by interpolating between $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$. An interpolation between $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$ can be obtained by adding any number of hyperplanes of $\mathcal{A}_n \setminus \mathcal{A}_{n-1}$ to $\mathcal{A}_{n-1}$ in any given order. If we take an ordered set of such hyperplanes, say $H_1, H_2, \ldots, H_n$, we obtain an interpolation of the form

$$\mathcal{A}_{n,k} = \mathcal{A}_{n-1} \cup \{H_1, H_2, \ldots, H_k\} \tag{5}$$

between $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$, where $1 \leq k \leq n$. Since the lattices of the corresponding interpolated subarrangements are isomorphic, the order in which we add the hyperplanes of $\mathcal{A}_n \setminus \mathcal{A}_{n-1}$ to $\mathcal{A}_{n-1}$ is irrelevant. Because we will compute the characteristic polynomials for these subspace arrangements simply by counting the lattice points in certain sets, the fact that $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$ are of different dimensions plays no role here. That the addition of hyperplanes of $\mathcal{B}_n \setminus \mathcal{B}_{n-1}$ to $\mathcal{B}_{n-1}$ when interpolating between $\mathcal{B}_{n-1}$ and $\mathcal{B}_n$ is independent of their order was verified in [3]. Such is also the case for $\{\mathcal{D}_n, \mathcal{B}_n\}$. For this reason, these interpolations are also called unordered interpolations.

Theorem 2.1 *The characteristic polynomial of $\mathcal{A}_{n,k}$ is*

$$\chi(\mathcal{A}_{n,k}, t) = t\ (t-1)(t-2)\cdots(t-n+1)(t-k).$$

Proof. It is clear that we can assume that

$$\mathcal{A}_{n,k} = \mathcal{A}_{n-1} \cup \{x_i = x_{n+1} : 1 \leq i \leq k\}$$

and count the points in $C_t \setminus \bigcup \mathcal{A}_{n,k}$. Now the first n coordinates of a point $\mathbf{x} = (x_1, x_2, \ldots, x_n, x_{n+1}) \in C_t \setminus \bigcup \mathcal{A}_{n,k}$ must distinct. So there are $t = 2s+1$ choices for x_1, $t-1$ choices for x_2, ..., and $t-n+1$ choices for x_n. Then there are $t-k$ choices for x_{n+1} since x_{n+1} can have any value except x_i where $1 \leq i \leq k$. This gives

$$|C_t \setminus \bigcup \mathcal{A}_{n,k}| = t\ (t-1)(t-2)\cdots(t-n+1)(t-k).$$

■

To proceed from $\mathcal{B}_{n-1}$ to $\mathcal{B}_n$, take any ordering of the hyperplanes of $\mathcal{B}_n \setminus \mathcal{B}_{n-1}$, say $H_1, H_2, \ldots, H_{2n-1}$. For each k with $1 \leq k \leq 2n-1$, we define

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-1} \cup \{H_1, H_2, \ldots, H_k\}$$

Theorem 2.2 *The characteristic polynomial of $\mathcal{B}_{n,k}$ is*

$$\chi(\mathcal{B}_{n,k}, t) = (t-1)(t-3)\cdots(t-2n+3)(t-k).$$

Proof. It suffices to consider the interpolation of the form

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-1} \cup \{x_i = x_n : 1 \leq i \leq k\}.$$

If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C_t \setminus \bigcup \mathcal{B}_{n,k}$, then the first $n-1$ coordinates of $\mathbf{x}$ must have different nonzero absolute values. So there are $t-1$ choices for x_1 since 0 is not allowed. Since the second coordinate can be anything except 0 and $\pm x_1$, there are $t-3$ choices for x_2. Continuing in this way, we see that $x_1, x_2, \ldots, x_{n-1}$ can be chosen in $(t-1)(t-3)\cdots(t-2n+3)$ ways, and then x_n can be chosen in $t-k$ ways since x_n can be anything except x_i where $1 \leq i \leq k$. Hence

$$|C_t \setminus \bigcup \mathcal{B}_{n,k}| = (t-1)(t-3)\cdots(t-2n+3)(t-k).$$

■

Next, we consider the interpolations between $\mathcal{B}_n$ and $\mathcal{D}_n$. Take an arbitrary order $H_1, H_2, \ldots, H_n$ of the hyperplanes of $\mathcal{B}_n \setminus \mathcal{D}_n$ and let

$$\mathcal{D}\mathcal{B}_{n,k} = \mathcal{D}_n \cup \{H_1, H_2, \ldots, H_k\},$$

where $1 \leq k \leq n$.

Theorem 2.3 *The characteristic polynomial of $\mathcal{DB}_{n,k}$ is*

$$\chi(\mathcal{DB}_{n,k}, t) = (t-1)(t-3)\cdots(t-2n+3)(t-n-k+1).$$

Proof. Now we consider the interpolation of the form

$$\mathcal{DB}_{n,k} = \mathcal{D}_n \cup \{x_i = 0 : 1 \le i \le k\}.$$

If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C_t \setminus \bigcup \mathcal{DB}_{n,k}$, then all coordinates of $\mathbf{x}$ must have different absolute values. Moreover, the first k coordinates of $\mathbf{x}$ are nonzero and the number of 0's in the remaining $n-k$ coordinates of $\mathbf{x}$ is at most one. This observation enables us to partition $C_t \setminus \bigcup \mathcal{DB}_{n,k}$ into $n-k+1$ parts, say $A_0, A_1, \ldots, A_{n-k}$, such that

$$A_0 = \{\mathbf{x} \in C_t \setminus \bigcup \mathcal{DB}_{n,k} : \ x_i \neq 0 \text{ for all } 1 \le i \le n\},$$

$$A_i = \{\mathbf{x} \in C_t \setminus \bigcup \mathcal{DB}_{n,k} : x_{k+i} = 0\}, \quad 1 \le i \le n-k.$$

It is clear that $A_0 = C_t \setminus \bigcup \mathcal{B}_n$ and so $|A_0| = \chi(\mathcal{B}_n, t)$. For each fixed i, $1 \le i \le n-k$, the first k coordinates of $\mathbf{x}$ can be chosen in $(t-1)(t-3)\cdots(t-2k+1)$ ways since they have different nonzero absolute values. Once $x_1, x_2, \ldots, x_k$ have been chosen, then the remaining coordinates $x_{k+1}, x_{k+2}, \ldots, x_{k+i}, \ldots, x_n$ can be chosen in $(t-2k-1)(t-2k-3)\cdots(t-2n+3)$ ways with x_{k+i} being 0. So we have

$$|A_i| = (t-1)(t-3)\cdots(t-2k+1)(t-2k-1)\cdots(t-2n+3),$$

which is independent of i. For simplicity, we put i equal to 1, and then the result follows from

$$|C_t \setminus \bigcup \mathcal{DB}_{n,k}| = |A_0| + (n-k)|A_1|.$$

■

3 Ordered Interpolations

When interpolating between $\mathcal{D}_{n-1}$ and $\mathcal{D}_n$, or $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$, or $\mathcal{A}_{n-1}$ and $\mathcal{D}_n$, the order in which the hyperplanes are added is important. One can find counterexamples that explain this point in [3]. So we form these interpolations by adding the hyperplanes between each pair of these arrangements in certain orders.

We first interpolate between $\mathcal{D}_{n-1}$ and $\mathcal{D}_n$. Let $H_1, H_2, \ldots, H_{2n-2}$ be any ordering of the hyperplanes of $\mathcal{D}_n \setminus \mathcal{D}_{n-1}$ such that all $\{x_i = x_n\}$ come before all $\{x_i = -x_n\}$. Then let

$$\mathcal{D}_{n,k} = \mathcal{D}_{n-1} \cup \{H_1, H_2, \ldots, H_k\}.$$

It is clear that if $1 \le k \le n-1$, then

$$\mathcal{D}_{n,k} = \mathcal{D}_{n-1} \cup \{x_i = x_n : 1 \le i \le k\};$$

and if $n \le k \le 2n-2$, then

$$\mathcal{D}_{n,k} = \mathcal{D}_{n-1} \cup \{x_i = x_n : 1 \le i \le n-1\} \cup \{x_i = -x_n : 1 \le i \le k-n+1\}.$$

Theorem 3.1 *The characteristic polynomial of $\mathcal{D}_{n,k}$ is as follows:*

1. *If $1 \le k \le n-1$, then*

$$\chi(\mathcal{D}_{n,k}, t) = (t-1)(t-3)\cdots(t-2n+5)(t-n+2)(t-k).$$

2. *If $n \le k \le 2n-2$, then*

$$\chi(\mathcal{D}_{n,k}, t) = (t-1)(t-3)\cdots(t-2n+5)(t-k+1)(t-n+1).$$

Proof. First, we suppose that $1 \le k \le n-1$. Then a point $\mathbf{x} = (x_1, \ldots, x_n) \in C_t \setminus \bigcup \mathcal{D}_{nk}$ has the following properties:

1. $x_i \ne \pm x_j$ for all i, j with $1 \le i < j \le n-1$,

2. $x_i \ne -x_n$ for all i with $1 \le i \le k$.

Note that any point of $C_t \setminus \bigcup \mathcal{D}_{n,k}$ has at most one zero coordinate. So we partition $C_t \setminus \bigcup \mathcal{D}_{n,k}$ into $A_0, A_1, \ldots, A_n$, where

$$A_0 = \{\mathbf{x} \in C_t \setminus \bigcup \mathcal{D}_{n,k} : \text{ all } x_i \ne 0\},$$

$$A_i = \{\mathbf{x} \in C_t \setminus \bigcup \mathcal{D}_{n,k} : x_i = 0\}, \quad 1 \le i \le n. \tag{6}$$

It is clear that $A_0 \cup A_n = C_t \setminus \bigcup \mathcal{B}_{n,k}$ and then $|A_0 \cup A_n| = \chi(\mathcal{B}_{n,k}, t)$. For $1 \le i \le n-1$, an argument similar to the one in the case $\mathcal{A} = \mathcal{B}_{n,k}$ shows that the number of choices for $x_1, x_2, \ldots, x_i, \ldots, x_n$ with x_i being zero is $(t-1)(t-3)\cdots(t-2n+5)(t-k)$ which is independent of i. This gives

$$\begin{aligned} |C_t \setminus \bigcup \mathcal{D}_{n,k}| &= |A_0 \cup A_n| + (n-1)|A_1| \\ &= (t-1)(t-3)\cdots(t-2n+5)(t-n+2)(t-k). \end{aligned}$$

Secondly, we suppose that $n \le k \le 2n-2$. Then $\mathbf{x} = (x_1, \ldots, x_n) \in C_t \setminus \bigcup \mathcal{D}_{nk}$ has the following properties:

1. $x_i \ne \pm x_j$ for all i, j with $1 \le i < j \le n-1$,

2. $x_i \ne x_n$ for all i with $1 \le i \le n-1$,

3. $x_i \ne -x_n$ for all i with $1 \le i \le k-n+1$,

We also partition $C_t \setminus \bigcup \mathcal{D}_{n,k}$ into $A_0, A_1, \ldots, A_n$ as in (6) and obtain $|A_0 \cup A_n| = \chi(\mathcal{B}_{n,k}, t)$. For the remaining parts, there are always $(t-1)(t-3)\cdots(t-2n+5)$ choices for $x_1, x_2, \ldots, x_{n-1}$. Then by (3), x_n can be chosen in $t-(n-1)-(k-n) = t-k+1$ ways if $1 \le i \le k-n+1$, and in $t-(n-1)-(k-n+1) = t-k$ ways if $k-n+2 \le i \le n-1$. So we see that if $1 \le i \le k-n+1$, then

$$|A_i| = (t-1)(t-3)\cdots(t-2n+5)(t-k+1);$$

and if $k-n+2 \le i \le n-1$, then

$$|A_i| = (t-1)(t-3)\cdots(t-2n+5)(t-k).$$

This produces

$$\begin{aligned} |C_t \setminus \bigcup \mathcal{D}_{n,k}| &= |A_0 \cup A_n| + (k-n+1)|A_1| + (2n-k-2)|A_{n-1}| \\ &= (t-1)(t-3)\cdots(t-2n+5)(t-k+1)(t-n+1). \end{aligned}$$

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Next, we interpolate between $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$, or $\mathcal{A}_{n-1}$ and $\mathcal{D}_n$. Also since in these cases the order in which the hyperplanes are added is important, we will add the hyperplanes of $\mathcal{B}_n \setminus \mathcal{A}_{n-1}$ (or of $\mathcal{D}_n \setminus \mathcal{A}_{n-1}$) to $\mathcal{A}_{n-1}$ in some special orders. First, we require some notation. We follow the notation in [3].

Consider the following sets of ordered pairs

$$\begin{aligned} &T = \{(i,j):\ 1 \le i < j \le n\}, \\ &T_c = T \cup \{(0,l) : 2 \le l \le n+1\}, \\ &T_r = T \cup \{(k,n+1) : 1 \le k \le n\}. \end{aligned}$$

We place a total order on T_c and T_r, respectively, by defining

$$(i,j) \le_c (k,l) \ \text{ if and only if } j < l, \text{ or } j = l \text{ and } i \le k; \tag{7}$$

and

$$(i,j) \le_r (k,l) \ \text{ if and only if } i < k, \text{ or } i = k \text{ and } j \le l. \tag{8}$$

Now we consider the interpolations between $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$. We define

$$\mathcal{AB}^c_{n,k,l} = \mathcal{A}_{n-1} \cup \{x_i = 0 : 1 \le i \le n\} \cup \{x_i = x_j : i < j \text{ and } (i,j) \le_c (k,l)\},$$

where $\le_c$ is as in (7), and

$$\mathcal{AB}^r_{n,k,l} = \mathcal{A}_{n-1} \cup \{x_i = 0 : 1 \le i \le n\} \cup \{x_i = x_j : i < j \text{ and } (i,j) \le_r (k,l)\}$$

where $\le_r$ is as in (8). Note that we always assume that $i < j$.

For the interpolations between $\mathcal{A}_{n-1}$ and $\mathcal{D}_n$, we let

$$\mathcal{AD}^c_{n,k,l} = \mathcal{A}_{n-1} \cup \{x_i = x_j : i < j \text{ and } (i,j) \leq_c (k,l)\},$$

where $\leq_c$ is defined by (7), and

$$\mathcal{AD}^r_{n,k,l} = \mathcal{A}_{n-1} \cup \{x_i = x_j : i \leq j \text{ and } (i,j) \leq_l (k,l)\},$$

where $\leq_r$ is defined by (8).

Since the remaining results in this section can be verified in a manner similar to those obtained above, we omit the proofs.

Theorem 3.2 *The characteristic polynomials of $\mathcal{AB}^c_{n,k,l}$ and $\mathcal{AB}^r_{n,k,l}$ are*

$$\chi(\mathcal{AB}^c_{n,k,l}, t) = (t-1)(t-3)\cdots(t-2l+3)(t-k-l)(t-l-1)\cdots(t-n),$$

$$\begin{aligned}\chi(\mathcal{AB}^r_{n,k,l}, t) &= (t-1)(t-3)\cdots(t-2u+1)(t-k-l) \\ &\cdot(t-2k-2)(t-2k-4)\cdots(t-2w),\end{aligned}$$

where $u = \lceil \frac{n+k-1}{2} \rceil$ *and* $w = \lfloor \frac{k+n-1}{2} \rfloor$.

Theorem 3.3 *The characteristic polynomial of $\mathcal{AD}^c_{n,k,l}$ and $\mathcal{AD}^r_{n,k,l}$ are*

$$\begin{aligned}\chi(\mathcal{AD}^c_{n,k,l}, t) &= (t-1)(t-3)\cdots(t-2l+5)(t-k-l+2) \\ &\cdot(t-l+1)(t-l)\cdots(t-n+1),\end{aligned}$$

$$\begin{aligned}\chi(\mathcal{AD}^r_{n,k,l}, t) &= (t-1)(t-3)\cdots(t-2u+5)(t-2u+3) \\ &\cdot(t-l-k+2)(t-2k)\ (t-2k-2)(t-2k-4) \\ &\cdots(t-2w+2)(t-n+1),\end{aligned}$$

where $u = \lceil \frac{n+k-1}{2} \rceil$ *and* $w = \lfloor \frac{k+n-1}{2} \rfloor$.

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